Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 22, 2016, No. 1, 33–41

Some Golden Ratio generalized Fibonacci and Lucas sequences

J. V. Leyendekkers[1] and A. G. Shannon[2,3]

[1] Faculty of Science, The University of Sydney, NSW 2006, Australia

[2] Emeritus Professor, University of Technology Sydney, NSW 2007, Australia
[3] Campion College, PO Box 3052, Toongabbie East, NSW 2146, Australia
e-mails: t.shannon@campion.edu.au, tshannon38@gmail.com

Received: 5 November 2015 **Accepted:** 28 January 2015

Abstract: Various characteristics of the ordinary Fibonacci and Lucas sequences, many known for centuries, are shown to be common to generalized sequences related to the Golden Ratio. Periodicity properties are also investigated.
Keywords: Unit digits (right-end-digits), Modular rings, Golden Ratio, Fibonacci and Lucas numbers, Pythagorean triples.
AMS Classification: 11B39, 11B50.

1 Introduction

The German mathematician Johannes Kepler (1571–1630) showed that the ratio of consecutive Fibonacci numbers converges to the Golden Ratio [7]. This is also the case for the members of the Golden Ratio Family [4] associated with generalized Fibonacci sequences. Another property discovered by Kepler, but often attributed to the Scottish mathematician, Robert Simson (1687–1768) is that for the sequence of ordinary Fibonacci numbers $\{F_n\}$:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$$

which can also be expressed iteratively and with some redundancy as

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{2n}\left(F_{n-2}^2 - F_{n-3}F_{n-1}\right)$$

and this can be generalized to

$$F_n^2(a) - F_{n-1}(a)F_{n+1}(a) = r_1^2\left(F_{n-2}^2(a) - F_{n-3}(a)F_{n-1}(a)\right) \tag{1.1}$$

in which a is in Class $\bar{1}_4 \in Z_4$ (a modular ring) (Table 1) with row r_1 and $\{F_n(a)\}$ represents a generalized Fibonacci sequence, where the sequence of ordinary Fibonacci numbers can be expressed as $\{F_n(5)\}$ in this notation [3, 5]. The purpose of the paper is to explore some of the properties of $\{F_n(a)\}$, including period properties [8].

2 The generalized sequence $\{F_n(a)\}$

In this notation, the corresponding generalized Fibonacci numbers satisfy

$$F_{n+1}(a) = F_n(a) + r_1 F_{n-1}(a) \tag{2.1}$$

and any Golden Ratio family member is given by

$$\frac{F_n(a)}{F_{n-1}(a)} \to \varphi_a$$

in which

$$\varphi_a = \frac{1+\sqrt{a}}{2} \tag{2.2}$$

and the generalized Binet formula in this notation is [10]

$$F_n(a) = \frac{\left(\frac{1+\sqrt{a}}{2}\right)^n - \left(\frac{1-\sqrt{a}}{2}\right)^n}{\sqrt{a}}. \tag{2.3}$$

which is well-known for the Fibonacci numbers as

$$F_n = \frac{\left(\frac{(1+\sqrt{5})}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}. \tag{2. 4}$$

Row $r_i \downarrow$	**Class** $i \rightarrow$	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$	**Comments**
0		0	1	2	3	$N = 4r_i + i$
1		4	5	6	7	even $\bar{0}_4$, $\bar{2}_4$
2		8	9	10	11	$\left(N^n, N^{2n}\right) \in \bar{0}_4$
3		12	13	14	15	odd $\bar{1}_4, \bar{3}_4$; $N^{2n} \in \bar{1}_4$

Table 1. Classes and rows for Z_4

We note in passing that the Binet formula for the Fibonacci numbers is also usually attributed incorrectly to Jacques Philippe Marie Binet (1786–1856), but it was previously known to such famous mathematicians as Abraham de Moivre (1667–1754), Daniel Bernoulli

(1700–1782), and Leonhard Euler (1707–1783) [12]. Another way of generalizing the Binet formula, related to (2.3), may be found in [2].

When $r_1 = 7$, $a = 29$, and Table 2 lists the first ten elements of $\{F_n(29)\}$.

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	8	15	71	176	673	1905	6616	19951

Table 2. Some terms of $\{F_n(29)\}$

Thus when $n = 8$, Equation (1.1) becomes

$$\begin{aligned} F_8^2(29) - F_7(29)F_9(29) &= 1905^2 - 673\times 6616 \\ &= -82354 \\ &= 49\left(176^2 - 71\times 673\right) \\ &= 7^2\left(F_6^2(29) - F_5(29)F_7(29)\right). \end{aligned}$$

When $r_1 = 10$, $a = 41$, and the first ten elements of the sequence $\{F_n(41)\}$ are set out in Table 3.

