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Embedding index in some classes of graphs

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Abstract: A Subset S of the vertex set of a graph G is called a dominating set of G if each vertex of G is either in S or adjacent to at least one vertex in S. A partition $D = \{D_1, D_2, \ldots, D_k\}$ of the vertex set of G is said to be a domatic partition or simply a d-partition of G if each class D_i of D is a dominating set in G. The maximum cardinality taken over all d-partitions of G is called the domatic number of G denoted by $d(G)$. A graph G is said to be domatically critical or d-critical if for every edge x in G, $d(G - x) < d(G)$, otherwise G is said to be domatically non d-critical. The embedding index of a non d-critical graph G is defined to be the smallest order of a d-critical graph H containing G as an induced subgraph denoted by $\theta(G)$. In this paper, we find the $\theta(G)$ for the Barbell graph, the Lollipop graph and the Tadpole graph.

Keywords: Domination number, Domatic partition, Domatic number, d-Critical graphs.
AMS Classification: 05C69.

1 Introduction

A subset S of the vertex set of a graph G is called a dominating set of G if each vertex of G is either in S or adjacent to at least one vertex in S. A partition $D = \{D_1, D_2, \ldots, D_k\}$ of the vertex set of G is said to be a domatic partition or simply a d-partition of G if each class D_i of D is a dominating set in G. The maximum cardinality taken over all d partitions of G is called the domatic number of G denoted by $d(G)$.

A graph G is said to be domatically critical or d-critical if for every edge x in G, $d(G - x) < d(G)$, otherwise G is said to be domatically non d-critical. This concept was introduced by E. J. Cockayne and S. T. Hedetniemi in [3]. Further study on this class of graphs was carried out by B. Zelinka [1], D. F. Rall [2], H. B. Walikar, A. P. Deshpande, Savita Basapur and L. Sudershan Reddy [7, 8, 9 10].

It is not difficult to see that $d(G)-1 \le d(G-x) \le d(G)$ for any edge x in G and thus, the d-critical graphs may also be defined as those graphs G for which $d(G-x)=d(G)-1$ holds for every edge x in G. Graphs that are critical with respect to a given property frequently play an important role in the investigation of that property. The critical concept in graph theory was introduced by Dirac [6] in 1952 with respect to chromatic number of a graph mainly to study the four color conjecture. In [7, 8] it was proved that "Every non d-critical graph can be embedded in some d-critical graph" by constructing a d-critical graph of order $2p$ containing a given non d-critical graph of order p. Also, the path P_{3n-1}, a non d-critical graph can be viewed as an induced subgraph of the cycle C_{3n}, a d-critical graph. The embedding index of a non d-critical graph G is defined to be the smallest order of d-critical graph H containing G as an induced subgraph denoted by $\theta(G)$, i.e., $\theta(G)=\min\{p(H)-p(G) : H \in F\}$ where F is the family of all d-critical graphs H containing G as an induced subgraph ($p(H)$ denotes the order of H).

It is obvious that $1 \le \theta(G) \le p(G)$ for any non d-critical graph G.

A graph is said to be domatically full if and only if $d(G)=\delta(G)+1$, where $\delta(G)$ denotes the minimum degree of G.

Throughout this paper, by a graph G we mean a finite, undirected graph without multiple edges or loops. By K_m & P_n we mean a complete graph of m vertices and a path of n vertices.

2 Preliminary results

Theorem 2.1 [1]. Let G be a domatically critical graph with domatic number $d(G) = d$. Then the vertex set $V(G)$ of G is the union of d pair wise disjoint sets V_1, V_2, V_3, …, V_d with the property that for any two distinct integers i, j ; $1 \le i, j \le d$, the subgraph G_{ij} of G induced by the set $V_i \cup V_j$ is a bipartite graph on the sets V_i, V_j all of whose connected components are stars.

Theorem 2.2 [1]. A regular domatically full graph G with n vertices and with a domatic number d exists if and only if d divides n. Such a graph is also domatically critical.

