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On a limit where appear prime numbers

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Abstract: Let p_n be the n-th prime number. The following limit is well-known

$$\lim_{n\to\infty} \frac{(p_1 p_2 \cdots p_n)^{\frac{1}{n}}}{p_n} = \frac{1}{e}$$

Let k be a fixed but arbitrary nonnegative integer. In this note we prove the more general limit

$$\lim_{n\to\infty} \frac{\left(p_1^{(1^k)} p_2^{(2^k)} \cdots p_n^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{p_n} = \frac{1}{\sqrt[k+1]{e}}.$$

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1 Main results

Let p_n be the n-th prime number. The following limit is well-known (see [1], [4] and [5])

$$\lim_{n\to\infty} \frac{(p_1 p_2 \cdots p_n)^{\frac{1}{n}}}{p_n} = \frac{1}{e}. \tag{1}$$

In this note we generalize this limit. We have the following theorem.

Theorem 1. *Let k be a fixed but arbitrary nonnegative integer. The following limit holds*

$$\lim_{n\to\infty} \frac{\left(p_1^{(1^k)} p_2^{(2^k)} \cdots p_n^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{p_n} = \frac{1}{\sqrt[k+1]{e}} \tag{2}$$

Proof: If $k = 0$, then the theorem is true (see (1)). Suppose that k is a positive integer. The prime number theorem is

$$p_i \sim i \log i$$

Therefore

$$\log p_i = \log i + \log\log i + f(i) \qquad (i \geq 1) \tag{3}$$

where

$$f(i) \to 0 \tag{4}$$

and we put $\log\log 1 = 0$.

Now, we have (see (3))

$$\log\left(p_1^{(1^k)} p_2^{(2^k)} \cdots p_n^{(n^k)}\right) = \sum_{i=1}^{n} i^k \log p_i = \sum_{i=1}^{n} \left(i^k \log i + i^k \log\log i + f(i) i^k\right)$$

$$= \sum_{i=1}^{n} i^k \log i + 2^k \log\log 2 + \sum_{i=3}^{n} i^k \log\log i + \sum_{i=1}^{n} f(i) i^k \tag{5}$$

The function $x^k \log x$ is nonnegative and strictly increasing on the interval $[1, \infty)$. Consequently

$$\sum_{i=1}^{n} i^k \log i = \int_1^n x^k \log x \, dx + O(n^k \log n) \tag{6}$$

Note that the sum in the left side is a sum of rectangles of basis 1 and hight $i^k \log i$.

The function $x^k \log\log x$ is nonnegative and strictly increasing on the interval $[e, \infty)$. Consequently

$$\sum_{i=3}^{n} i^k \log\log i = \int_3^n x^k \log\log x \, dx + O(n^k \log\log n) \tag{7}$$

Note that the sum in the left side is a sum of rectangles of basis 1 and hight $i^k \log\log i$.

We have (use integration by parts)

$$\int_1^n x^k \log x \, dx = \frac{n^{k+1}}{k+1} \log n - \frac{1}{(k+1)^2} n^{k+1} + \frac{1}{(k+1)^2} \tag{8}$$

On the other hand, we have (use integration by parts)

$$\int_3^n x^k \log\log x \, dx = \frac{n^{k+1}}{k+1} \log\log n - \frac{3^{k+1}}{k+1} \log\log 3 - \frac{1}{k+1} \int_3^n \frac{x^k}{\log x} \, dx \tag{9}$$

The L'Hospital's rule gives

$$\lim_{x\to\infty} \frac{\int_3^x \frac{t^k}{\log t} \, dt}{x^{k+1}} = \lim_{x\to\infty} \frac{\frac{x^k}{\log x}}{(k+1)x^k} = 0 \tag{10}$$

