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Catalan triangles and Finucan's hidden folders

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Abstract: This note is to capture and extend some of Finucan's ideas for further exploration by students. These ideas connect with several elementary concepts in combinatorial analysis which lend themselves to undergraduate research projects.
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1 Introduction

Henry Finucan (1917–1983) was Reader in Mathematics at the University of Queensland for many years, having studied there and at Oxford and Cambridge [19]. He was an inspirational teacher, with unorthodox insights, and a great encourager of young mathematicians (which is how I met him fifty years ago).

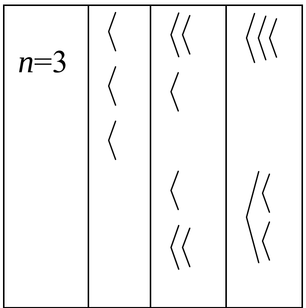
The purpose of this note is to capture, extend and expound some of his ideas for further development by interested researchers. In particular, his manila folder realisation [7, 8, 9] is an excellent start for studying the Catalan numbers, Young tableaux and other number triangles which can then be extended into projects to investigate non-negative zero-return random walks. Finucan's approach was very much in the hands-on spirit of the Belgian mathematician, Eugène Catalan [5].

2 Context

We are asked to imagine n manila folders with h folders within them: dated I realise but quite concrete! Each situation produces various configurations (Figures 1 and 2). Furthermore, suppose v spines of the folders are visible from the rear and that u folders are unoccupied as in the tableaux of Figures 3 and 4. Note that $h = n - v$.

The reader is invited to try the cases for $n = 6$ and 7 to get a feeling for what is going on; compare your results with Figures 5 and 6. In the lower triangular arrays of Figures 3, 4, 5, 6, the range of values of (u, v) is $u = 1$ to n, and $v = 1$ to u.

We shall use Finucan's symbol ${}^{n}F_{h}$ to represent the number of such configurations, so that the total number summed over all v is the Catalan number c_n, as we shall see. In the two-way decompositions of Figures 3, 4, 5, 6, these Catalan numbers appear in the bottom right hand (total) cell. Now continue to unravel these patterns and you should finish up with the Catalan triangle of hidden folders (Figure 7) in which the Catalan numbers appear along the backward sloping boundary diagonal.

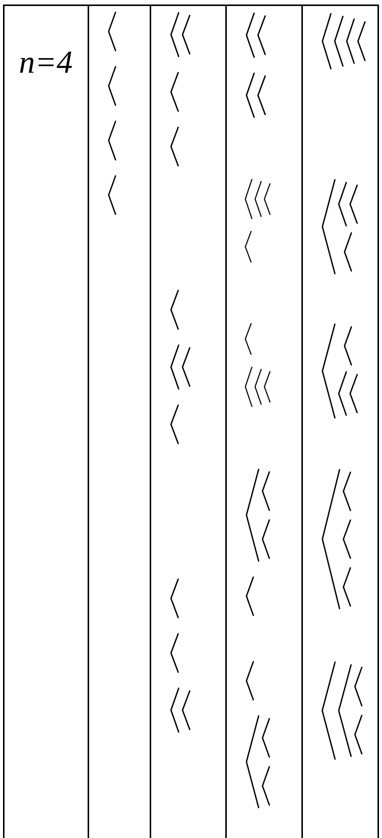


h	0	1	2
c	1	2	2

Figure 1. Spines

↑ v					
3			1	‖	1
2		2	0	‖	2
1	1	1	0	‖	2
	1	2	3	‖	5
	u →				

Figure 3. F values for $n = 3$



h	0	1	2	3
c	1	3	5	5

Figure 2. Visible spines

c represents the number of configurations;

h represents the number of internal folders ($h \geq 0$);

n represents the total number of folders ($n > 0$).

4				1	1
3			3	0	3
2		3	2	0	5
1	1	3	1	0	5
	1	2	3	4	14
		u →			

Figure 4. F values for $n = 4$

v ↑							
6						1	1
5					5	0	5
4				10	4	0	14
3			10	15	3	0	28
2		5	20	15	2	0	42
1	1	10	20	10	1	0	42
u →	1	2	3	4	5	6	132

Figure 5. F values for $n = 6$

v ↑								
7								1
6						6	0	6
5					15	5	0	20
4				20	24	4	0	48
3			15	45	27	3	0	90
2		6	40	60	24	2	0	132
1	1	15	50	50	15	1	0	132
u →	1	2	3	4	5	6	7	429

Figure 6. F values for $n = 7$

1								
1	1							
1	2	2						
1	3	5	5					
1	4	9	14	14				
1	5	14	28	42	42			
1	6	20	48	90	132	132		
1	7	27	75	165	297	429	429	
1	8	35	110	275	572	1001	1430	1430

Figure 7. Catalan triangle of hidden folders

Other features include

- row sums: Catalan numbers (also on last and second last backward diagonals) [22],
- sums along forward diagonals: 1, 1, 2, 3, 6, 10, 20,
- partial column sums appear in [25] .