Theorem 2.3 [9]. If G is regular and domatically full, then G is domatically critical, however the converse is not true.

Theorem 2.4 [8]. If G is a domatically full graph in which every edge is incident with a vertex of minimum degree, then G is d-critical.

Theorem 2.5 [12]. Let $G = C_n$. Then, $d(G)=\begin{Bmatrix} 3 & n \equiv 0 \bmod 3 \\ 2 & Otherwise \end{Bmatrix}$, $n>2$, $n \in N$.

2.1 Examples

Figure 1 below shows a non d-critical graph G and it is easy to see that there exists no d-critical graph of order six containing G as an induced subgraph, thus $\theta(G) \ge 2$.

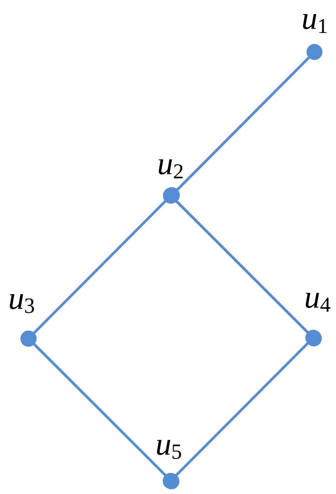

Figure 1. Non d-Critical graph G

Next, we consider the graph H in Figure 2 which contains graph G as an induced subgraph, whose vertex set $V(H) = V(G) \cup \{u, v\}$ and the edge set $E(H) = E(G) \cup \{uv, uu_4, vu_1\}$. The partition $D = \{D_1 = \{u_2, u\}, D_2 = \{u_1, u_4\}, D_3 = \{u_3, u_5, v\}\}$ is a d-partition of H and thus $d(H) \geq 3$ with $d(H) \leq \delta(H) + 1$. Hence $d(H) = 3 = \delta(H) + 1$, i.e., H is domatically full. Also it can be seen from Figure 2 that every edge of H is incident with vertex of minimum degree 2.

By Theorem 2.4, H is d-critical and it contains G as an induced subgraph. Hence, $\theta(G) = 2$.

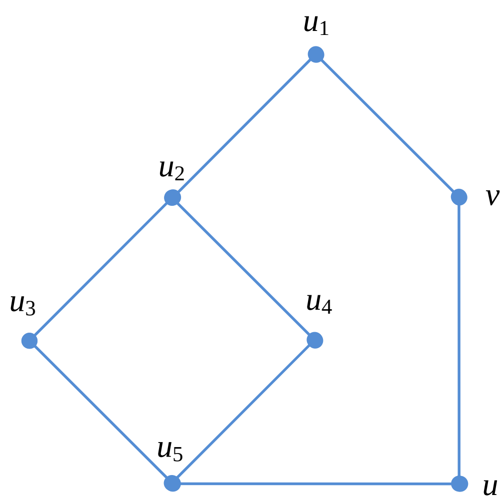

Figure 2. d-Critical graph H (containing G)

3 Definitions

Definition 3.1: Barbell graphs. A (N, n) Barbell graph is the graph obtained by connecting N copies of the complete graph K_n by a bridge. It is denoted by $B(N, n)$.

Definition 3.2: Lollipop Graph. A (m, n) Lollipop graph is the graph obtained by joining a complete graph K_m to a path P_n with a bridge. It is denoted by $L(m, n)$.

Definition 3.3: Tadpole graph. A (m, n) Tadpole graph is the graph obtained by joining a cycle C_m to a path P_n with a bridge. It is denoted by $T(m, n)$.

Definition 3.4: A graph G is called **indominable** if its vertex set can be partitioned into independent dominating sets.

Definition 3.5: A dominating set D of a graph G is called an **independent dominating set** of G if D is independent in G.

