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The definitive solution of Gauss's lattice points problem in the circle

Aldo Peretti

Murillo 1131, 9) "D" (1414) Buenos Aires, Argentina
e-mail: aldopperetti@gmail.com

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Abstract: By means of Bienayme's theorem of Statistics is found that the remainder term in Gauss's problem about the lattice points in the circle is a function normally distributed with mean value zero and standard deviation 1,10368 multiplied by the fourth root of x. This result can not be improved.

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Our starting point of departure will be the known formula

$$A(x) = \pi x + n\sqrt{x}\sum_{n=1}^{N}\frac{N(n)}{\sqrt{n}}J_1(2\pi\sqrt{nx} + O\left(\frac{x}{n}\right),$$

where $A(x)$ is their quantity of lattice points in the entire circle of diameter $\sqrt{x}$, $N(n)$ is the quantity of solutions of the Diophantine equation $x_1^2 + x_2^2 = n$, and $J_1(x)$ is the Bessel function of the first order. Here, any point in the border of the circle is counted with weight $\frac{1}{2}$.

Replacing the Bessel functions by their asymptotic approximations

$$J_1(2\pi\sqrt{nx}) \approx \frac{1}{\pi\sqrt[4]{nx}}\cos(2\pi\sqrt{nx} - \frac{3}{4}\pi),$$

we obtain that

$$A(x) \approx \pi x + \frac{4\sqrt[4]{x}}{\pi}\sum_{n=1}^{N}\frac{N(n)}{\cos}(2\pi\sqrt{nx} - \frac{3}{4}\pi). \qquad (A)$$

We formulate three hypotheses.

Hypothesis 1: The function $y = \cos a_0 x$ has mean value zero, whatever n and a_n.

Hypothesis 2: It varies at random between limits ± 1, as a_n and x have different values.

Hypothesis 3: $\cos(a_n x)$ has a distribution probability function that is constant.

Now, the variance of a function of constant probability in the interval (a, b) is known to be

$$\frac{(b-a)^2}{12}$$

and

$$\sigma = \frac{b-a}{\sqrt{12}}.$$

In our case, as $b = 1$ and $a = -1$, it turns out that

$$\sigma = \frac{2}{\sqrt{12}} = \frac{1}{\sqrt{3}} = 0.577.... \tag{B}$$

Besides, we have Bienayme's theorem concerning the variable sum of random independent variables, which states.

If $x = a_1 x_1 + a_2 x_2 + ... + a_m x_m$ (where x_i are independent random variables), then holds:

$$\overline{x} = a_1 \overline{x}_1 + a_2 \overline{x}_2 + ... + a_m \overline{x}_m, \tag{C}$$

where $\overline{x}_i$ represents mean values, and the standard deviation σ_x of the sum is given by

$$\sigma_x^2 = a_1^2 \sigma_{x_1}^2 + a_2^2 \sigma_{x_2}^2 + ... + a_m^2 \sigma_{x_m}^2. \tag{D}$$

We apply this theorem to the series

$$\sum_{n=1}^{N} \frac{N(n)}{n^{3/4}} \cos(2\pi\sqrt{nx} - \frac{3}{4}\pi),$$

choosing

$$a_n = \frac{N(n)}{n^{3/4}} \text{ and } x_n = \cos(2\pi\sqrt{nx} - \frac{3}{4}\pi).$$

The sum $\sum\limits_{n=1}^{N}$ is a sample of the universe $\sum\limits_{n=1}^{\infty}$, so that by the Central Limit Theorem has a normal distribution, defined by its mean value and its standard deviation (STD).

Then, by Hypothesis 1 and formula (A), the mean value is zero and the STD, according to (A), (B) and (D), is:

$$STD = \frac{1}{\sqrt{3}} \left\{ \sum_{n=1}^{N} \frac{N^2(n)}{n^{3/2}} \right\}^{1/2}.$$

The series between brackets is convergent, and its value, when $N \to \infty$, is (according to [1], Ch. 8, p. 251)

$$\frac{16\zeta^2(\frac{3}{2})L^2(\frac{3}{2})}{\zeta(3)(1+2^{3/2})}.$$

But

$$\zeta(\frac{3}{2}) = 2.6124..., \ \zeta(3) = 1.20206..., \ 1 + 2^{-3/2} = 1.13355...,$$

$$L(\frac{3}{2}) = \prod_{p\equiv 1 (\text{mod } n)} \frac{1}{1-\frac{1}{p^{3/2}}} \prod_{p\equiv 3 (\text{mod } n)} \frac{1}{1-\frac{1}{p^{3/2}}} = 0.8645...$$

from which follows

$$\sum_{n=1}^{\infty} \frac{N^2(n)}{n^{3/2}} = 2.2543...$$

and

$$STD \leq \frac{1}{\sqrt{3}} \left\{ \sum_{n=1} \frac{N^2(n)}{n^{3/2}} \right\}^{1/2} = 0.86683...$$

Returning to (A), we deduce that the probability that $A(x) - \pi x$ be comprised between the values $\pm t\frac{4}{\pi} STD \sqrt[4]{x}$ (or $\pm 1.1068 t \sqrt[4]{x}$), i.e.,

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-u^2/2} du.$$

In this theorem, neither the value $\frac{1}{4}$ for x, nor the value $1.1068...$ for the coefficient can be improved, so that the result is definitive. Figure 1 shows the excellent agreement between the preceding calculation and the numerical computations.

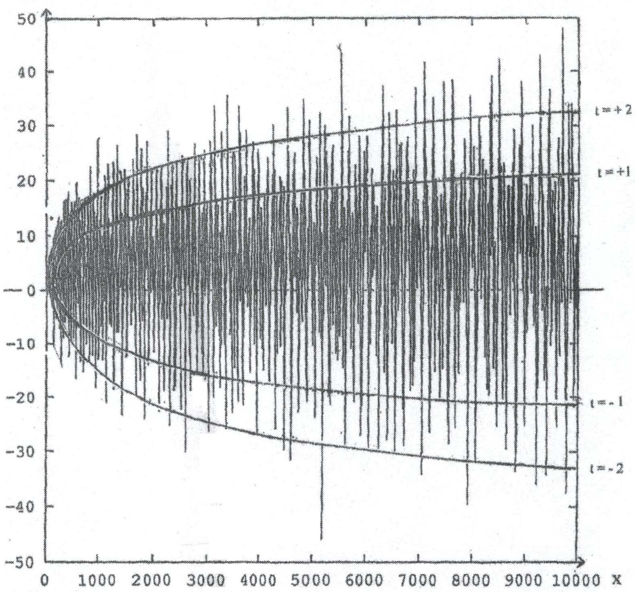

Figure 1: The function $R(x) = A(x) - \pi x$ in the range $1 \leq x \leq 10^5$. Between the two curves $t = \pm 1$ are comprised 68.268% of the values and between the two curves $t = \pm 2$ are comprised 95.450% of the values.

References

[1] Landau, E. (1947) Vorlesungen Über Zahlentheorie, Chelsea Publishing Company, New York.