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	11	31	131	341	1651	5061	21571	72181

Table 3. Some terms of $\{F_n(41)\}$

Thus when $n = 4$, Equation (1.1) becomes

$$\begin{aligned} F_4^2(41) - F_3(41)F_5(41) &= 31^2 - 11\times 131 \\ &= -1000 \\ &= 100\left(1^2 - 1\times 11\right) \\ &= 10^2\left(F_2^2(41) - F_1(41)F_3(41)\right). \end{aligned}$$

However, not all the ordinary Fibonacci properties seem to generalize to other Golden Ratio Fibonacci sequences. For instance, an interesting characteristic of φ_a is that the sum of any ten consecutive ordinary Fibonacci numbers (S_{10}) is always equal to 11 times the seventh number $F_7(5)$ [7]. This does not seem to extend to generalized Fibonacci sequences, though there are some “near misses” as seen in Table 4. This provokes the questions: are there analogues of the integer 11 and $F_7(5)$ results with these other sequences? What are the deeper underlying patterns?

r_1	a	S_{10}	i	Factors of $S_{10}+i$
1	5	143	0	11, 13 (=$F_7(5)$)
2	9	682	0	11, 2, 31
3	13	2053	4	11, 187
4	17	4826	3	11, 439
5	21	9715	–2	11, 883
6	25	17578	0	11, 2, 799
7	29	29417	–3	11, 2, 1337
8	33	46378	–2	11, 2^3, 527
9	37	69751	0	11, 17, 373
10	41	100970	–1	11, 9179
11	45	141613	1	11, 2, 6437
12	49	395602	2	11, 2^2, 3^5, 37

Table 4. Factors of $S_{10} + i$

3 Unit digit periods

Joseph Louis Lagrange (1786–1813) showed that the unit digit (right-end-digit, RED) for φ_5 repeats itself with a period of 60 [7]. The periodicity of these generalized Fibonacci numbers is more complex but is uniform for sequences with the same RED for all φ_a (Table 5).

r_1^*	a	b	c	d	e	f	g	h	i
0	5	1	1	–	–	–	–	–	–
1	9	60	60	20	19	14	3	3	9
2	13	4	4	–	–	–	–	–	–
3	17	24	24	8	4	–	–	–	–
4	21	6	6	–	–	–	–	–	–
5	25	3	3	–	–	–	–	–	–
6	29	20	20	13	3	–	–	–	–
7	33	12	12	8	4	–	–	–	–
8	37	21	23	7	8	–	–	–	–
9	5	6	6	–	–	–	–	–	–

Table 5. Periods for unit digits $F_n^* = F_m^* = 1$ (Superscript * for REDs)

Periods are 2-fold $(r_1^* = 2,4,5,9)$, 4-fold $(r_1^* = 3,6,7,8)$, and 6-fold $(r_1^* = 1,11,21,31,...)$. Their actual periodicities are generated by the formulas in Table 6. For instance, the period 2-fold periods when $a = 21$ are given by $n = 1 + 6t$, $n = 2 + 6t$, $t = 1, 2, 3, ...$.

j	n_j	m_j
1	$1 + bt$	$2 + ct$
2	$n_1 - d$	$m_1 - e$
3	$n_2 - e$	$1 + ct$
4	$n_3 - f$	$m_3 - g$
5	–	$m_4 - h$
6	–	$m_5 - i$

Table 6. Periodicity formulas for Table 5 (t = 1, 2,3, ...)

This structure applies to all other REDs. For example, for $F_n^*(a) = 7$, the starting numbers are 14 and 16, so the first seven numbers are, respectively, 14, 24, 34, ..., and 16, 26, 36, ..., with $F_n^*(a) = 7$ occurring at 74 and 76. The specific values of d, e, f, g, h, i will vary but will satisfy the expressions in Tables 5 and 6.

4 Pythagorean and Fibonacci triples

When members of the family of prime subscripted Fibonacci numbers belong to the class $\bar{1}_4$ and when F_p is itself prime, F_p equals a sum of squares as in (4.2) below. If F_p is composite and its factors do not all belong to the class $\bar{3}_4$ then the composite F_p also equals a sum of squares. Thus any of these F_p can form a Pythagorean triple [5, 9]. These triples can be represented by

$$c^2 = a^2 + b^2 \tag{4.1}$$

in which $c \in \bar{1}_4, c = x^2 + y^2, a = 2xy$, and $b = x^2 - y^2$. For the ordinary Fibonacci numbers we may talk about associated 'Fibonacci triples' (F_{2n+1}, F_{n+1}, F_n) when $2n + 1$ is prime

$$F_{2n+1} = F_{n+1}^2 + F_n^2 \tag{4.2}$$

For example, when $r_1 = 1$ and $n = 5$ this becomes

$$\begin{aligned} F_{11}(5) &= 89 \\ &= 64 + 25 \\ &= F_6(5)^2 + 1 \times F_5(5)^2 \end{aligned}$$

and the Fibonacci triple is $(F_{2n+1}(5), F_{n+1}(5), F_n(5))$ = (89, 8, 5). We seek an analogue to (4.2) for the other Golden Ratio Fibonacci numbers, $F_n(a)$, displayed in Table 7.

