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On directed pathos line cut vertex digraph of an arborescence

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Abstract: In this paper we define the digraph valued function(digraph operator), namely the line cut vertex digraph $n(D)$ of a digraph D and the directed pathos line cut vertex digraph $DPn(T)$ of an arborescence T. Planarity, outer planarity, maximal outer planarity, minimally non-outer planarity, and crossing number one properties of $DPn(T)$ are discussed. Also, the problem of reconstructing an arborescence from its directed pathos line cut vertex digraph is presented.
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1 Introduction

We shall assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [3, 6]. The concept of $pathos$ of a graph G was introduced by Harary [4] as a collection of minimum number of edge disjoint open paths whose union is G. The path number of a graph G is the number of paths in any pathos. The path number of a tree T equals k, where $2k$ is the number of odd degree vertices of T. Harary [5] and Stanton [8] have calculated the path number of certain classes of graphs like trees and complete graphs.

For a tree T with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$, edge set $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$ and cut vertex set $C(T) = \{C_1, C_2, \ldots, C_r\}$, detailed by R. Chandrasekhar, et al. [2], gave the following definition. The *pathos line cut vertex graph of a tree* T, written $Pn(T)$, is the graph whose vertices are the edges, paths of pathos and cut vertices of T, with two vertices of $Pn(T)$ adjacent whenever the corresponding edges of T are adjacent, the edge lie on the corresponding path of pathos and the edge incident with the cut vertex.

In this paper, we extend the definition of the pathos line cut vertex graph of a tree to an arborescence. Furthermore, some of its characterizations such as the planarity, outer planarity, etc., are discussed.

M.Aigner[1] defines the *line digraph* of a digraph as follows. Let D be a digraph with n vertices $v_1, v_2, \ldots, v_n$ and m arcs and $L(D)$ its associated *line digraph* with $n^{'}$ vertices and $m^{'}$ arcs. We immediately have $n^{'} = m$ and $m^{'} = \sum_{i=1}^{n} d^-(v_i){\cdot}d^+(v_i)$. Furthermore, the in-respectively out-degree of a vertex $v^{'} = (v_i, v_j)$ in $L(D)$ are $d^-(v^{'}) = d^-(v_i)$, $d^+(v^{'}) = d^+(v_j)$.

We need some concepts and notations on directed graphs. A *directed graph*(or just $digraph$) D consists of a finite non-empty set $V(D)$ of elements called $vertices$ and a finite set $A(D)$ of ordered pair of distinct vertices called $arcs$. Here $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. If (u, v) or uv is an arc in D, then we say that u dominates v(v is dominated by u) and denote it by $u \rightarrow v$. A digraph D is $semicomplete$ if for each pair of distinct vertices u and v, at least one of the arcs (u, v) and (v, u) exists in D. A semicomplete digraph of order n is denoted by D_n.

For a connected digraph D, a vertex z is called a *cut vertex* if $D - \{z\}$ has more than one connected component. A $block$ of a digraph D is a maximal connected subgraph B of D such that no vertex of B is a cut vertex of D. The *out-degree* of a vertex v, written $d^+(v)$, is the number of arcs going out from v and the *in-degree* of a vertex v, written $d^-(v)$, is the number of arcs coming into v. The *total degree* of a vertex v, written $td(v)$, is the number of arcs incident with v. We immediately have $td(v) = d^-(v) + d^+(v)$.

A vertex with an in-degree(out-degree) zero is called a *source(sink)*. An *out-star*(*in-star*) in a digraph D is a star in the underlying undirected graph of D such that all arcs are directed out of(into) the center. The directed path on $n \geq 2$ vertices is the digraph $\vec{P}_n = \{V(\vec{P}_n), E(\vec{P}_n), \eta\}$, where $V(\vec{P}_n) = \{u_1, u_2, \ldots, u_n\}$, $E(\vec{P}_n) = \{e_1, e_2, \ldots, e_{n-1}\}$, where η is given by $\eta(e_i) = (u_i, u_{i+1})$, for all $i \in \{1, 2, \ldots, (n-1)\}$.

