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On (M, N)-convex functions

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Abstract: We consider certain properties of functions $f : J \to I$ (I, J intervals) such that $f(M(x,y)) \leq N(f(x), f(y))$, where M and N are general means. Some results are extensions of the case $M = N = L$, where L is the logarithmic mean.
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1 Introduction

Let $I \subset \mathbb{R}$ be an open interval. An application $M : I \times I \to I$ is called a *mean*, if

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\} \text{ for } x, y \in I.$$

Let $J \subset I$ be another open interval and $M, N : J \times J \to J$ be two given means such that $N(J \times J) \subset I$.

A function $f : J \to I$ is called an (M, N)-*convex (concave)* function, if

$$f(M(x, y)) \leq (\geq) N(f(x), f(y)) \text{ for all } x, y \in J.$$

When $M = N$, then f is called an M-*convex* function. M-convex functions have been introduced and studied for the first time in 1997 by J. Matkowski and J. Rätz ([2, 3]), while the particular case $M = L$, where L is the logarithmic mean have been studied later by J. Matkowski [4], Z. Kominek and T. Zgraja [9], and T. Zgraja [10].

The general (M, N)-convex functions were introduced in 1998 by J. Sándor [7], who studied also the case when N is an integral mean.

The aim of this paper is to point out some new properties of general (M, N)-convex functions, when M and N satisfy certain conditions. Particularly, when $N = A$ = arithmetic mean, or $N = G$ = geometric mean, continuity properties will be offered. These extend earlier known results.

2 Definitions and notations

Let $a, b > 0$. Then the *arithmetic* and *geometric means* of a and b are defined by $A = A(a,b) = \frac{a+b}{2}$; $G = G(a,b) = \sqrt{ab}$. The *logarithmic mean* is $L = L(a,b) = \frac{b-a}{\log b - \log a} (a \neq b)$; $L(a,a) = a$; while the *identric mean* is $I = \frac{1}{e}\left(b^b/a^a\right)^{1/(b-a)} (a \neq b)$; $I(a,a) = a$. For many properties of the logarithmic and identric means, see [1], [6], [8] (we note that $I(a,b)$ should not be confused with the interval I).

A common generalization of A and G are the *power means*, defined by $A_p = A_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}$ $(p \neq 0)$; $A_0 = G$. The *integral mean* of a function $f : I \to \mathbb{R}$ is given by $\mathfrak{I}(a,b) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$ $(a \neq b)$, $\mathfrak{I}(a,a) = a$; where $a, b \in I$.

A mean is called *subadditive (superadditive)*, if

$$M(x_1 + x_2, y_1 + y_2) \leq (\geq) M(x_1, y_1) + M(x_2, y_2); x_i, y_i > 0 (i = \overline{1,2}),$$

and M is called *subhomogenous (superhomogenous)* if

$$M(tu, tv) \leq (\geq) t \cdot M(u, v); \forall t, u, v > 0.$$

Examples of (M, N)-convex functions

1. $M = N = A$ = arithmetic mean (**Jensen convexity**)
2. $M = A$, $N = G$ ($J = (0, \infty)$) (**log–convex functions**)
3. $M = N = G$ (**multiplicative (or geometric) convex functions**)
4. $M = G$, $N = A$ ((G, A)**-convex functions**)
5. $M = A$, $N = A_p$ ((A, A_p)**-convex functions**)
6. $M = N = L$ (L**-convex functions**)
7. M = arbitrary, $N = \mathfrak{I}$ (integral mean) (**convex functions with respect to an integral mean**)

The function $f(x) = x^k, x \in (0, \infty)$ is L-convex iff $k \in \mathbb{R}\backslash(0, 1)$ (see [10]). The function $f(x) = a^x, x \in (0, \infty)$ is L-convex for $a > 1$ and neither L-convex, nor L-concave for any $a \in (0, 1)$. Finally, the function $f(x) = e^x$ is *identric convex* (i.e. convex with respect to the identric mean $I(a, b)$).

3 Some basic properties

Theorem 1. *Assume that N is a symmetrical subhomogenous mean such that $G \leq N$, where G is the geometric mean. Let $f : (a,b) \to (0,\infty)$ be (M,N)-concave function (here $0 < a < b$). Then the application*

$$F(x) = \frac{1}{f(x)}, x \in (a,b)$$

is (M,N)-convex.

