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On irrationality of some distances between points on a circle

Mladen Vassilev–Missana

5 Victor Hugo Str, Ap. 3
1124 Sofia, Bulgaria
e-mail: missana@abv.bg

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Abstract: Let $n > 3$ be an arbitrary integer. In the present paper, it is shown that if K is an arbitrary circle and M_i, $i = 1, \ldots, n$, are points on K, dividing K into n equal arcs, then for each point M on K, different from the mentioned above, at least $\lfloor \frac{n}{3} \rfloor$ of the distances $|MM_i|$ are irrational numbers.
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1 Preliminaries

Lemma. *Let $n > 3$ be an integer. Then the number $\cos \frac{\pi}{n}$ is irrational.*

Proof. According to a well-known formula of de Moivre, [2], for $0 < \varphi < \frac{\pi}{2}$:

$$(\cos \varphi + i \sin \varphi)^n = \cos(n\varphi) + i \sin(n\varphi)$$

$$(\cos \varphi - i \sin \varphi)^n = \cos(n\varphi) - i \sin(n\varphi)$$

By pairwise addition of the above we get:

$$\cos(n\varphi) = \frac{(\cos \varphi + i \sin \varphi)^n + (\cos \varphi - i \sin \varphi)^n}{2} \tag{1}$$

Putting

$$\cos \varphi = x \tag{2}$$

we have: $0 < x < 1, \sin\varphi = \sqrt{1-x^2}, \varphi = \arccos x$ and (1) gives

$$\cos(n \arccos x) = \frac{(x + i\sqrt{1-x^2})^n + (x - i\sqrt{1-x^2})^n}{2} \tag{3}$$

The right hand-side of (3) is polynomial of x of power n, known as the n-th Tchebychev polynomial of the first kind [3] and usually denoted by $T_n(x)$. Obviously

$$T_n(x) = \cos(n \arccos x) = \frac{(x + i\sqrt{1-x^2})^n + (x - i\sqrt{1-x^2})^n}{2}. \tag{4}$$

From (4) we obtain the representation

$$T_n(x) = \cos(n \arccos x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} (1-x^2)^k \tag{5}$$

Comparing (1) and (5) we have:

$$\cos(n\varphi) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} (1-x^2)^k, \tag{6}$$

where φ and x are related through (2).

Let $n > 3$ and let us put in (6)

$$\varphi = \frac{\pi}{n}, \tag{7}$$

assuming that the number $x = \cos\frac{\pi}{n}$ is rational.

Let

$$x = \frac{p}{q}, \tag{8}$$

where p and q are integers such that:

$$0 < p < q;\ \gcd(p, q) = 1.$$

From (6), (7) and (8) we find:

$$-1 = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} \left(\frac{p}{q}\right)^{n-2k} \left(1 - \frac{p^2}{q^2}\right)^k$$

or in other form:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} p^{n-2k} \left(q^2 - p^2\right)^k = -q^n. \tag{9}$$

Expanding $(q^2 - p^2)^k$ by the Newton binomial we obtain:

$$(q^2 - p^2)^k = \begin{cases} (-1)^k p^{2k} + \alpha q^2, \text{ for } k \geq 1 \\ 1, \text{ for } k = 0 \end{cases}, \tag{10}$$

where α is an integer constant.

Equation (10) substituted in (9) gives:

$$p^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} + \beta q^2 = -q^n, \tag{11}$$

where β is integer constant.

From the equations:

$$2^n = (1+1)^n = \sum_{t=0}^{n} \binom{n}{t}$$

$$0 = (1-1)^n = \sum_{t=0}^{n} (-1)^t \binom{n}{t},$$

it follows

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = 2^{n-1}, \tag{12}$$

Substituting (12) in (11) we find

$$2^{n-1} p^n + \beta q^2 = -q^n. \tag{13}$$

Since $\gcd(p, q) = 1$, from (13) it follows that q divides 2^{n-1}. Hence,

$$q = 2^s, \tag{14}$$

where $1 \leq s \leq n-1$.