3 Recurrence relations

It can be observed that the elements, ${}^{n}F_{h}$, seem to be related by the partial recurrence relation

$$ {}^{n}F_{h} = {}^{n}F_{h-1} + {}^{n-1}F_{h} \tag{3.1} $$

with boundary conditions – a topic for further research. This recurrence relation also appears in the context of coloured trees [24] which leads on to spanning trees in general [3]. A solution of this recurrence relation can be confirmed to be

$$ {}^{n}F_{h} = \frac{n-h}{n}\begin{bmatrix} n \\ h \end{bmatrix} $$

in which $\begin{bmatrix} n \\ h \end{bmatrix}$ is the negative binomial coefficient defined by

$$ \begin{bmatrix} n \\ h \end{bmatrix} = \frac{n(n+1)(n+2)...(n+h-1)}{h!} $$

with the rising factorial function in the numerator [12]. For example,

$$^4F_2 = \frac{2}{4}\left(\frac{4\times 5}{1\times 2}\right) = 5.$$

Finucan also explored further related ideas about $F(n;\, u,\, v)$, the number of configurations at $(u,\, v)$ for which he explained the physical meaning of the recurrence relation

$$F(n;u,v) = F(n-1;u-1,v-1) + F(n-1;u,v) + \left(F(n:u,v+1) - F(n-1;u-1,v)\right), \quad (3.2)$$

which has boundary conditions of the Catalan numbers and zeroes and which has a solution

$$F(n;u,v) = \frac{v}{n-u}\binom{n-1}{u}\binom{n-v-1}{u-v}, \quad (3.3)$$

which can be readily confirmed with any of the previous examples. The parentheses in the recurrence relation are there to help in explaining the physical significance of the terms in relation to the folders.

Larcombe [18] has used the definition of the $(n + 1)^{\text{th}}$ Catalan polynomial, $P_n(x)$, $n \geq 0$,

$$P_n(x) = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor}\binom{n-i}{i}(-x)^i, \quad (3.4)$$

from which we can obtain another type of Catalan triangle from the absolute values of the coefficients if powers of x in the polynomial:

1				
1				
1	1			
1	2			
1	3	1		
1	4	3		
1	5	6	1	
1	6	10	4	
1	7	15	10	1
1	8	21	20	5

Figure 8. Catalan polynomial coefficients triangle

Elements in the n^{th} column and j^{th} row are related by the partial recurrence relation

$$\binom{n}{j} = \binom{n}{j-1} + \binom{n-1}{j-2}$$

By analogy with some of the approaches to properties of the Pascal triangle we observe that

- The sum along the leading diagonals is the sequence {1, 1, 2, 3, 4, 6, 9, 13, 19, 28, …}, which is Narayana's sequence [6, 21], named after a 14th century Indian mathematician and is in practice actually defined by the third order homogenous linear recurrence relation [15, 20]

$$v_n = v_{n-1} + v_{n-3},\ v_1 = v_2 = v_3 = 1;$$

- The sums along the rows are the elements of the Fibonacci sequence {1, 1, 2, 3, 5, 8, …};
- The partial sums of the n^{th} column, $n = 1, 2, 3, \ldots$, are sequences $\{u_{n,j}\}$ defined by the partial recurrence relation with unit boundaries

$$u_{n,j} = u_{n,j-1} + u_{n-1,j-2}.$$

4 Concluding comments

The recurrence relations in this note can be extended along the lines of the binomial coefficient numbers suggested in [11]. Proofs need to be developed for the partial recurrence relation (3.2), the solution to which in (3.3) has other combinatorial interpretations [3]. The Catalan numbers also appear on diagonals of Young Tableaux [10] in a landmark paper by Leonard Carlitz and John Riordan [4] which can stimulate advanced undergraduate projects with q-generalizations [1]. The tracking of the number relations leads naturally into the study of patterns of lattice paths in combinatorics. The Catalan numbers can be related to Stirling numbers [17, 22] which, in turn, can lead back to the well-known Pascal triangle [27]. An encyclopaedic source of results to stimulate further related ideas may be found in Henry Gould's bibliography [12].

It can be seen in this note that these ideas have the attraction of lending themselves to concrete counting of elements, before trying to generalize: that is, guess and test your guess – a necessary, but not a sufficient, part of research in mathematics and an important ingredient in the learning of undergraduate mathematics [23]. A further step, not pursued here, is to try to prove that the results are generally verifiable [26]. Ideas for further development of the triangles in this paper may be found in Vern Hoggatt [14]. More particularly, Koshy has explored somewhat similar triangles associated with another Cambridge mathematician, another 'Henry' (Henry Forder), who, like Finucan, was also an excellent teacher and who also worked in the antipodes (albeit New Zealand) [16]!

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