Proof. Applying the property $N(tu,tv) \leq tN(u,v)$ for $u = \frac{1}{f(x)}$, $v = \frac{1}{f(y)}$, $t = f(x)f(y)$, we get the inequality $N\left(\frac{1}{f(x)}, \frac{1}{f(y)}\right) \geq \frac{1}{f(x)f(y)}. N\left(f(y), f(x)\right) = \frac{1}{f(x)f(y)} \cdot N\left(f(x), f(y)\right).$

Therefore, we can write $(F(x), F(y)) = N\left(\frac{1}{f(x)}, \frac{1}{f(y)}\right) \geq \frac{1}{f(x)f(y)} \cdot N(f(y), f(x)) = \frac{N^2(f(x), f(y))}{f(x)f(y)} \cdot \frac{1}{N(f(x), f(y))} \geq \frac{1}{N(f(x), f(y))} \geq \frac{1}{f(M(x,y))} = F(M(x,y))$. Here we have taken into account the fact that N is symmetrical, and that $\frac{N^2(f(x), f(y))}{f(x)f(y)} \geq 1$, which is a consequence of $N \geq G$. □

Theorem 2. *Let $f, g : (a,b) \to (0,\infty)$ such that $f(x) < g(x)$ for all $x \in (a,b)$ and suppose that f is (M,N)- convex, while g is (M,N)-concave. If the mean N is superadditive then the application $h = g - f$ is an (M,N)-concave function.*

Proof. One has $h(M(x,y)) = g(M(x,y)) - f(M(x,y)) \geq N(g(x), g(y)) - N(f(x), f(y)) = N(h(x) + f(x), h(y) + f(y)) - N(f(x), f(y))$.

On the other hand, by the superadditivity of N, we can write

$$N(h(x) + f(x), h(y) + f(y)) \geq N(h(x), h(y)) + N(f(x), f(y)),$$

so we get

$$h(M(x,y)) \geq N(h(x), h(y)),$$

and the result follows. □

Remark 1. *Let $f(x) = x^{-\alpha}$, $g(x) = k \cdot x$, where α, k, p are selected such that $k \cdot p^{\alpha+1} > 1$ (i.e. $kx > \frac{1}{x^\alpha}$ for $x > p$.) When $M = N = L$, then L being superadditive (see [4]), so if $\alpha > 0$, then $-\alpha \notin (0,1)$, thus $f(x) = x^{-\alpha}$ is L-convex function. By Theorem 2 we get that $h(x) = kx - x^{-\alpha}$ is L-concave.*

Theorem 3. *If $f, g : (a,b) \to (0,\infty)$ are (M,N)-convex (concave) functions, and N is superadditive (subadditive), then $k = f + g$ is (M,N)-convex (concave), too.*

Proof. We prove the case when f, g are (M, N)-convex, and N is superadditive. The second case can be proved in an analogous way.

$h(M(x, y)) = f(M(x, y)) + g(M(x, y)) \leq N(f(x), f(y)) + N(g(x), g(y)) \leq N(f(x) + g(x), f(y) + g(y)) = N(h(x), h(y))$. □

The following theorem is almost immediate, and we state it here for the sake of completeness:

Theorem 4. *If f is (M, N)-convex, N is subhomogeneous and $\alpha > 0$, then the function $g = \alpha \cdot f$ is (M, N)-convex, too.*

4 Continuity properties

Theorem 5. *Assume that for all $x > 0$, the application $M(x, \cdot) : (0, \infty) \to (0, \infty)$ is an increasing homeomorphism. Let f be an (M, A)-convex function, where A is the arithmetic mean. If $f : (a, b) \to (0, \infty)$ is monotonic, then f is continuous.*

Proof. The proof uses ideas from [5] and [10]. Suppose that the function f is increasing and let $z \in (a, b)$ be fixed. If $\lim\limits_{x \nearrow z} f(z) = f(z-), \lim\limits_{x \searrow z} f(z) = f(z+)$ are standard notations in what follows, then a basic property gives the inequality $f(z-) \leq f(z+)$.