Let us first look at the case

I. n is odd.

Then we have

$$n - 2k \geq 1.$$

Therefore, for $p > 1$ the left hand side of (9) is divisible by p, while the right hand side is not. Hence, for $p > 1$ (9) is impossible.

Let $p = 1$. Then

$$\cos\frac{\pi}{n} = \frac{p}{q} = \frac{1}{2^s}, \tag{15}$$

according to (14). Since $n > 3$, then we should have

$$\cos\frac{\pi}{n} > \cos\frac{\pi}{3} = \frac{1}{2}. \tag{16}$$

From (15) and (16) it follows:

$$\frac{1}{2^s} > \frac{1}{2}.$$

But the last is obviously false. Hence we proved that if n is odd and $n > 3$, then $\cos\frac{\pi}{n}$ is an irrational number.

II. n is even.

Then $n = 2T$ and $T \geq 2$, because $n \geq 4$. Substituting in (9) we have

$$\sum_{k=0}^{T}(-1)^k \binom{2T}{2k} (p^2)^{T-k} \left(q^2 - p^2\right)^k = -q^{2T}. \tag{17}$$

Now we rewrite (17) in the form

$$\sum_{k=0}^{T-1}(-1)^k \binom{2T}{2k} (p^2)^{T-k} \left(q^2 - p^2\right)^k = -q^{2T} - (-1)^T (q^2 - p^2)^T. \tag{18}$$

Let T be even. Then if we denote by L, the left hand side of (18), we have

$$L \equiv 0 \pmod{p}$$

If we denote by R, the right hand side of (18), we have

$$R \equiv -2q^{2T} \pmod{p}$$

But $q = 2^s$ and $\gcd(p, q) = 1$, which makes the last congruence impossible. Therefore, we have $p = 1$ and as in **I.**, we conclude that $\cos \frac{\pi}{n}$ is an irrational number.

Let T be odd. Then (18) may be rewritten as

$$\sum_{k=0}^{T-1}(-1)^k \binom{2T}{2k} (p^2)^{T-k} \left(q^2 - p^2\right)^k = (q^2 - p^2)^T - (q^2)^T. \tag{19}$$

Let R^* be the right hand side of (19), we have:

$$R^* = (q^2 - p^2)^T - (q^2)^T = \left((q^2 - p^2) - q^2\right) R^{**}$$

where

$$R^{**} = \sum_{k=0}^{T-1}(q^2 - p^2)^{T-k-1}(q^2)^k.$$

Thus, finally we find:

$$R^* = -p^2 \sum_{k=0}^{T-1}(q^2 - p^2)^{T-k-1}(q^2)^k.$$

Then, dividing by p^2 both sides of (19), we obtain:

$$\sum_{k=0}^{T-1}(-1)^k \binom{2T}{2k} (p^2)^{T-k-1} \left(q^2 - p^2\right)^k = -\sum_{k=0}^{T-1}(q^2 - p^2)^{T-k-1}(q^2)^k. \tag{20}$$

On the left hand side of (20) we single out the term corresponding to $k = 0$, and on the right hand side the one corresponding to $k = T - 1$, which allows us to rewrite (20) in the form.

$$(p^2)^{T-1} + (q^2)^{T-1} = \gamma. \left(q^2 - p^2\right), \tag{21}$$

where γ is an integer constant.

From (21) it follows that

$$2(q^2)^{T-1} = \gamma.\left(q^2 - p^2\right) + (q^2)^{T-1} - (p^2)^{T-1}.$$

The last equality means, that there exists an integer constant δ (since $(q^2)^{T-1} - (p^2)^{T-1} \equiv 0 \pmod{(q^2 - p^2)}$ for T odd), for which

$$2(q^2)^{T-1} = \delta.\left(q^2 - p^2\right), \tag{22}$$

We rewrite (22) in the form:

$$2^{2sT-2s+1} = \delta.\left(q^2 - p^2\right). \tag{23}$$

Equality (23) is however impossible, since $q = 2^s$, p is odd number and therefore $q^2 - p^2$ is an odd number greater than 1. Therefore, there exist no p and q, for which $\cos\frac{\pi}{n} = \frac{p}{q}$. Therefore, $\cos\frac{\pi}{n}$ is an irrational number.