Figure 3. $B(2, 4)$ Barbell graph

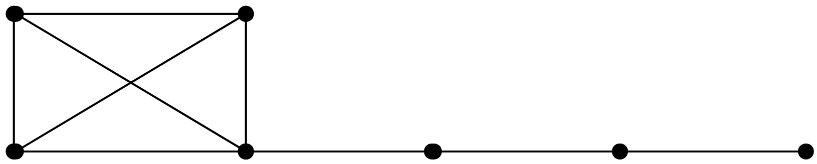

Figure 4. $L(4, 3)$ Lollipop graph

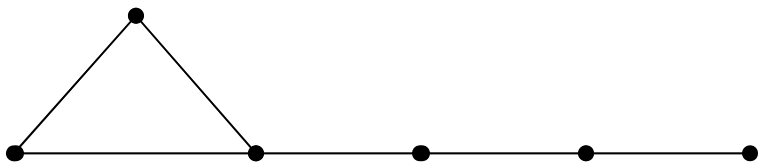

Figure 5. $T(3, 3)$ Tadpole graph

4 Main results

Theorem 4.1. Let $G = B(2, n)$. Then, $\theta(G) = 2$.

Proof. Let H be a d-critical graph containing G has an induced subgraph with $d(H) = k \geq 2$. Then by Theorem 2.1 the vertex set $V(H)$ can be partitioned into dominating sets $D_1, D_2, \ldots, D_k$ such that D_i is independent and $\langle D_i \cup D_j \rangle$ is union of stars for $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$. Define $D_i^1 = D_i \cap V(G)$ for $i = 1, 2, \ldots, k$. Then D_i^1 $i = 1, 2, \ldots, k$ are independent and $\langle D_i^1 \cup D_j^1 \rangle$ is a union of either stars or isolated vertices or both or empty graphs.

Therefore the vertices of G are distributed in D_i for $i = 1, 2, \ldots, k$. Thus each D_i will contain at least one vertex other than the vertices of G. Thus $p(H) \geq 2n + k \geq 2n + 4$ since $k \geq 2$. Therefore $\theta(G) \geq 2$.

Label the vertices of the two copies of K_n in G as $u_1, u_2, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ where u_n and v_n are of degree n. Let H be a super graph containing G as an induced subgraph. The graph H is obtained from G by adding two new vertices w_1 and w_2 such that vertices $u_1, u_2, u_3, \ldots, u_{n-1}, w_2$ are adjacent to w_1 and vertices $v_1, v_2, v_3, \ldots, v_{n-1}, w_1$ are adjacent to w_2. Now, the graph H will be n regular with $D_1 = \{u_n, w_2\}$, $D_2 = \{v_n, w_1\}$, $D_3 = \{u_1, v_1\}$, $D_4 = \{u_2, v_2\}$, …, $D_{n+1} = \{u_{n-1}, v_{n-1}\}$ as its domatic partitions.

Thus $d(H) \geq 2$ but $d(H) \leq \delta(H) + 1 = n + 1$. Furthermore H is d-critical since every edge in H is incident with a vertex of minimum degree ($\delta(H) = n$) and it is domatically full. Therefore we have $\theta(G) \leq 2$. Hence $\theta(G) = 2$. □

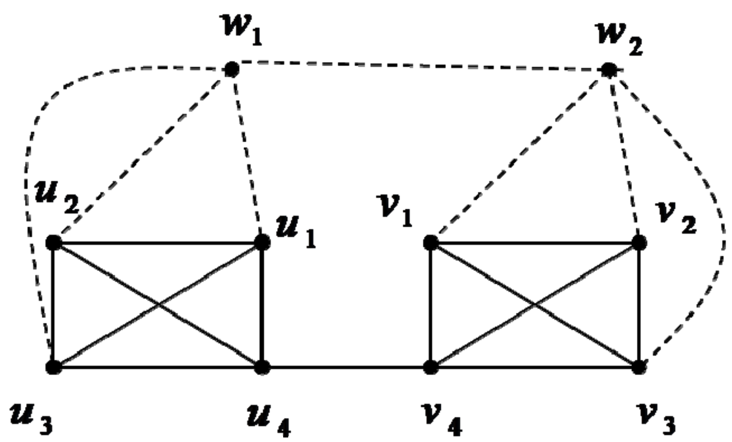

Figure 6. Graph H containing (2, 4) Barbell graph as an induced subgraph

Corollary 4.1.1. Let $G = L(n, 1)$. Then $\theta(G) \leq n + 1$.