An *arborescence* T is a directed graph in which, for a vertex u called the $root$ (a vertex of in-degree zero) and any other vertex v, there is exactly one directed path from u to v. A *root arc* of T is an arc which directed out from the root of T, i.e., an arc whose tail is the root of T.

Since most of the results and definitions for undirected planar graphs are valid for planar digraphs also, the following definitions holds good for planar digraphs.

If D is a planar digraph, then the *inner vertex number* $i(D)$ of D is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of D in the plane. A digraph D is an $outerplanar$ if $i(D) = 0$ and *minimally non-outerplanar* if $i(D) = 1$, [7]. The *crossing number* of a digraph D, denoted by $\mathrm{cr}(D)$, is the minimum number of crossings of its arcs when the digraph D is drawn in the plane.

2 Definition of $n(D)$ and $DPn(T)$

If a directed path $\vec{P}_n$ starts at one vertex and ends at a different vertex, then $\vec{P}_n$ is called an *open directed path*. A directed path is said to be *non-empty* if it has at least one arc. The *directed pathos* of an arborescence T is defined as the minimum number of arc disjoint open directed paths whose union is T. The *directed path number* k' of T is the number of open directed paths in any directed pathos of T, and is equal to the number of sinks in T. Finally, we assume that the direction of the directed pathos is along the direction of the arcs in T.

We now define the digraph operators "line cut vertex digraph of a digraph" and "directed pathos line cut vertex digraph of an arborescence" as follows:

For a connected digraph D, the *line cut vertex digraph* $Q = n(D)$ has vertex set $V(Q) = A(D) \cup C(D)$ and the arc set

$$A(Q) = \begin{cases} ab : a, b \in A(D), \text{the head of } a \text{ coincides with the tail of } b, \\ Cd : C \in C(D), d \in A(D), \text{the tail } d \text{ is } C, \\ dC : C \in C(D), d \in A(D), \text{the head of } d \text{ is } C, \end{cases}$$

where $C(D)$ is the cut vertex set of D.

For an arborescence T, the *directed pathos line cut vertex digraph* $Q = DPn(T)$ has vertex set $V(Q) = A(T) \cup C(T) \cup P(T)$ and the arc set

$$A(Q) = \begin{cases} ab : a, b \in A(T), \text{the head of } a \text{ coincides with the tail of } b, \\ Cd : C \in C(T), d \in A(T), \text{the tail } d \text{ is } C, \\ dC : C \in C(T), d \in A(T), \text{the head of } d \text{ is } C, \\ Pa : a \in A(T), P \in P(T), \text{the directed path } P \text{ lie on arc } a, \\ P_iP_j : P_i, P_j \in P(T), \text{the directed paths } P_i(v_s, v_t) \text{ and } P_j(v_k, v_l) \\ \qquad\qquad\qquad \text{have a common vertex, say } v_c \text{ in } T. \end{cases}$$

Here $C(T)$ is the cut vertex set and $P(T)$ is the directed path set of T.

It should be noted from the definition that, in $A(Q)$, it suffices to write an arc P_iP_j (instead of P_jP_i) as it is possible to reach the head of P_j from the tail of P_i through a common vertex in T. Furthermore, if the out-degree of the root of T is more than one, then $DPn(T)$ is disconnected. Hence we consider the arborescences having out-degree of the root exactly one.

See Fig. 1 and Fig. 2 for an example of an arborescence and its directed pathos line cut vertex digraph.

