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A generalized recurrence formula for Stirling numbers and related sequences

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Abstract: In this note, we provide a combinatorial proof of a generalized recurrence formula satisfied by the Stirling numbers of the second kind. We obtain two extensions of this formula, one in terms of r-Whitney numbers and another in terms of q-Stirling numbers of Carlitz. Modifying our proof yields analogous formulas satisfied by the r-Stirling numbers of the first kind and by the r-Lah numbers.
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1 Introduction

By a *partition* of the set $[n] = \{1, 2, \dots, n\}$, we will mean a collection of pairwise disjoint subsets, called *blocks*, whose union is $[n]$. The cardinality of the set of partitions of $[n]$ having exactly k blocks is given by the Stirling number of the second kind (see, e.g., [9, p. 33]), which will be denoted here by $S(n, k)$. The numbers $S(n, k)$ satisfy a variety of identities and the reader is referred to [5, Section 6.1] and [1, Chapter 7]. The $S(n, k)$ have the following generalized recurrence, which we were unable to find in the literature.

Theorem 1.1. *If* $n, k \geq 1$ *and* $0 \leq \ell \leq k-1$, *then*

$$S(n,k) = \sum_{i=\ell+1}^{n-k+\ell+1} \sum_{j=k-\ell-1}^{n-i} \binom{n-i}{j} \ell^{n-i-j} S(i-1,\ell) S(j+1,k-\ell). \tag{1.1}$$

Note that when $\ell = 1$, formula (1.1) reduces to

$$
\begin{aligned}
S(n,k) &= \sum_{i=2}^{n-k+2} \sum_{j=k-2}^{n-i} \binom{n-i}{j} S(j+1,k-1) = \sum_{j=k-2}^{n-2} S(j+1,k-1) \sum_{i=2}^{n-j} \binom{n-i}{j} \\
&= \sum_{j=k-1}^{n-1} \binom{n-1}{j} S(j,k-1),
\end{aligned}
$$

which is a well-known recurrence for $S(n,k)$ (see, e.g., [5, Equation 6.15]). In this note, we provide a combinatorial proof of formula (1.1). Furthermore, we establish two generalizations of (1.1): namely, one involving the recently introduced r-Whitney number [4] and another involving a q-Stirling number considered originally by Carlitz [3]. Modifying our proof yields analogous formulas satisfied by the r-Stirling numbers of the first kind [2] and by the recently studied r-Lah numbers [8].

2 An identity for r-Whitney numbers

The r-Whitney numbers of the second kind (see, e.g., [4, 7]), denoted by $W_{m,r}(n,k)$, are defined as the connection constants in the polynomial identities

$$
(mx+r)^n = \sum_{k=0}^{n} W_{m,r}(n,k) m^k (x)_k, \qquad n \geq 0, \tag{2.1}
$$

where $(x)_k = x(x-1)\cdots(x-k+1)$ if $k \geq 1$, with $(x)_0 = 1$. Here, $r \geq 0$ and $m \geq 1$ are integers, but may also taken to be indeterminates. Note that $W_{1,0}(n,k) = S(n,k)$ for all n and k. Equivalently, the r-Whitney numbers are defined by the recurrence

$$
W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (r+mk)W_{m,r}(n-1,k), \qquad n,k \geq 1, \tag{2.2}
$$

with $W_{m,r}(n,0) = r^n$ and $W_{m,r}(0,k) = \delta_{k,0}$ for all $n,k \geq 0$. The formula for the ordinary generating function is given in [4]:

$$
\sum_{n \geq k} W_{m,r}(n,k) x^n = \frac{x^k}{(1-rx)(1-(r+m)x)\cdots(1-(r+mk)x)}, \qquad k \geq 0. \tag{2.3}
$$

We now establish the following recurrence formula for the r-Whitney numbers.

Theorem 2.1. *If $n,k \geq 1$ and $0 \leq \ell \leq k-1$, then*

$$
W_{m,r}(n,k) = \sum_{i=\ell+1}^{n-k+\ell+1} \sum_{j=k-\ell-1}^{n-i} m^{j-k+\ell+1} \binom{n-i}{j} (r+m\ell)^{n-i-j} W_{m,r}(i-1,\ell) S(j+1,k-\ell). \tag{2.4}
$$

