Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 22, 2016, No. 2, 58–63

Sequences obtained from $x^2 \pm y^2$

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Received: 15 July 2015 **Accepted:** 6 May 2016

Abstract: Integers in class $\bar{3}_4$ of the modular ring Z_4 equal $x^2 - y^2$ but not $x^2 + y^2$ whereas integers in class $\bar{1}_4$ can equal both $x^2 + y^2$ and $x^2 - y^2$. This structure generates an infinity of sequences with neat curious patterns.
Keywords: Modular rings, Golden Ratio, Infinite series, Binet formula, Right-end-digits, Fibonacci sequence, Meta-Fibonacci sequences.
AMS Classification: 11B39, 11B50.

1 Introduction

We have previously shown that primes in class $\bar{1}_4$ of the modular ring Z_4 (Table 1) equal $x^2 + y^2$ and that there is only one such (x, y) couple [2, 3] and this couple has no common factors.

Row $r_i \downarrow$	**Class** $i \rightarrow$	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$	**Comments**
0		0	1	2	3	• $N = 4r_i + i$
1		4	5	6	7	• even $\bar{0}_4, \bar{2}_4$
2		8	9	10	11	• $(N^n, N^{2n}) \in \bar{0}_4$
3		12	13	14	15	• odd $\bar{1}_4, \bar{3}_4$; $N^{2n} \in \bar{1}_4$

Table 1. Classes and rows for Z_4

On the other hand, composites in this class have either one couple with common factors or multiple couples; the same number as the factors. These couples can be predicted via the simple equation [2]

$$x, y = \tfrac{1}{2}\left(A \pm \sqrt{2N - A^2}\right) \tag{1.1}$$

in which N is the integer and $A = x + y$ with x odd and y even. For Fibonacci primes the (x, y) couple is given by [4]:

$$F_p = F^2_{\frac{p+1}{2}} + F^2_{\frac{p-1}{2}}. \tag{1.2}$$

This identity itself follows from Equation 8 of [5]. Here we show that integers in both $\bar{1}_4$ and $\bar{3}_4$ equal a difference of squares, $x^2 - y^2$.

- In class $\bar{3}_4$ there are no even powers (Table 1) so that x is always even
  - $\bar{0}_4 - \bar{1}_4 = 4(r_0 - r_1 - 1) + 3$.
  - only the primes have one couple for $x^2 - y^2$.
- In class $\bar{1}_4$ primes and squares have only one (x, y) couple for $x^2 - y^2$.

2 Class $\bar{3}_4$

For prime, p, it can be shown that $(x + y) = A = p$, and with

$$p = (x^2 - y^2) = (x - y)A,$$

then $x - y = 1$, so that x and y are more easily calculated from

$$x = \tfrac{1}{2}(p + 1), \tag{2.1}$$

and

$$y = \tfrac{1}{2}(p - 1). \tag{2.2}$$

For composites there are the same number of (x, y) couples as factors. One (x, y) couple satisfies Equations (2.1) and (2.2). If there are two factors, then there will be another (x, y) couple apart from the one which satisfies Equations (2.1) and (2.2). If there are five factors, then there will be another four (x, y) couples.

The A values equal a factor of N or multiples of them as in Tables 2 and 3 for $N^* = 1$ and 7 (in which N^* indicates the right-end-digit (RED) of N). Similar results may be found for $N^* = 3$, 5 and 9. The others are calculated from

$$x = \tfrac{1}{2}(A + \tfrac{N}{A}), \tag{2.3}$$

and

$$y = \tfrac{1}{2}(A - \tfrac{N}{A}). \tag{2.4}$$

A is taken from the largest factor (or multiples of factors). Hence with

$$(x + y) = A$$

and

$$(x - y) = N/A,$$

we have (Tables 2 and 3)