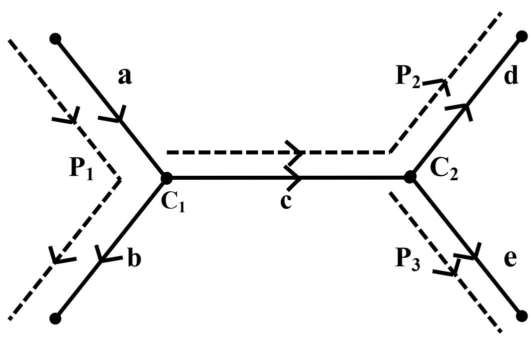

Figure 1. Arborescence T

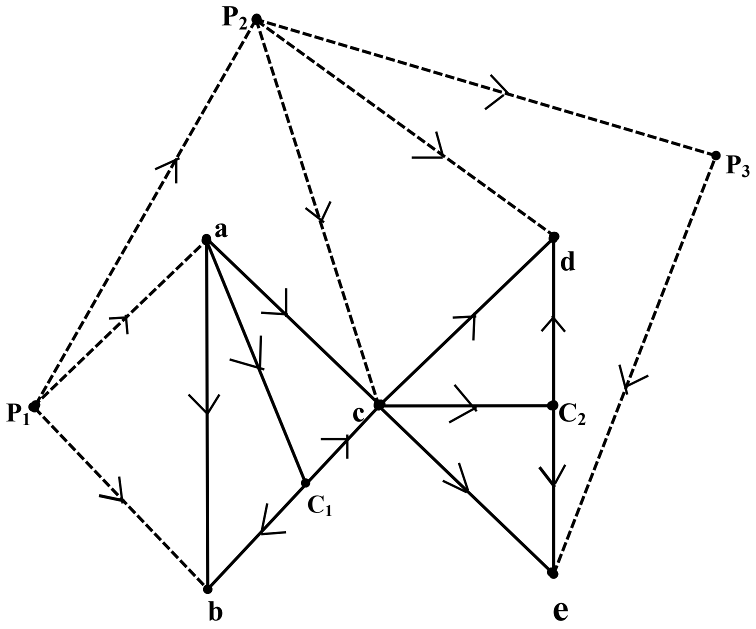

Figure 2. $DPn(T)$

The following existing Theorems are required to prove further results:
Theorem A[3]: Every maximal outerplanar graph G with p vertices has $(2p-3)$ edges.
Theorem B[6]: A directed multi digraph D is Eulerian if and only if D is connected and $d^{-}(x) = d^{+}(x)$, for every vertex $x \in D$.

3 Observations

The following observations are easily justified

Observation 3.1. *If T is an arborescence with $n \geq 3$ vertices, then $L(T) \subseteq n(T) \subseteq DPn(T)$. Here $\subseteq$ is the subdigraph notation.*

Observation 3.2. *The number of arcs whose end vertices are the directed path vertices in $DPn(T)$ is $(k^{'}-1)$, where $k^{'}$ is the directed path number of T.*

Observation 3.3. *A connected digraph D with n vertices $v_1, v_2, \ldots, v_n$ and m arcs is a non-empty directed path if $\sum_{i=1}^{n} d^{-}(v_i) \cdot d^{+}(v_i) + m - 2n + 3 = 0$, where d^{-} and d^{+} be the in-degree and out-degree of vertices of D, respectively.*

4 Decomposition and reconstruction

One of the major challenges in the study of digraph operators is to reproduce the original digraph from the digraph operator, i.e., when is an acyclic digraph the directed pathos line cut vertex digraph of some arborescence T and is T reconstructible from $DPn(T)$?

A digraph is a *complete bipartite digraph* if its vertex set can be partitioned into two sets A, B in such a way that every arc has its initial vertex in A and its terminal vertex in B and any two vertices $a \in A$ and $b \in B$ are joined by an arc. Let T be an arborescence with vertex set $V(T) = \{v_1, v_2, \dots, v_n\}$, cut vertex set $C(T) = \{C_1, C_2, \dots, C_r\}$ and directed path set $P(T) = \{P_1, P_2, \dots, P_s\}$. We consider the following five cases.

Case 1: Let v be a vertex of T with $d^-(v) = \alpha$ and $d^+(v) = \beta$. Then α arcs coming into v and the β arcs going out of v give rise to a complete bipartite subdigraph with α tails and β heads and $\alpha \cdot \beta$ arcs joining each tail with each head. This is the decomposition of $L(T)$ into mutually arc disjoint complete bipartite subdigraphs.

Case 2: Let C be a cut vertex of T with $d^-(C) = \alpha^{'}$. Then $\alpha^{'}$ arcs coming into C give rise to a complete bipartite subdigraph with $\alpha^{'}$ tails and a single head(i.e., C itself) and $\alpha^{'}$ arcs joining each tail with C.