Proof. We compute the generating function of the quantity on the right-hand side of (2.4). Recalling the well-known formulas

$$
\sum_{n \geq k} S(n,k) x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}, \qquad k \geq 1,
$$

and

$$\sum_{n\geq j}\binom{n}{j}x^n=\frac{x^j}{(1-x)^{j+1}},\qquad j\geq 0,$$

we have

$$\sum_{n\geq k}x^n\left(\sum_{i=\ell+1}^{n-k+\ell+1}\sum_{j=k-\ell-1}^{n-i}m^{j-k+\ell+1}\binom{n-i}{j}(r+m\ell)^{n-i-j}W_{m,r}(i-1,\ell)S(j+1,k-\ell)\right)$$
$$=\sum_{i\geq\ell+1}W_{m,r}(i-1,\ell)\sum_{j\geq k-\ell-1}S(j+1,k-\ell)m^{j-k+\ell+1}(r+m\ell)^{-j}x^i\cdot\sum_{n\geq i+j}\binom{n-i}{j}((r+m\ell)x)^{n-i}$$
$$=\sum_{i\geq\ell+1}W_{m,r}(i-1,\ell)\sum_{j\geq k-\ell-1}S(j+1,k-\ell)m^{j-k+\ell+1}(r+m\ell)^{-j}x^i\cdot\frac{((r+m\ell)x)^j}{(1-(r+m\ell)x)^{j+1}}$$
$$=m^{-k+\ell}\sum_{i\geq\ell+1}W_{m,r}(i-1,\ell)x^{i-1}\cdot\sum_{j\geq k-\ell-1}S(j+1,k-\ell)\frac{(mx)^{j+1}}{(1-(r+m\ell)x)^{j+1}}$$
$$=m^{-k+\ell}\frac{x^\ell}{\prod_{i=0}^{\ell}(1-(r+mi)x)}\cdot\frac{(mx/(1-(r+m\ell)x))^{k-\ell}}{\prod_{i=1}^{k-\ell}\left(1-\frac{mix}{1-(r+m\ell)x}\right)}$$
$$=\frac{x^\ell}{\prod_{i=0}^{\ell}(1-(r+im)x)}\cdot\frac{x^{k-\ell}}{\prod_{i=1}^{k-\ell}(1-(r+(i+\ell)m)x)}=\frac{x^k}{\prod_{j=0}^{k}(1-(r+mj)x)},$$

by (2.3), which implies (2.4). □

Note that formula (2.4) reduces to (1.1) when $m=1$ and $r=0$. We can also provide a combinatorial proof of (2.4), but will need first to introduce some terminology.

Definition 2.2. *Given $0\leq r\leq m$, by an r-partition of $[m]$, we will mean one in which the elements $1,2,\ldots,r$ belong to distinct blocks. If $n,k,r\geq 0$, then let $\Pi_r(n,k)$ denote the set of all r-partitions of $[n+r]$ having $k+r$ blocks.*

Recall that the cardinality of $\Pi_r(n,k)$ is given by the r-Stirling number of the second kind introduced by Broder [2] and that $W_{m,r}(n,k)$ reduces to the r-Stirling number when $m=1$, whence $|\Pi_r(n,k)|=W_{1,r}(n,k)$. To obtain the combinatorial interpretation for $W_{m,r}(n,k)$ that will be used in the proof below, we color certain elements of $[n+r]$ within a member of $\Pi_r(n,k)$ based off of their relative size within a block. In what follows, let $[m,n]=\{m,m+1,\ldots,n\}$ if $m\leq n$ are positive integers, with $[m,n]=\varnothing$ if $m>n$.

Given a member of $\Pi_r(n,k)$, we will refer to the blocks containing an element of $[r]$ as *special* and the remaining blocks comprised exclusively of elements of $[r+1,r+n]$ as *non-special*. Furthermore, we will refer to an element within a member of $\Pi_r(n,k)$ that is the smallest within its block as *minimal*, and to all other elements as *non-minimal*.

Definition 2.3. *Given an integer $m\geq 1$, let $\Pi_{m,r}(n,k)$ denote the set of r-partitions of $[n]$ having k blocks wherein within each non-special block, every non-minimal element is assigned one of m colors.*

We now give a combinatorial proof of formula (2.4) above.

Combinatorial proof of Theorem 2.1.

Proof. Let $A(n,k)$ ($= A_{m,r}(n,k)$) denote the cardinality of the set $\Pi_{m,r}(n,k)$. We first show that $W_{m,r}(n,k) = A(n,k)$ for all n and k. Note that $A(0,k) = \delta_{k,0}$ follows from the definitions and that $A(n,0) = r^n$ since each element of $[r+1, r+n]$ within $\pi \in \Pi_{m,r}(n,0)$ belongs to one of r special blocks of π. We now count members of $\Pi_{m,r}(n,k)$ where $n,k \geq 1$. Observe that there are $A(n-1,k-1)$ members of $\Pi_{m,r}(n,k)$ in which the element $n+r$ occupies a (non-special) block by itself. On the other hand, if $n+r$ belongs to a block containing at least one member of $[n+r-1]$, then there are $rA(n-1,k)$ possibilities in which $n+r$ belongs to a special block and $mkA(n-1,k)$ possibilities in which $n+r$ belongs to a non-special block (note that $n+r$ being non-minimal implies it is assigned one of m colors in the latter case). Combining the three previous cases yields $A(n,k) = A(n-1,k-1) + (r+mk)A(n-1,k)$ if $n,k \geq 1$, and thus $A(n,k) = W_{m,r}(n,k)$, by (2.2).