r_1	n	1	2	3	4	5	6	7	8	9	10	11	12
0	$F_n(1)$	1	1	1	1	1	1	1	1	1	1	1	1
1	$F_n(5)$	1	1	2	3	5	8	13	21	34	55	89	144
2	$F_n(9)$	1	1	3	5	11	21	43	85	171	341	683	1365
3	$F_n(13)$	1	1	4	7	19	40	97	217	508	1159	2683	6160
4	$F_n(17)$	1	1	5	9	29	65	181	441	1165	2929	7589	19305
5	$F_n(21)$	1	1	6	11	41	96	301	781	2286	6191	17621	48576
6	$F_n(25)$	1	1	7	13	55	133	463	1261	4039	11605	35839	105469
7	$F_n(29)$	1	1	8	15	71	176	673	1905	6616	19951	66263	205920

Table 7. $F_n(a)$, $n = 1,2,\ldots,12$

For example, when $r_1 = 2$, and n is odd, $F_n(a) \in \bar{3}_4$, but when n is even, $F_n(a) \in \bar{1}_4$, so that only F_{2m} can be used for c in (4.1). For example, when $n = 4$, we have

$$\begin{aligned} F_8(9)^2 &= 85^2 \\ &= 7225 \\ &= 7056 + 169 \\ &= 84^2 + 13^2. \end{aligned}$$

The Pythagorean triple is (85, 84, 13) but there is no associated Fibonacci triple in the even subscripted case. If, however, we consider $n = 5$, then we have

$$\begin{aligned} F_{11}(9) &= 683 \\ &= 441 + 242 \\ &= 21^2 + 2 \times 11^2 \\ &= F_6(9)^2 + 2 \times F_5(9)^2 \end{aligned}$$

and there is a Fibonacci triple (683, 21, 11).

When $r_1 = 3$, each triple of the sequence $\{F_n(13)\}$ has two odd and one even numbers. The numbers in class $\bar{1}_4$ have the form $n = 1 + 6t$ and $n = 2 + 6t$. The sequence is {1, 1, 4, 7, 19, 40, 97, …} (Table 7). For instance, $F_7(13) = 97$, which may be used for c in (4.1). We then use the equation

$$x, y = \frac{A \pm \sqrt{2F_7(13) - A^2}}{2}, \tag{4.3}$$

in which x is odd and y is even with $A = x + y$ [6], so that in this case $x = 9$ and $y = 4$ and the Pythagorean triple is (97, 65, 72); that is,

$$\begin{aligned} F_7(13)^2 &= 97^2 \\ &= 9409 \\ &= 4225 + 5184 \\ &= 65^2 + 72^2 \\ &= \left(F_5(5)^2 F_7(5)\right)^2 + \left(F_3(5) F_7(5)\right)^2. \end{aligned}$$

For the corresponding Fibonacci triple, we have

$$\begin{aligned} F_7(13) &= 97 \\ &= 49 + 48 \\ &= 7^2 + 3 \times 4^2 \\ &= F_4(13)^2 + 3 \times F_3(13)^2. \end{aligned}$$

The Fibonacci triple is then (97, 7, 4) and a picture of the analogue we seek is emerging, namely

$$F_{2n+1}(a) = F_{n+1}(a)^2 + r_1 F_n(a)^2. \tag{4.4}$$

This can be readily proved with the corresponding Binet formula (2.3) (as in the case of the ordinary Fibonacci numbers). We provide another illustration.

When $r_1 = 7$, for each triple of the sequence $\{F_n(29)\}$ the structure is similar. For instance, $F_{13}(29) = 669761$, which may be used for c in (4.1). In this case, $x = 505$ and $y = 644$ (from (4.3)) and the Pythagorean triple is (669761, 159711, 650440); that is,

$$\begin{aligned} F_{13}(29)^2 &= 669761^2 \\ &= 448579797121 \\ &= 25507603521 + 423072193600 \\ &= 159711^2 + 650440^2. \end{aligned}$$

The corresponding Fibonacci triple will be found from

$$\begin{aligned} F_{13}(29) &= 669761 \\ &= 452929 + 216832 \\ &= 673^2 + 7 \times 176^2 \\ &= F_7(29)^2 + 7 \times F_6(29)^2. \end{aligned}$$

The Fibonacci triple is (669761, 673, 176).