$$(x - y)(x + y) = N.$$

		(2.3), (2.4)		**(2.1), (2.2)**	
N	**Factors**	***x*, *y***	***A***	***x*, *y***	***A***
231	3, 7, 11	(20, 13)	33	(116, 115)	231
		(16, 15)	21		
1011	3, 337	(170, 167)	337	(506, 505)	1011
2751	3, 7, 131	(200, 193)	393	(1376, 1375)	2751
		(76, 55)	131		
3291	3, 1097	(550, 547)	1097	(1646, 1645)	3291
4431	3, 7, 211	(320, 313)	633	(2216, 2215)	4431
		(116, 95)	211		
6511	17, 383	(200, 183)	383	(3256, 3255)	6511
20031	3, 11, 607	(916, 905)	1821	(10016, 10013)	20031
		(320, 287)	607		
58811	23, 2557	(1290, 1267)	2557	(29406, 29405)	58811
109311	3, 83, 439	(700, 617)	1317	(54656, 54655)	109311
		(344, 95)	439		
389871	3, 3, 43319	(21664, 21655)	43319	(194936, 194935)	389871
		(64980, 64977)	129957		
535971	3, 19, 9403	(4730, 46723)	9403	(267986, 267985)	535971
		(14114, 14995)	28209		

Table 2. $N \in \bar{3}_4$

		(2.3), (2.4)		**(2.1), (2.2)**	
N	**Factors**	***x*, *y***	***A***	***x*, *y***	***A***
87	3, 29	(16, 13)	29	(44, 43)	87
187	11, 17	(14, 3)	17	(94, 93)	187
427	7, 61	(34, 27)	61	(214, 213)	427
1687	7, 241	(124, 117)	241	(844, 843)	1687
4047	3, 19, 71	(116, 97)	213	(2024, 2023)	4047
		(64, 7)	71		
6187	23, 269	(146, 123)	269	(3094, 3093)	6187
24027	3, 8009	(4006, 4003)	8009	(12014, 12013)	24027
53367	3, 17789	(8896, 8893)	17789	(26684, 26683)	53367
110667	3, 37, 997	(1514, 1477)	2991	(55334, 55333)	110667
		(554, 443)	997		
389187	3, 3, 43243	(64866, 64863)	129729	(94594, 94593)	389187
		(21626, 216517)	43243		
533867	149, 3583	1866, 1717)	3583	(266934, 266933)	533867

Table 3. $N \in \bar{3}_4$

3 Class $\bar{1}_4$

For class $\bar{1}_4$ primes and squares have only one (x, y) couple for $(x^2 - y^2)$. Composites have the same number of couples as there are factors. Some examples are given in Table 4 for $N^* = 7$; similar results can be obtained for $N^* = 1, 3, 5$ and 9.

To calculate $(x^2 - y^2)$ from the factors, we use Equations (2.3) and (2.4), while for primes and for first couples for composites, we use Equations (2.1) and (2.2). For $(x^2 + y^2)$ we use Equation (1.1). Here we use the (x, y) couples from a $\bar{3}_4$ integer for $(x^2 + y^2)$ integers.

		(2.3), (2.4)		**(2.1), (2.2)**	
N	**Factors**	***x, y***	***A***	***x, y***	***A***
77	7, 11	(9, 2)	11	(39, 38)	77
117	3, 39	(16, 13)	39	(59, 58)	117
357	3, 119	(61, 58)	119	(179, 178)	357
1677	3, 13, 43	(71, 58)	129	(839, 838)	1677
		(41, 2)	43		
4017	3, 13, 103	(161, 148)	309	(2009, 2008)	4017
		(71, 32)	103		
6177	3, 7, 19, 161	(805, 322)	1127	(3089, 3088)	6177
		(308, 175)	483		
		(280, 117)	397		
26057	71, 367	(219, 148)	367	(13029, 13028)	26057
53357	233, 229	(231, 2)	233	(26679, 26678)	53357
121017	3, 13, 3103	(20171, 20168)	40339	(60509, 60508)	121017
		(1571, 1532)	3103		
389917	11, 35447	(17729, 17718)	35447	(194959, 194958)	389917
535997	7, 11, 6961	(24367, 24360)	48727	(267999, 267998)	535997
		(3519, 3442)	6961		

Table 4. $N \in \bar{1}_4$

4 Sequences from $(x^2 \pm y^2)$

Starting with a class $\bar{3}_4$ integer N_1 with $\left(x_1^2 - y_1^2\right)$, the $(x_1\, y_1)$ couple may be used to generate an integer from class $\bar{1}_4$, N_2 say, using $\left(x_1^2 + y_1^2\right)$. N_2 will also have an x_2, y_2 couple for N_3, $\left(x_2^2 - y_2^2\right)$, and so on.