Hence the Lemma is proved. □

2 Main results

Theorem 1. *Let $n > 3$ be an integer and K is an arbitrary circle. Let the points M_i, $i = 1, \ldots, n$, (taken clockwise by ascending magnitude of the indices) lie on K splitting it to n equal arcs. Let M be a point on K different from the mentioned above. Then at least $\lfloor\frac{n}{3}\rfloor$ of the distances $|MM_i|$, $i = 1, \ldots, n$, are irrational numbers.*

Proof. Without loss of generality we will assume that $M \in \widehat{M_1M_n}$ where this arc does not contain any of the points M_j, $j = 2, \ldots, n-1$. We consider the following triples of distances, as long as it is possible:

$$(|MM_1|, |MM_2|, |MM_3|), (|MM_4|, |MM_5|, |MM_6|), (|MM_7|, |MM_8|, |MM_9|), \ldots$$

Obviously, there are exactly $\lfloor\frac{n}{3}\rfloor$ such triples. And this triples share no common distance.

Let the triple $(|MM_i|, |MM_{i+1}|, |MM_{i+2}|)$ be one of the mentioned above. Consider the convex quadrilateral $MM_iM_{i+1}M_{i+2}$. It is inscribed in K. By Ptolemy theorem ([1]) we have:

$$|MM_i||M_{i+1}M_{i+2}| + |MM_{i+2}||M_iM_{i+1}| = |M_iM_{i+2}||MM_{i+1}|$$

Since the consecutive arcs are equal (from the conditions of the Theorem) we have

$$a \stackrel{\text{def}}{=} |M_iM_{i+1}| = |M_{i+1}M_{i+2}|$$

From the above we obtain

$$\frac{|MM_i| + |MM_{i+2}|}{|MM_{i+1}|} = \frac{|M_iM_{i+2}|}{a} \tag{24}$$

But the angles in the base of the isosceles triangle $M_iM_{i+1}M_{i+2}$ equal $\frac{\pi}{n}$. Then a trivial calculation shows

$$\frac{|M_iM_{i+2}|}{a} = 2\cos\frac{\pi}{n}. \tag{25}$$

From (24) and (25) follows

$$\frac{|MM_i| + |MM_{i+2}|}{|MM_{i+1}|} = 2\cos\frac{\pi}{n}. \qquad (26)$$

Now the assumption that all three distances: $|MM_i|$, $|MM_{i+1}|$, $|MM_{i+2}|$, are rational numbers leads to the conclusion that $\cos\frac{\pi}{n}$ is also rational. But according to the Lemma this is not true. Therefore, at least one of the the distances $|MM_i|$, $|MM_{i+1}|$, $|MM_{i+2}|$ is an irrational number.

Hence, in all of the considered triples, there is at least one distance which is an irrational number. The number of all triples is $\lfloor\frac{n}{3}\rfloor$. Therefore, at least $\lfloor\frac{n}{3}\rfloor$ of the distances $|MM_i|$, $i = 1, \ldots, n$, are irrational numbers.

The Theorem is proved. □

Remark. *The condition $n > 3$ is necessary for the validity of the Theorem. Indeed, if $n = 3$, and $M \in \widehat{M_1M_3}$ is the arc such that M_2 does not lie on it, then (from Ptolemy theorem)*

$$|MM_2| = |MM_1| + |MM_3|$$

and from this equality it may be easily seen, that there are infinitely many in number points M on K such that the distances: $|MM_1|$, $|MM_2|$, $|MM_3|$ are simultaneously rational.

References

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