Case 3: Let C be a cut vertex of T with $d^+(C) = \beta^{'}$. Then $\beta^{'}$ arcs going out of C give rise to a complete bipartite subdigraph with a single tail (i.e., C itself) and $\beta^{'}$ heads and $\beta^{'}$ arcs joining C with each head.

Case 4: Let P be a directed path that lie on $\alpha^{''}$ arcs in T. Then $\alpha^{''}$ arcs give rise to a complete bipartite subdigraph with a single tail (i.e., P itself) and $\alpha^{''}$ heads and $\alpha^{''}$ arcs joining P with each head.

Case 5: Let P be a directed path and $\beta^{''}$ be the number of directed paths whose heads are reachable from the tail of P through the common vertex in T. Then $\beta^{''}$ arcs give rise to a complete bipartite subdigraph with a single tail (i.e., P itself) and $\beta^{''}$ heads and $\beta^{''}$ arcs joining P with each head.

Hence by all above cases, $H = DPn(T)$ is decomposed (see Fig. 3) into mutually arc-disjoint complete bipartite subdigraphs with $V(H) = A(T) \cup C(T) \cup P(T)$ and arc sets:

(i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i be the sets of in-coming and out-going arcs at v_i of T, respectively,

(ii) $\cup_{j=1}^{r} \cup_{k=1}^{r} Z_j^{'} \times C_k$ such that $Z_j^{'} \times C_k = 0$ for $j \neq k$,

(iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} C_k \times Z_j$ such that $C_k \times Z_j = 0$ for $k \neq j$, where $Z_j^{'}$ and Z_j be the sets of in-coming and out-going arcs at C_j of T, respectively,

(iv) $\cup_{k=1}^{s} \cup_{j=1}^{s} P_k \times Y_j$ such that $P_k \times Y_j = 0$ for $k \neq j$,

(v) $\cup_{k=1}^{s} \cup_{j=1}^{s} P_k \times Y_j^{'}$ such that $P_k \times Y_j^{'} = 0$ for $k \neq j$,

where Y_j is the set of arcs on which P_k lies and $Y_j^{'}$ is the set of directed paths whose heads are reachable from the tail of P_k through the common vertex in T.

Conversely, let H be an acyclic digraph of the type described above. We first reconstruct T without directed pathos. For that, let us denote each complete bipartite subdigraphs obtained by Case (1) by $T_1, T_2, \dots, T_l$.

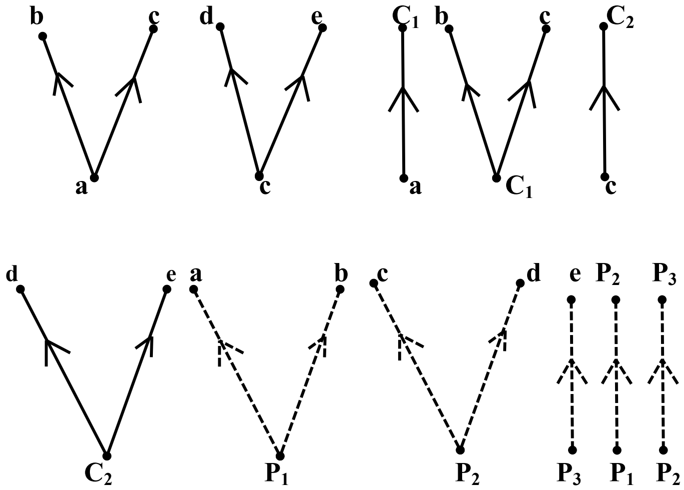

Figure 3. Decomposition of $DPn(T)$ into complete bipartite subdigraphs.

Let the vertex set of T be $V(T) = \{t_0, t_1, \ldots, t_l, t_{l+1}, t_1^{'}, t_2^{'}, \ldots, t_s^{'}\}$, where $t_1^{'}, t_2^{'}, \ldots, t_s^{'}$ are the vertices corresponding to pendant arcs $e_1^{'}, e_2^{'}, \ldots, e_s^{'}$ of T, respectively. The arcs of T are obtained by the following procedure. For each vertex $v \in L(T)$, we draw an arc, say a_v to T as follows.