Let now $z < z_n < b(n \in \mathbb{N})$, $z_n \to z(n \to \infty)$ be a sequence converging to z from the right. By definition of f, we can write

$$f(M(z, z_n)) \leq A(f(z), f(z_n)) = \frac{f(z) + f(z_n)}{2} \tag{$*$}$$

Since $z \leq M(z, z_n) \leq z_n$ and $M(z, z_n) \to z + (n \to \infty)$, by $(*)$ we get

$$f(z_+) \leq \frac{f(z) + f(z_+)}{2},$$

giving $f(z_+) \leq f(z)$.

But f is increasing, so it is well-known that

$$f(z-) \leq f(z) \leq f(z+) \tag{1}$$

Therefore, we get

$$f(z+) = f(z) \tag{2}$$

Now, for given $z_n \in (z, b)$ we can construct a $w_n \in (a, z)$ such that $z = M(w_n, z_n)$, where $w_n \to z(n \to \infty)$. □

Thus, one has

$$f(z) = f(M(w_n, z_n)) \leq \frac{f(w_n) + f(z_n)}{2}$$

and letting $n \to \infty$ we get by (2):

$$f(z+) = f(z) \le \frac{f(z-) + f(z+)}{2},$$

so $f(z+) \le f(z-)$.

By (1) and (2) we have

$$f(z) = f(z+) = f(z-),$$

i.e. f is continuous at z.

When f is decreasing, a similar argument applies, by selecting now $z_n \in (a, z)$.

Remark 2. *If N is a mean with the property that $N \le A$, then any (M, N)-convex function is (M, A)-convex, too. For example, for $M = N = L$ we obtain that an L-convex function, is (L, A)- convex, too. Thus we reobtain a result by T. Zgraja [10].*

Theorem 6. *Let M satisfy the same property as in Theorem 5, and suppose that f is (M, G)-concave, where G is the geometric mean. If $f : (a, b) \to (0, \infty)$ is a monotone function, then it is continuous.*

Proof. The same method can be applied as in the proof of Theorem 5, by changing only relation $(*)$ to

$$f(M(z, z_n)) \ge G(f(z), f(z_n)) = \sqrt{f(z)f(z_n)} \tag{$**$}$$

and repeating the arguments. □

Remark 3. *If f is (M, N)-concave, and $N \ge G$, then clearly f will be an (M, G)-concave function, too. E.g. for $M = N = L$. Thus the class of (L, G)-concave functions is larger than that of L-concave functions.*

In what follows, a function $f : (a, b) \to \mathbb{R}$ will be called *intervally monotone* if there exist a finite number of points $a = x_0 < x_1 < \ldots < x_n = b$ such that f is monotone on each intervals (x_{i-1}, x_i) $(i = 1, 2, \ldots, n)$.

Theorem 7. *Suppose that M satisfies the conditions of Theorem 5, and let $f : (a, b) \to (0, \infty)$ be intervally monotonic, and (M, A)-convex function (or (M, G)-concave). Then f is continuous.*

Proof. On base of Theorem 5 (Th. 6) it is sufficient to prove the continuity in such a point z, where the monotonicity property is changed. Let us suppose, e.g. that f is decreasing in a left vecinicity (α, z) and right vecinicity (z, β) of z. Let (w_n) such that $\alpha < w_n < z$ and $w_n \to z$ $(n \to \infty)$.

Then we can write

$$f(M(w_n, z)) \le A(f(w_n), f(z)) = \frac{f(w_n) + f(z)}{2} \tag{$*$}$$

Since $\alpha \leq M(w_n, z) \leq z$ and $M(w_n, z) \to z$- (as $n \to \infty$), from (∗) we get $f(z-) \leq \frac{f(z-) + f(z)}{2}$, so $f(z-) \leq f(z)$.