Proof. By adding the $n - 1$ vertices to $L(n, 1)$, a $(2, n)$ Barbell graph can be constructed from $L(n, 1)$. Hence the $(2, n)$ Barbell graph is a super graph containing G as an induced subgraph. Therefore, $\theta(L(n,1)) \leq n - 1 + 2 \leq n + 1$, Hence the proof. □

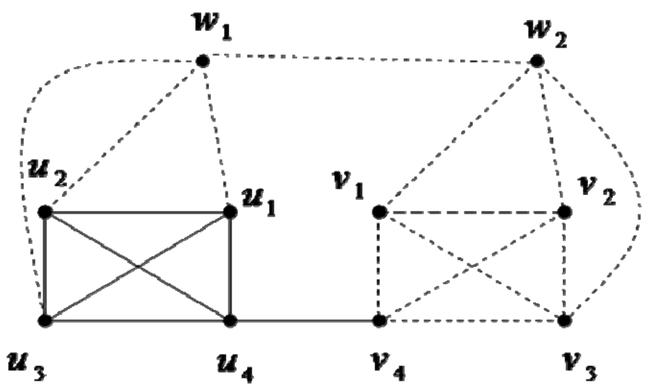

Figure 7. Graph H containing $L(4, 1)$ Lollipop graph
as an induced subgraph

Theorem 4.2. Let $G = T(3,3n-1)$. Then, $\theta(G) = 2$, $n \in N$.

***Proof*.** Let H be a d-critical graph containing G has an induced subgraph with $d(H) = k \geq 2$. Then by Theorem 2.1 the vertex set $V(H)$ can be partitioned into dominating sets $D_1, D_2, \ldots, D_k$ such that D_i is independent and $\langle D_i \cup D_j \rangle$ is union of stars for $i \neq j$ and $i, j \in \{1, 2, ..., k\}$. Define $D_i^1 = D_i \cap V(G)$ for $i = 1, 2, \ldots, k$. Then D_i^1 $i = 1, 2, \ldots, k$ are independent and $\langle D_i^1 \cup D_j^1 \rangle$ is a union of either stars or isolated vertices or both or empty graphs.

Therefore the vertices of G are distributed in D_i for $i = 1, 2, \ldots, k$. Thus each D_i will contain at least one vertex other than the vertices of G. Thus $p(H) \geq 3n + 2 + k \geq 3n + 4$ since $k \geq 2$. Therefore $\theta(G) \geq 2$.

Label the vertices of the cycle as u_1, u_2, u_3 and vertices of the path as $v_1, v_2, v_3, ..., v_{3n-1}$. Let H be a super graph containing G as an induced subgraph which is obtained from G by adding two new vertices w_1 and w_2 such that u_1, w_1, w_2, v_{3n-1} forms a path of length three in G.

Then $D_1 = \{u_3, v_3, v_6, v_9, ..., w_2\}$, $D_2 = \{u_1, v_2, v_8, ..., v_{3n-1}\}$, $D_3 = \{u_2, v_1, v_4, v_7, ..., w_1\}$ are the domatic partitions of H. Thus $d(H) \geq 2$ but $d(H) \leq \delta(H) + 1 = 3$. Further H is d-critical since every edge in H is incident with a vertex of minimum degree ($\delta(H) = 2$) except the edge u_1u_3. Since $H - u_1u_3$ is a cycle of length $3n + 4$, by Theorem 2.5, $d(H - u_1u_3) = 2$ and also domatically full. Therefore, we have $\theta(G) \leq 2$. Hence $\theta(G) = 2$. □