Thus, a sequence of integers can be generated to infinity (Table 5). For example, in the first row of Table 2, the composite 231 has three factors, (3, 7, 11), so that there will be three (x, y)

couples (16, 5), (20, 13) and (116, 115). Thus three sequences can be generated from 231. Such a sequence with $N_1 = 231$ is shown in Table 6. An infinity of sequences can be generated in this way as in Figure 1. Of course, the $(x^2 + y^2)$ derived sequences will all be in class $\bar{1}_4$.

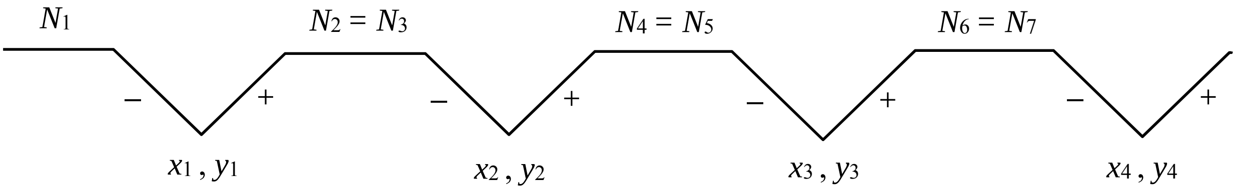

Figure 1. General oscillating sequence

N_1	x_1, y_1	A	N_2	FACTORS	x_2, y_2	A
231	16, 5	21	281(p)	—	141, 140	281
	20, 13	33	569(p)	—	285, 284	569(p)
1011	170, 167	2337	56789	521, 109	315, 206	521
2751	76, 55	131	8801	13, 677	345, 331	677
	200, 193	393	77249(p)	—	38625, 38624	77249
3291	550, 547	1097	601709	53, 11353	5703, 5650	11353
4431	116, 95	211	22481(p)	—	11241, 11240	22481
	320, 313	633	200369	13, 15413	7713, 7700	15413
6511	200, 183	383	73489	13, 5653	2833, 2820	5653
20031	916, 905	1821	1658081	37, 41, 1093	20241, 20200	40441
					1305, 2121	1517
	320, 287	607	184819	421, 439	430, 9	439
58811	1290, 1267	2557	3269389	17, 192317	96167, 96150	192317
109311	344, 95	439	137361	13, 97, 101	705, 608	1313
	700, 617	1317	870689			
535971	14114, 14095	281209	397874021	53, 7507057	3753555, 3753502	7507057
	4730, 4673	9403	44209829(p)	—	22104915, 22104915	44209829

Table 5. Primes: Equations (2.1, 2.2);
Composites: Equations (2.3, 2.4)

5 Concluding example

i	j	N_i	x_j, y_j	$\mathrm{Sgn}(x^2 \pm y^2)$
1	1	231	16, 5	–1
2	1	281	16, 5	+1
3	2	281	141, 140	–1
4	2	39481	141, 140	+1
5	3	39481	1525, 1512	–1
6	3	4611769	1525, 1512	+1
7	4	4611769	10013, 9780	–1
8	4	195908569	10013, 9780	+1

Table 6. Example of an oscillating sequence of the type:
{231, 281, 281, 39481, 39481, 4611769, 4611769, 195908569, 195908569, ...}

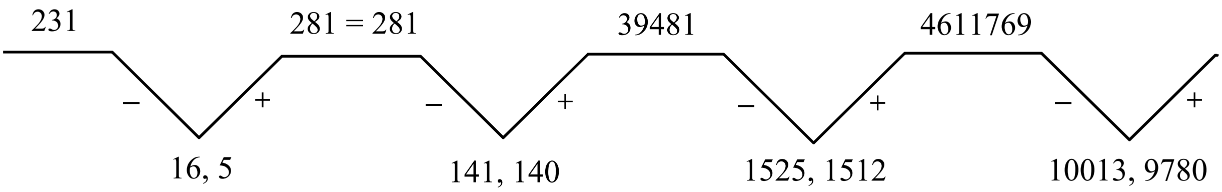

Figure 2. Oscillating sequence for $N_1 = 231$

There is in a sense in which these oscillating sequences are also related to pulsated Fibonacci recurrence sequences [1].

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