(a) If $d_{L(T)}^{+}(v) > 0$, $d_{L(T)}^{-}(v) = 0$, then $a_v = (t_0, t_i)$, where i is the base (or index) of T_i such that $v \in X_i$;

(b) If $d_{L(T)}^{+}(v) = 0$, $d_{L(T)}^{-}(v) > 0$, then $a_v = (t_j, t_n^{'})$, $n = 1, 2, \ldots, s$, where j is the base of T_j such that $v \in Y_j$;

(c) If $d_{L(T)}^{+}(v) > 0$, $d_{L(T)}^{-}(v) > 0$, then $a_v = (t_i, t_j)$, where i and j are the indices of T_i and T_j such that $v \in X_j \cap Y_i$.

We now mark the cut vertices of D as follows. From Case (2) and Case (3), we observe that for every cut vertex C, there exists at most two complete bipartite subdigraphs, one containing C as the tail, and other as head. Let it be $C_j^{'}$ and $C_j^{''}$, for $1 \leq j \leq r$ such that $C_j^{'}$ contains C as the tail and $C_j^{''}$ as head. If the heads of $C_j^{'}$ and tails of $C_j^{''}$ are the heads and tails of a single T_i, for $1 \leq i \leq l$, then the vertex t_i is a cut vertex in the reconstruction, where i is the index of T_i. If D has an end arc, then a vertex of an end arc whose total degree at least two is a cut vertex in the reconstruction.

Now, the number of directed pathos of T equals the number of subdigraphs obtained by Case (4), i.e., P_z, for $z = 1, 2, \ldots, s$. Let ϕ be the number of tails in each P_z. We mark the directed pathos of T as follows. Suppose we mark the open directed path P_1 in T. For this we choose any ϕ (i.e., number of tails of subdigraph P_1 in Case (4)) number of arcs in T and mark P_1 on them. Similarly, we choose any ϕ (i.e., number of tails of subdigraph P_2 in Case (4)) number of arcs in T and mark P_2 on them. This process is repeated till every open directed path is marked in T. An acyclic digraph T thus constructed apparently has H as its directed pathos line cut vertex digraph. Hence we have the following Theorem.

Theorem 4.1. *H is the directed pathos line cut vertex digraph of an arborescence T if and only if $V(H) = A(T) \cup C(T) \cup P(T)$ and arc set $A(H)$ equals:*

(i) $\cup_{i=1}^{n} X_i \times Y_i$,

(ii) $\cup_{j=1}^{r} \cup_{k=1}^{r} Z_j^{'} \times C_k$ *such that* $Z_j^{'} \times C_k = 0$ *for* $j \neq k$,

(iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} C_k \times Z_j$ *such that* $C_k \times Z_j = 0$ *for* $k \neq j$,

(iv) $\cup_{k=1}^{s} \cup_{j=1}^{s} P_k \times Y_j$ *such that* $P_k \times Y_j = 0$ *for* $k \neq j$,

(v) $\cup_{k=1}^{s} \cup_{j=1}^{s} P_k \times Y_j^{'}$ *such that* $P_k \times Y_j^{'} = 0$ *for* $k \neq j$.

5 Basic properties of $DPn(T)$

In this section, we establish some basic relationships between an arborescence and its directed pathos line cut vertex digraph.

Proposition 5.1. *Let T be an arborescence with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and cut vertex set $C(T) = \{C_1, C_2, \ldots, C_r\}$. Then the number order and size of $PBn(T)$ are $m + k^{'} + \sum_{j=1}^{r} C_j$ and $\sum_{i=1}^{n} d^{-}(v_i) \cdot d^{+}(v_i) + \sum_{j=1}^{r} \{d^{-}(C_j) + d^{+}(C_j)\} + m + k^{'} - 1$, respectively, where m is the size and $k^{'}$ is the directed path number of T.*