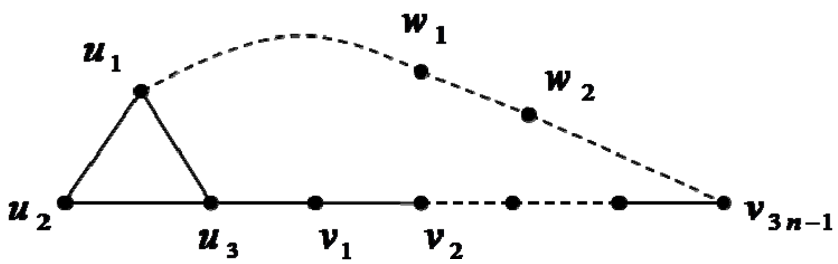

Figure 8. Graph H containing Tadpole graph $T(3, 3n - 1)$
as an induced subgraph

Theorem 4.3. Let $G = T(3,3n)$. Then, $\theta(G) = 1$, $n \in N$.

***Proof*.** Let H be a d-critical graph containing G has an induced subgraph with $d(H) = k \geq 1$. Then by Theorem 2.1 the vertex set $V(H)$ can be partitioned into dominating sets $D_1, D_2, \ldots, D_k$ such that D_i is independent and $\langle D_i \cup D_j \rangle$ is union of stars for $i \neq j$ and $i, j \in \{1, 2, ..., k\}$. Define $D_i^1 = D_i \cap V(G)$ for $i = 1, 2, \ldots, k$. Then D_i^1 $i = 1, 2, \ldots, k$ are independent and $\langle D_i^1 \cup D_j^1 \rangle$ is a union of either stars or isolated vertices or both or empty graphs.

Therefore the vertices of G are distributed in D_i for $i = 1, 2, \ldots, k$. Thus each D_i will contain at least one vertex other than the vertices of G. Thus $p(H) \geq 3n + 3 + k \geq 3n + 4$ since $k \geq 1$. Therefore $\theta(G) \geq 1$.

Label the vertices of the cycle as u_1, u_2, u_3 and vertices of the path as $v_1, v_2, v_3, ..., v_{3n}$. Let H be a super graph containing G as an induced subgraph which is obtained from G by adding one new vertex w_1 such that u_1, w_1, v_{3n} forms a path of length two in G. Then $D_1 = \{u_3, v_3, v_6, ..., v_{3n}\}$, $D_2 = \{u_1, v_2, v_5, ..., v_{3n-1}\}$, $D_3 = \{u_2, v_1, v_4, v_7, ..., v_{3n-2}, w_1\}$ are the domatic partitions of H. Thus $d(H) \ge 1$ but $d(H) \le \delta(H) + 1 = 3$. Further H is d-critical since every edge in H is incident with a vertex of minimum degree ($\delta(H) = 2$) except edge u_1u_3. Since $H - u_1u_3$ is a cycle of length $3n + 4$, by Theorem 2.5, $d(H - u_1u_3) = 2$ and also domatically full. Therefore we have $\theta(G) \le 1$. Hence $\theta(G) = 1$. □

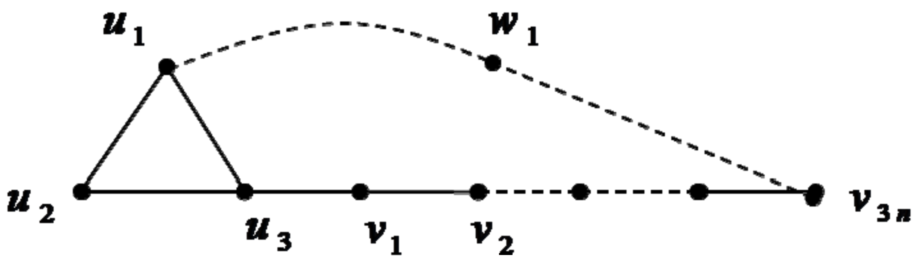

Figure 9. Graph H containing Tadpole graph $T(3,3n)$ as an induced subgraph

Theorem 4.4. Let $G = T(3, 3n-2)$. Then, $\theta(G) = 3$, $n > 1, n \in \mathbf{N}$.