To complete the proof of (2.4), we argue that the right-hand side also counts the members of $\Pi_{m,r}(n,k)$. Assume that the blocks within a member of $\Pi_{m,r}(n,k)$ are arranged from left to right in ascending order of smallest elements. Given $\pi \in \Pi_{m,r}(n,k)$, suppose that the smallest element of the $(\ell+1)$-st non-special block from the left is $r+i$; note that $\ell+1 \leq i \leq n-k+\ell+1$ in order for such a π to exist. Then there are $W_{m,r}(i-1,\ell)$ possibilities concerning placement of the elements of $[r+i-1]$. Suppose further that there are exactly j elements of $I = [r+i+1, r+n]$ lying within the non-special blocks of π to the right of and including the one containing $r+i$. Once they have been selected in one of $\binom{n-i}{j}$ ways, these elements, together with $r+i$, may be arranged in $k-\ell$ non-special blocks of π in $m^{j-k+\ell+1}S(j+1,k-\ell)$ ways, where the factor of m accounts for the coloring of the $j-k+\ell+1$ non-minimal elements within these blocks. Furthermore, there are $(r+m\ell)^{n-i-j}$ ways in which to position and color the remaining $n-i-j$ elements of I, which may go in either special blocks or in the first ℓ non-special blocks. Considering all possible i and j gives the cardinality of all members of $\Pi_{m,r}(n,k)$, which implies (2.4). $\square$

3 q-Stirling generalization

If k is a positive integer, then let $[k]_q = 1 + q + \cdots + q^{k-1}$, where q is an indeterminate, with $[0]_q = 0$. Define the q-Stirling number $S_q(n,k)$ by the recurrence

$$S_q(n,k) = S_q(n-1,k-1) + [k]_q S_q(n-1,k), \qquad n,k \geq 1, \tag{3.1}$$

with initial values $S_q(n,0) = \delta_{n,0}$ and $S_q(0,k) = \delta_{k,0}$ for all $n,k \geq 0$. The numbers $S_q(n,k)$ were originally considered by Carlitz [3] and reduce to $S(n,k)$ when $q = 1$. Equivalently, there is the generating function formula (see, e.g., [10]):

$$\sum_{n \geq k} S_q(n,k)x^n = \frac{x^k}{(1-[1]_q x)(1-[2]_q x)\cdots(1-[k]_q x)}, \qquad k \geq 1. \tag{3.2}$$

We have the following q-generalization of formula (1.1).

Theorem 3.1. *If* $n, k \geq 1$ *and* $0 \leq \ell \leq k-1$, *then*

$$S_q(n,k) = \sum_{i=\ell+1}^{n-k+\ell+1} \sum_{j=k-\ell-1}^{n-i} q^{\ell(j-k+\ell+1)} \binom{n-i}{j} [\ell]_q^{n-i-j} S_q(i-1,\ell) S_q(j+1,k-\ell). \tag{3.3}$$

Proof. Computing the generating function of the quantity on the right-hand side of (3.3), we have

$$\begin{aligned}
&\sum_{n\geq k} x^n \left(\sum_{i=\ell+1}^{n-k+\ell+1} \sum_{j=k-\ell-1}^{n-i} q^{\ell(j-k+\ell+1)} \binom{n-i}{j} [\ell]_q^{n-i-j} S_q(i-1,\ell) S_q(j+1,k-\ell) \right) \\
&= \sum_{i\geq \ell+1} S_q(i-1,\ell) \sum_{j\geq k-\ell-1} q^{\ell(j-k+\ell+1)} S_q(j+1,k-\ell) [\ell]_q^{-j} x^i \cdot \sum_{n\geq i+j} \binom{n-i}{j} ([\ell]_q x)^{n-i} \\
&= \sum_{i\geq \ell+1} S_q(i-1,\ell) \sum_{j\geq k-\ell-1} q^{\ell(j-k+\ell+1)} S_q(j+1,k-\ell) [\ell]_q^{-j} x^i \cdot \frac{([\ell]_q x)^j}{(1-[\ell]_q x)^{j+1}} \\
&= q^{\ell(\ell-k)} \sum_{i\geq \ell+1} S_q(i-1,\ell) x^{i-1} \cdot \sum_{j\geq k-\ell-1} S_q(j+1,k-\ell) \frac{(q^\ell x)^{j+1}}{(1-[\ell]_q x)^{j+1}} \\
&= q^{\ell(\ell-k)} \frac{x^\ell}{\prod_{i=1}^{\ell}(1-[i]_q x)} \cdot \frac{(q^\ell x/(1-[\ell]_q x))^{k-\ell}}{\prod_{i=1}^{k-\ell}\left(1-\frac{q^\ell [i]_q x}{1-[\ell]_q x}\right)} \\
&= \frac{x^\ell}{\prod_{i=1}^{\ell}(1-[i]_q x)} \cdot \frac{x^{k-\ell}}{\prod_{i=1}^{k-\ell}(1-[i+\ell]_q x)} = \frac{x^k}{\prod_{j=1}^{k}(1-[j]_q x)},
\end{aligned}$$