Proof. Let T be an arborescence with $V(T) = \{v_1, v_2, \ldots, v_n\}$ and $C(T) = \{C_1, C_2, \ldots, C_r\}$. Then the order of $PBn(T)$ equals the sum of arcs, cut vertices, and open directed paths of T. Hence the order of $PBn(T)$ is $m + k^{'} + \sum_{j=1}^{r} C_j$. Now, the size of $PBn(T)$ equals the sum of sizes of T and $L(T)$, total degree of cut vertices of T, and the number of arcs whose end vertices are the directed pathos vertices. By Observation 3.2, the size of $PBn(T)$ is $\sum_{i=1}^{n} d^{-}(v_i) \cdot d^{+}(v_i) + \sum_{j=1}^{r} \{d^{-}(C_j) + d^{+}(C_j)\} + m + k^{'} - 1$. This completes the proof. □

Proposition 5.2. *For any arborescence T, $DPn(T)$ is non-Eulerian.*

Proof. By definition of $DPn(T)$, there exists an one directed path vertex $P \in DPn(T)$ such that $d^{-}(P) = 0$ and $d^{+}(P) > 0$. Thus, $d^{-}(P) \neq d^{+}(P)$. By Theorem[B], $DPn(T)$ is non-Eulerian. □

Proposition 5.3. *Every $DPn(T)$ is weakly connected digraph and an acyclic. Hence it is non-Hamilton.*

Proof. The proof is obvious. □

6 Characterization of $DPn(T)$

We will give a constructive proof to show the planarity, outer planarity, and crossing number one properties of $DPn(T)$.

Theorem 6.1. *For an arborescence T, $DPn(T)$ is planar if and only if $td(v) \leq 3$, for every vertex $v \in T$.*

Proof. Suppose $DPn(T)$ is planar. Assume that $td(v) \geq 4$. Suppose that there exists a vertex v of total degree four in T. Let $V(T) = \{a, b, c, d, e\}$ and $A(T) = \{(a, c), (c, b), (c, d), (c, e)\}$ such that a and (a, c) be the root and root arc of T, respectively. Then $A(L(T)) = \{(ac, cb), (ac, cd), (ac, ce)\}$. Since $v = c$ is the cut vertex of T, it is the tail of arcs $T = \{(c, b), (c, d), (c, e)\}$ and head of arc $H = (a, c)$. Then the arcs incident into c from the vertices corresponding to arc of H, and the arcs incident out of v reaches the vertices corresponding to arcs of T in $L(T)$ gives $n(T)$ such that cr$(n(T)) = 0$. Let $P = \{P_1, P_2, P_3\}$ be the directed path set of T such that P_1 lie on arcs (a, c) and (c, d), P_2 lie on (c, b) and P_3 lie on (c, e) in T. On embedding $DPn(T)$ for the dominancy of directed pathos vertices(i.e., vertices corresponding to directed pathos of T) with the vertices of $L(T)$ gives $DPn(T)$ such that the cr$(DPn(T)) = 1$, a contradiction.

For the sufficiency, we consider the following two cases.

Case 1: If T is a directed path $\vec{P}_n$ on $n \geq 3$ vertices, then every subdigraph of $DPn(T)$ is $D_4 - e$. Clearly, cr$(DPn(T)) = 0$. Thus $DPn(T)$ is planar.

Case 2: Suppose T is not a directed path such that $td(v) \leq 3$. Let $V(T) = \{v_1, v_2, \ldots, v_n\}$ and $A(T) = \{e_1, e_2, \ldots, e_{n-1}\}$ such that v_1 and $e_1 = v_1v_2$ be the root and root arc of T, respectively. Then $L(T)$ is an out-tree of order $(n-2)$. The number of cut vertices of T equals the number of vertices of T whose total degree is at least two. Then each block of $n(T)$ is either D_2 or D_3 or $D_4 - e$. Furthermore, the directed path number k' of T is the number of sinks in T. On embedding $DPn(T)$ for the dominancy of directed pathos vertices with the vertices of $L(T)$ and the dominancy of directed pathos vertices gives $DPn(T)$ such that cr$(DPn(T)) = 0$. Hence $DPn(T)$ is planar. □

Theorem 6.2. *For an arborescence T, $DPn(T)$ is an outerplanar if and only if T is a directed path on $n \geq 3$ vertices.*