Letting in a similar way $z < z_n < \beta$, we get that $f(z+) \leq f(z)$. Let now $\alpha < w_n < z$ and $z < z_n < \beta$ be sequences converging to z and $z = M(w_n, z_n)$ for each $n \in \mathbb{N}$. Since $f(z) = f(M(w_n, z_n)) \leq \frac{f(w_n) + f(z_n)}{2}$, one gets

$$f(z) \leq \frac{f(z-) + f(z+)}{2} \tag{3}$$

But $f(z-) \leq f(z)$ and $f(z+) \leq f(z)$, so $\frac{f(z-) + f(z+)}{2} \leq f(z)$, and by (3) we get

$$f(z) = \frac{f(z-) + f(z+)}{2} \tag{4}$$

Therefore, it cannot be true at the same time that $f(z) > f(z-)$ and $f(z) > f(z+)$. Suppose that $f(z-) \geq f(z)$. Then we have $f(z-) = f(z)$, and from (4) we get $f(z+) = f(z) = f(z-)$.

If f is (M, G)- concave, the similar method may be applied, by taking into account that $f(M(w_n, z)) \geq \sqrt{f(w_n) \cdot f(z)}$. □

5 Other properties

The following surprising property shows that decreasing (M, A)-convex functions for $M \leq A$ are in fact the classical convex functions.

Theorem 8. *Let us assume that $f : (a, b) \to (0, \infty)$ is (M, A)-convex function, and decreasing function. Suppose that the conditions of Theorem 5 are satisfied, and that $M \leq A$. Then f is a convex function.*

Proof. By $f(M(x, y)) \leq \frac{f(x) + f(y)}{2}$ and f being decreasing, we obtain $f(M(x, y)) \geq f(A)$ as $M \leq A$. So we get

$$f(A) = f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2},$$

which means in fact that f is a Jensen–convex function [5].

On the other hand, by Theorem 5, f is continuous. It is well-known that, a Jensen–convex function which is continuous, coincides with a convex function, so the result follows. □

Theorem 9. *Let $f : (a, b) \to \mathbb{R}$ be an (M, N)-convex function, and satisfying the following properties:*

i) for all $x, y \in (a, b)$ there exists $z \in (a, b)$ such that

$$M(y, z) = x;$$

ii) there exists $\lambda \in (0, 1)$ such that for all $x, y \in (a, b)$ one has

$$M(x, y) \leq \lambda \max\{x, y\} + (1 - \lambda) \min\{x, y\}$$

If f is bounded from above, then f must be constant.

Proof. Let $c = \sup\{f(t) : t \in (a, b)\}$. Thus $f(t) \leq c$. Let $\epsilon > 0$ be arbitrary, and select $y \in (a, b)$ such that

$$f(y) > c - (1 - \lambda)\epsilon \qquad (*)$$

We will show that $f(x) \geq c - \epsilon$ for all $x \in (a, b)$. Then, by letting $\epsilon \to 0$, we get $f(x) \geq c$, which along with $f(x) \leq c$ gives $h(x) = c$.

Let us suppose that there is an $x_0 \in (a, b)$ with $f(x_0) < c - \epsilon$.

Then by i) $\exists z \in (a, b) : y = M(y_0, z)$.

Thus

$$\begin{aligned} c - (1 - \lambda)\epsilon < f(y) = f(M(y_0, z)) &\leq N(f(y_0), f(z)) \\ &\leq \lambda \max\{f(y_0), f(z)\} + (1 - \lambda) \min\{f(y_0), f(z)\} \\ &< \lambda c + (1 - \lambda)(c - \epsilon) = c - (1 - \lambda)\epsilon. \end{aligned}$$

This contradicts relation $(*)$. □

Remark 4. *1. If $M = N = A$, we get the following classical result: If $f : (a, b) \to \mathbb{R}$ is Jensen-convex, and bounded from above, then it is constant ([5]).*

2. For $M = N = L, \lambda = \frac{1}{2}$ we get that if f is L-convex, and bounded from above, then it is constant (Z. Kominek, T. Zgraja [9]).

Remark 5. *Finally, we mention two results wich have been proved in 1998 by the first author [7]:*

Theorem 10. *Let $f \in C[a, b]$ be strictly increasing such that $1/f^{-1}$ is a convex function (where f^{-1} is the inverse function of f). Then f is an $(L, \mathfrak{I})$-convex function, where L is the logarithmic mean and $\mathfrak{I}$ is the integral mean.*

Theorem 11. *Let $f \in C[a, b]$ be strictly increasing such that f^{-1} is log–convex. Then f is $(I, \mathfrak{I})$-concave function, where I is the identric mean, and $\mathfrak{I}$ is the integral mean.*

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