by (3.2), where we have used the fact that $[i+\ell]_q = [\ell]_q + q^\ell [i]_q$. $\square$

4 Two related formulas

Let $S(n,k;r) = |\Pi_r(n,k)|$ denote the r-Stirling number of the second kind, where we are using the parametrization of Merris [6]. Letting $m=1$ in formula (2.4) above implies

$$S(n,k;r) = \sum_{i=\ell+1}^{n-k+\ell+1} \sum_{j=k-\ell-1}^{n-i} \binom{n-i}{j} (r+\ell)^{n-i-j} S(i-1,\ell;r) S(j+1,k-\ell), \qquad r \geq 0. \tag{4.1}$$

Analogous formulas can also be given for other r-numbers. Let $s(n,k;r)$ denote the r-Stirling number of the first kind [2, 6], which counts the permutations of $[n+r]$ into $k+r$ disjoint cycles where the elements of $[r]$ belong to different cycles. Let $L(n,k;r)$ denote the r-Lah number [8], which counts the number of partitions of $[n+r]$ into $k+r$ contents-ordered blocks where the elements of $[r]$ belong to different blocks. Let $s(n,k) = s(n,k;0)$ be the (signless) Stirling number of the first kind and $L(n,k) = L(n,k;0)$ be the Lah number.

We have the following recurrence formulas satisfied by $s(n,k;r)$ and $L(n,k;r)$.

Theorem 4.1. *If* $n, k \geq 1$, $r \geq 0$, *and* $0 \leq \ell \leq k-1$, *then*

$$s(n,k;r) = \sum_{i=\ell+1}^{n-k+\ell+1} \sum_{j=k-\ell-1}^{n-i} \binom{n-i}{j} (n+r-j-2)_{(n-i-j)} s(i-1,\ell;r) s(j+1,k-\ell) \tag{4.2}$$

and

$$L(n,k;r) = \sum_{i=\ell+1}^{n-k+\ell+1} \sum_{j=k-\ell-1}^{n-i} \binom{n-i}{j} (n+2r+\ell-j-2)_{(n-i-j)} L(i-1,\ell;r) L(j+1,k-\ell). \quad (4.3)$$

Proof. These results may be obtained by making suitable modifications to the combinatorial proof of Theorem 2.1 above so as to allow for ordering within blocks which we now describe. Applying the same terminology as before, assume that the non-special cycles or ordered blocks are arranged from left to right in ascending order of minimal elements. Then consider the smallest element, $r+i$, of the $(\ell+1)$-st non-special cycle or ordered block from the left. For (4.2), note that then there are $s(i-1,\ell;r)$ possibilities concerning the placement of the elements of $[r+i-1]$ and $s(j+1,k-\ell)$ possibilities regarding the positions of the j elements of I that comprise the rightmost $k-\ell$ non-special cycles, together with $r+i$.

The other $n-i-j$ elements of I, which we denote by $a_1 < a_2 < \cdots < a_{n-i-j}$, must go in the special cycles or in the first ℓ non-special cycles. As there are $r+i-1$ elements already within these cycles, there are $r+i-1$ choices regarding the position of a_1. Subsequently adding the elements $a_1 < a_2 < \cdots$ to these cycles, it follows that there are $r+i+t-2$ choices regarding the position of a_t for $1 \leq t \leq n-i-j$. Thus, there are

$$\prod_{t=1}^{n-i-j} (r+i+t-2) = (n+r-j-2)_{(n-i-j)}$$

possibilities concerning the placement of these elements. Considering all possible i and j then gives (4.2). Similar reasoning applies to (4.3) except that there are $2r+\ell+i+t-2$ choices regarding the position of the element a_t for each t since now all possible orderings of elements within a block are allowed. □

Taking $r=0$ in (4.2) and (4.3) gives analogues of (1.1) for $s(n,k)$ and $L(n,k)$, respectively.

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