Proof. Suppose $DPn(T)$ is an outerplanar. Assume that T is an arborescence whose underlying graph is a star graph $K_{1,3}$. Let $V(T) = \{a, b, c, d\}$ and $A(T) = \{(a, b), (b, c), (b, d)\}$ such that a and (a, b) be the root and root arc of T, respectively. Then $A(L(T)) = \{(ab, bc), (ab, bd)\}$. Since b is the cut vertex of T, it is the tail of arcs $T = \{(b, c), (b, d)\}$ and head of arc $H = (a, b)$. Then the arcs incident into v from the vertices corresponding to arc of H, and the arcs incident out of b reaches the vertices corresponding to arcs of T in $L(D)$ gives $n(T)$, i.e, $n(T) = D_4 - e$. Thus $i(n(T)) = 0$. Let $P = \{P_1, P_2\}$ be the directed path set of T such that P_1 lie on arcs (a, b), (b, c) and P_2 lie on (b, d) in T. On embedding $DPn(T)$ for the dominancy of directed pathos vertices with the vertices of $L(T)$ and the dominancy of directed pathos vertices gives $DPn(T)$ such that $i(DPn(T)) = 1$, a contradiction.

Conversely, suppose T is a directed path $\vec{P}_n$ on $n \geq 3$ vertices. Then every subdigraph of $DPn(T)$ is $D_4 - e$. Clearly, $i(DPn(T)) = 0$. Hence $DPn(T)$ is an outerplanar. □

Theorem 6.3. *For an arborescence T, $DPn(T)$ is maximal outerplanar if and only if T is a directed path $\vec{P}_n$ on $n \geq 3$ vertices.*

Proof. Suppose $DPn(T)$ is maximal outerplanar. Then $DPn(T)$ is connected. Hence T is connected. If $DPn(T)$ is D_2, then obviously T is also D_2. Let T be an arborescence with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and cut vertex set $C(T) = \{C_1, C_2, \ldots, C_r\}$. By Proposition 5.1, the order of $DPn(T)$ is $m + k^{'} + \sum_{j=1}^{r} C_j$ and the size is $\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + \sum_{j=1}^{r}(d^-(C_j) + d^+(C_j)) + m + k^{'} - 1$. Since $DPn(T)$ is maximal outerplanar, by Theorem [A], the size of $PBn(T)$ is $(2n-3)$, i.e., $2(m + k^{'} + \sum_{j=1}^{r} C_j) - 3$. Hence,

$$\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + \sum_{j=1}^{r}(d^-(C_j) + d^+(C_j)) + m + k^{'} - 1 = 2(m + k^{'} + \sum_{j=1}^{r} C_j) - 3,$$

$$\Rightarrow \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + \sum_{j=1}^{r}(d^-(C_j) + d^+(C_j)) + m + k^{'} - 1 = 2(n - 1 + k^{'} + \sum_{j=1}^{r} C_j) - 3,$$

$$\Rightarrow \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + \sum_{j=1}^{r}(d^-(C_j) + d^+(C_j)) + m = 2n - 3 + k^{'} - 1 + 2\sum_{j=1}^{r} C_j.$$

It is known that for an arborescence which is a directed path, the directed path number $k^{'} = 1$.

$$\therefore \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + \sum_{j=1}^{r}(d^-(C_j) + d^+(C_j)) + m = 2n - 3 + 2\sum_{j=1}^{r} C_j.$$

Since every cut vertex is of total degree two in a directed path,

we have $\sum_{j=1}^{r}(d^-(C_j) + d^+(C_j)) = 2\sum_{j=1}^{r} C_j$.

$$\therefore \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + m = 2n - 3.$$

$\Rightarrow \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + m - 2n + 3 = 0$. By Observation 3.3, T is a non-empty directed path. The necessity is thus proved.

Sufficiency. Suppose that T is a non-empty directed path $\vec{P}_n$ on $n \geq 3$ vertices. We show that $DPn(T)$ is maximal outerplanar by the induction on the number of vertices (≥ 3) of T. It is easy to observe that $DPn(T)$ of a directed path $\vec{P}_3$ is $D_4 - e$, which is a maximal outerplanar.