*Proof***.** Let H be a d-critical graph containing G has an induced subgraph with $d(H) = k \ge 3$. Then by Theorem 2.1 the vertex set $V(H)$ can be partitioned into dominating sets $D_1, D_2, \ldots, D_k$ such that D_i is independent and $\langle D_i \cup D_j \rangle$ is union of stars for $i \ne j$ and $i, j \in \{1, 2, ..., k\}$. Define $D_i^1 = D_i \cap V(G)$ for $i = 1, 2, \ldots, k$. Then D_i^1 $i = 1, 2, \ldots, k$ are independent and $\langle D_i^1 \cup D_j^1 \rangle$ is a union of either stars or isolated vertices or both or empty graphs.

Therefore the vertices of G are distributed in D_i for $i = 1, 2, \ldots, k$. Thus each D_i will contain at least one vertex other than the vertices of G. Thus $p(H) \ge 3n + 1 + k \ge 3n + 4$ since $k \ge 3$. Therefore $\theta(G) \ge 3$.

Label the vertices of the cycle as u_1, u_2, u_3 and vertices of the path as $v_1, v_2, v_3, ..., v_{3n-2}$. Let H be a super graph containing G as an induced subgraph which is obtained from G by adding three new vertices w_1, w_2 and w_3 such that $u_1, w_1, w_2, w_3, v_{3n-2}$ forms a path of length four. Then $D_1 = \{u_3, v_3, v_6, ..., v_{3n-3}, w_2\}$, $D_2 = \{u_1, v_2, v_5, ..., v_{3n-4}, w_3\}$, $D_3 = \{u_2, v_1, v_4, v_7, ..., v_{3n-2}, w_1\}$ are the domatic partitions of H. Thus $d(H) \ge 3$ but $d(H) \le \delta(H) + 1 = 3$. Further H is d-critical since every edge in H is incident with a vertex of minimum degree ($\delta(H) = 2$) except edge u_1u_3. Since $H - u_1u_3$ is a cycle of length $3n + 4$, by Theorem 2.5, $d(H - u_1u_3) = 2$ and also domatically full. Therefore we have $\theta(G) \le 3$. Hence $\theta(G) = 3$. □

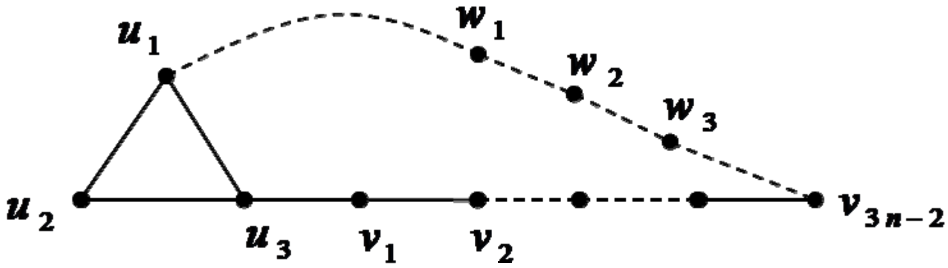

Figure 10. Graph *H* containing Tadpole graph *T*(3, 3*n* – 2)
as an induced subgraph

For *n* = 1, $\theta(T(3,1)) = 1$. Let *H* be a super graph containing *T*(3, 1) as an induced subgraph. The graph *H* is obtained from *T*(3, 1) by adding one new vertex w_1 such that w_1 is adjacent to u_3 and v_1. Then $D_1 = \{u_3\}$, $D_2 = \{u_1, v_1\}$, $D_3 = \{u_2, w_1\}$ is the domatic partition of *H*. Removal of any edge decreases the domatic partition.

Hence *H* is *d*-critical. □

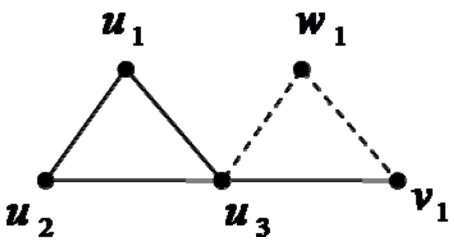

Figure 11. Graph *H* containing Tadpole graph *T*(3, 1)
as an induced subgraph

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