5 Final comments

Further observation of the cells in Table 7 can reveal other properties of the generalized Fibonacci numbers which are analogous to the well-known properties of the ordinary Fibonacci numbers as well as relations among the sequences for varying values of a, such as intersections [1, 11]. That is, there are very likely many special characteristics for each a of the Golden Ratio family as well as the shared ones. For instance, when $r_1 = 2$,

$$\sum_{m=1}^{n} F_m(9) = F_{m+1}(9) - \delta(2, n) \tag{5.1}$$

in which $\delta(2,n)$ is the divisor function

$$\delta(2,n)=\begin{cases}1, & 2\mid n,\\ 0, & 2\nmid n.\end{cases} \tag{5.2}$$

$a \in \bar{1}_4$ always and only integers in this class can be a sum of squares (all primes, for example, and composites with some factors$\in \bar{1}_4$). Other examples include considering Golden Ratio Lucas numbers $L_n(a)$ which also satisfy (2.1) (Table 8). We see from this table that equivalent to (4.4) we have

$$aF_{2n+1}(a) = L_{n+1}(a)^2 + r_1 L_n(a)^2. \tag{5.3}$$

This may be compared with the known [13, Equation (25)]:

$$5F_{2n+1}(5) = L_{n+1}(5)^2 + L_n(5)^2. \tag{5.4}$$

r_1	n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	$L_n(1)$	1	1	1	1	1	1	1	1	1	1	1
1	$L_n(5)$	1	3	4	7	11	18	29	47	76	123	199
2	$L_n(9)$	1	5	7	17	31	65	127	257	511	1025	2047
3	$L_n(13)$	1	7	10	31	61	154	337	799	1810	4207	9637
4	$L_n(17)$	1	9	13	49	101	297	701	1889	4693	12249	31021
5	$L_n(21)$	1	11	16	71	151	506	1261	3791	10096	29051	79531
6	$L_n(25)$	1	13	19	97	211	793	2059	6817	19171	60073	175099
7	$L_n(29)$	1	15	22	127	281	1170	3137	11327	33286	112575	345577

Table 8. $L_n(a)$, $n = 1, 2, \ldots, 11$

Similarly, the well-known [13, Equation (6)]

$$L_n(5) = F_{n+1}(5) + F_{n-1}(5) \tag{5.5}$$

becomes

$$L_n(a) = F_{n+1}(a) + r_1 F_{n-1}(a) \tag{5.6}$$

for the generalized Golden Ratio Fibonacci and Lucas sequences.

References

[1] Horadam, A. F. (1966) Generalizations of two theorems of K. Subba Rao. *Bulletin of the Calcutta Mathematical Society*, 58(1), 23–29.

[2] Kilic, E., & D. Tasci (2006) The generalized Binet formula, representation and sums of the generalized order-*k* Pell numbers. *Taiwanese Journal of Mathematics*, 10(6), 1661–1670.

[3] Leyendekkers, J. V., J. M. Rybak, & A. G. Shannon (1995) Integer class properties associated with an integer matrix. Notes on Number Theory and Discrete Mathematics. 1(2), 53–59.

[4] Leyendekkers, J. V., & A. G. Shannon (2015) The Golden Ratio Family and Generalized Fibonacci Numbers. *Journal of Advances in Mathematics*. 10(1), 3130–3137.

[5] Leyendekkers, J.V., A.G Shannon, J.M. Rybak. 2007. *Pattern Recognition: Modular Rings and Integer Structure*. North Sydney: Raffles KvB Monograph No.9.

[6] Leyendekkers, J.V., A.G Shannon. The sum of squares for primes. *Notes on Number Theory and Discrete Mathematics.* 21(4), 17–21.

[7] Livio, M. (2002) *The Golden Ratio*. New York: Golden Books.

[8] Shannon, A. G., A. F. Horadam, & S. N. Collings (1974) Some Fibonacci congruences. *The Fibonacci Quarterly*, 12(4), 351–354.

[9] Shannon, A. G., & A. F. Horadam (1994) Arrowhead curves in a tree of Pythagorean triples. *International Journal of Mathematical Education in Science and Technology*, 25(2), 255–261.

[10] Shannon, A. G., & J. V. Leyendekkers (2015) The Golden Ratio Family and the Binet equation. *Notes on Number Theory and Discrete Mathematics*, 21(2), 35–42.

[11] Subba Rao, K. (1959) Some properties of Fibonacci numbers – II. *The Mathematics Student*, 27(1), 19–23.

[12] Tee, G. J. (1991) Proof of Filipponi's Conjecture on Non-Integer Sums of Fibonacci-Type Series. *Gazette of the Australian Mathematical Society*, 18, 161–164.

[13] Vajda, S. (1989) *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications*. Chichester: Ellis Horwood.