As the inductive hypothesis, let the directed pathos line cut vertex digraph of a non-empty directed path T with n vertices be maximal outerplanar.

We now prove that the directed pathos line cut vertex digraph of a directed path $T^{'}$ with $(n+1)$ vertices is maximal outerplanar. We first prove that it is an outerplanar.

Let the vertex and arc sequence of a directed path $T^{'}$ be $v_1e_1v_2e_2 \ldots v_{n-1}e_{n-1}v_ne_nv_{n+1}$, and $C(T^{'}) = \{C_1, C_2, \ldots, C_{n-2}, C_{n-1}\}$. Without loss of generality, let $T^{'} - v_{n+1} = T$. By the inductive hypothesis, $DPn(T)$ is maximal outerplanar. Now, the vertex v_{n+1} is one vertex more in $DPn(T^{'})$ than in $DPn(T)$. Also, there are only four arcs $e_{n-1}e_n$, $e_{n-1}C_{n-1}$, $C_{n-1}e_n$ and Pe_n more in $DPn(T^{'})$. Clearly, the induced subdigraph on the vertices P, e_{n-1}, C_{n-1} and e_n is not D_4. Hence $DPn(T^{'})$ is an outerplanar.

We now prove that $DPn(T')$ is maximal outerplanar. Since $DPn(T)$ is maximal outerplanar, by Theorem[A] it contains $2(m + \sum_{j=1}^{r} C_j + 1) - 3$ arcs. The outerplanar digraph $DPn(T')$ has $2(m+\sum_{j=1}^{r} C_j+1)-3+4 = 2[(m+1)+\{\sum_{j=1}^{r} C_j+1\}+1]-3$ arcs. By Theorem[A], $DPn(T')$ is maximal outerplanar. This completes the proof. □

Theorem 6.4. *For an arborescence T, $DPn(T)$ is minimally non-outerplanar if and only if the underlying graph of T is a star graph $K_{1,3}$.*

Proof. Suppose $DPn(T)$ is minimally non-outerplanar. Assume that T is an arborescence whose underlying graph is a star graph $K_{1,4}$. By Necessity part of Theorem 6.1, cr$(DPn(T)) = 1$, but $i(DPn(T)) \geq 2$, a contradiction.

Conversely, suppose that T is an arborescence whose underlying graph is a star graph $K_{1,3}$.

By Necessity part of Theorem 6.2, $i(DPn(T)) = 1$. Hence $PBn(T)$ is minimally non-outerplanar. □

Theorem 6.5. *For an arborescence T, $DPn(T)$ has the crossing number one if and only if the underlying graph of T is a star graph $K_{1,4}$.*

Proof. Suppose $DPn(T)$ has the crossing number one. Assume that T is an arborescence whose underlying graph is a star graph $K_{1,5}$. Let $V(T) = \{a, b, c, d, e, f\}$ and $A(T) = \{(a, c), (c, b), (c, d), (c, e), (c, f)\}$ such that a and (a, c) be the root and root arc of T, respectively. Then $A(L(T)) = \{(ac, cb), (ac, cd), (ac, ce), (ac, cf)\}$. Since c is the cut vertex of T, it is the tail of arcs $T = \{(c, b), (c, d), (c, e), (c, f)\}$ and head of arc $H = (a, c)$. Then the arcs incident into c from the vertices corresponding to arc of H, and the arcs incident out of v reaches the vertices corresponding to arcs of T in $L(T)$ gives $n(T)$ such that cr$(n(T)) = 0$. Let $P = \{P_1, P_2, P_3, P_4\}$ be the directed path set of T such that P_1 lie on arcs (a, c) and (c, b), P_2 lie on (c, d), P_3 lie on (c, e) and P_4 lie on (c, f) in T. On embedding $DPn(T)$ for the dominancy of directed pathos vertices with the vertices of $L(T)$ and the dominancy of directed pathos vertices gives $DPn(T)$ such that cr$(DPn(T)) \geq 2$, a contradiction.

Conversely, suppose T is an arborescence whose underlying graph is a star graph $K_{1,4}$. By Necessity part of Theorem 6.1, cr$(DPn(T)) = 1$. This completes the proof. □

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