Therefore (10) gives

$$\lim_{n\to\infty} \frac{\int_3^n \frac{x^k}{\log x}\, dx}{n^{k+1}} = 0$$

That is

$$\int_3^n \frac{x^k}{\log x}\, dx = o\left(n^{k+1}\right) \tag{11}$$

Equations (9) and (11) give

$$\int_3^n x^k \log\log x\, dx = \frac{n^{k+1}}{k+1} \log\log n + o\left(n^{k+1}\right) \tag{12}$$

Given $\epsilon > 0$, there exist n_0 such that if $n \geq n_0$ we have $|f(i)| < \epsilon$ (see (4)). Therefore

$$\left| \sum_{i=1}^{n} f(i) i^k \right| \leq \sum_{i=1}^{n} |f(i)|\, i^k \leq \sum_{i=1}^{n_0-1} |f(i)|\, i^k + \epsilon \sum_{i=n_0}^{n} i^k \leq \sum_{i=1}^{n_0-1} |f(i)|\, i^k + \epsilon \sum_{i=1}^{n} i^k \tag{13}$$

Now

$$\sum_{i=1}^{n} i^k = \int_1^n x^k\, dx + O(n^k) = \frac{n^{k+1}}{k+1} + o(n^{k+1}) \tag{14}$$

Therefore (see (13) and (14)) from a certain value of n we have

$$\left| \frac{\sum_{i=1}^{n} f(i) i^k}{n^{k+1}} \right| \leq \frac{\sum_{i=1}^{n_0-1} |f(i)|\, i^k}{n^{k+1}} + \epsilon \frac{\sum_{i=1}^{n} i^k}{n^{k+1}} \leq \epsilon$$

where ϵ is arbitrarily small. That is

$$\sum_{i=1}^{n} f(i) i^k = o(n^{k+1}) \tag{15}$$

Equations (5), (6), (8), (7), (12) and (15) give

$$\log\left(p_1^{(1^k)} p_2^{(2^k)} \cdots p_n^{(n^k)}\right) = \frac{n^{k+1}}{k+1} \log n + \frac{n^{k+1}}{k+1} \log\log n$$
$$- \quad \frac{n^{k+1}}{(k+1)^2} + o(n^{k+1}) \tag{16}$$

Therefore (see (16) and (3)) we have

$$\log\left(\frac{\left(p_1^{(1^k)} p_2^{(2^k)} \cdots p_n^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{p_n} \right) = \frac{k+1}{n^{k+1}} \log\left(p_1^{(1^k)} p_2^{(2^k)} \cdots p_n^{(n^k)}\right)$$
$$- \quad \log p_n = -\frac{1}{k+1} + o(1) \tag{17}$$

That is, limit (2). $\square$

The following limit is well known (see [1])

$$\lim_{n\to\infty} \frac{(12\cdots n)^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

We have the following generalization.

Theorem 2. *Let k be a fixed but arbitrary nonnegative integer. The following limit holds*

$$\lim_{n\to\infty} \frac{\left(1^{(1^k)}2^{(2^k)}\cdots n^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{n} = \frac{1}{\sqrt[k+1]{e}} \tag{18}$$

Proof: The proof is as Theorem 1, but simpler and shorter. □

Let P_n be the n-th perfect power. In a previous article [2] we prove the limit

$$\lim_{n\to\infty} \frac{(P_1 P_2 \cdots P_n)^{\frac{1}{n}}}{P_n} = \frac{1}{e^2}. \tag{19}$$

In the following theorem we generalize this limit.

Theorem 3. *Let k be a fixed but arbitrary nonnegative integer. The following limit holds*

$$\lim_{n\to\infty} \frac{\left(P_1^{(1^k)}P_2^{(2^k)}\cdots P_n^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{P_n} = \frac{1}{e^{\frac{2}{k+1}}} \tag{20}$$

Proof: Note that (see [3]) $P_i \sim i^2$ and consequently $\log P_i = 2\log i + f(i)$ where $f(i) \to 0$. Now, the proof is as Theorem 1, but simpler and shorter. □

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