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On some Pascal's like triangles. Part 10

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Abstract: In a series of papers, Pascal's like triangles with different forms have been described. Here, a new type of triangles is discussed. In the formula for their generation, operation summation is changed with operation subtraction. Some of their properties are studied.
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1 Introduction

Following the series of seven papers [1–3, 5–7], in which we discussed a new type of Pascal's Like Triangles (PLTs), in [4, 8, 9] we have shown their 3-dimensional analogues in the form of 3- or 4-face pyramids.

Now, we modify the form of operation (by the moment, it was summation) is the formula that generates the members of the triangle with other operation – subtraction.

The examples, that we give here are analogues of these from [1, 2, 3], but with a changed opeartion. For difference from the previous research, now there are two different forms of triangles.

Because, it is interesting to compare the members of the triangles from [1, 2, 3] and the new ones, in general, here, we follow the orger of their description from the previous publications.

Let us start with the following examples of an infinite triangle:

$$\mathbf{Q}_1: \quad \begin{array}{ccccccccccccccc}
 & & & & & & & \mathbf{1} & & & & & & & \\
 & & & & & & 1 & \mathbf{0} & 1 & & & & & & \\
 & & & & & 1 & 0 & \mathbf{0} & 0 & 1 & & & & & \\
 & & & & 1 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & & & & \\
 & & & 1 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 1 & & & \\
 & & 1 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 1 & & \\
 & 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 1 & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & & & & & & \vdots & \vdots & \vdots & & & & & &
\end{array}$$

$$\mathbf{Q}_2: \quad \begin{array}{ccccccccccccccc}
 & & & & & & & \mathbf{1} & & & & & & & \\
 & & & & & & 2 & -\mathbf{1} & 2 & & & & & & \\
 & & & & & 4 & -2 & \mathbf{1} & -2 & 4 & & & & & \\
 & & & & 8 & -4 & 2 & -\mathbf{1} & 2 & -4 & 8 & & & & \\
 & & & 16 & -8 & 4 & -2 & \mathbf{1} & -2 & 4 & -8 & 16 & & & \\
 & & 32 & -16 & 8 & -4 & 2 & -\mathbf{1} & 2 & -4 & 8 & -16 & 32 & & \\
 & 64 & -32 & 16 & -8 & 4 & -2 & \mathbf{1} & -2 & 4 & -8 & 16 & -32 & 64 & \\
128 & -64 & 32 & -16 & 8 & -4 & 2 & -\mathbf{1} & 2 & -4 & 8 & -16 & 32 & -64 & 128 \\
 & & & & & \vdots & & \vdots & & \vdots & & & & &
\end{array}$$

$$\mathbf{Q}_3: \quad \begin{array}{cccccccccccc}
 & & & & & & & \mathbf{1} & & & & \\
 & & & & & & 3 & -\mathbf{2} & 3 & & & \\
 & & & & & 9 & -6 & \mathbf{4} & -6 & 9 & & \\
 & & & & 27 & -18 & 12 & -\mathbf{8} & 12 & -18 & 27 & \\
 & & & 81 & -54 & 36 & -24 & \mathbf{16} & -24 & 36 & -54 & ... \\
 & & 243 & -162 & 108 & -72 & 48 & -\mathbf{32} & 48 & -72 & 108 & ... \\
 & 729 & -486 & 324 & -216 & 144 & -96 & \mathbf{64} & -96 & 144 & -216 & ... \\
2187 & -1458 & 972 & -648 & 432 & -288 & 192 & -\mathbf{128} & 192 & -288 & 432 & ... \\
 & & & & & \vdots & & \vdots & & \vdots & &
\end{array}$$

As mentioned above, we discuss two different schemes for constructing of PLTs.

To construct the first scheme, let the elements of the above infinite triangles be

$$\begin{array}{ccccccc}
 & & & a_{1,1} & & & \\
 & & a_{2,1} & a_{2,2} & a_{2,3} & & \\
 & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

where $a_{i,j}$ are arbitrary real (complex) numbers and for every natural number i and

1. for every natural number j for which $2 \leq j \leq i$ it will be valid:

$$a_{i,j} = a_{i-1,j-1} - a_{i,j-1};$$

2. for every natural number j for which $i \leq j \leq 2i-1$ it will be valid:

$$a_{i,j} = a_{i-1,j-1} - a_{i,j+1}.$$

We can prove, e.g., by induction

Lemma 1: For every natural number n:

$$\begin{array}{ccccccc} & & & \mathbf{1} & & & \\ & & n & -(\mathbf{n-1}) & n & & \\ & n^2 & -n^2+n & (\mathbf{n-1})^{\mathbf{2}} & -n^2+n & n & \\ n^3 & -n^3+n^2 & n^3-2n^2+n & -(\mathbf{n+1})^{\mathbf{3}} & n^3-2n^2+n & -n^3+n^2 & n^3 \\ & \vdots & & \vdots & & \vdots & \end{array}$$

Other interesting examples are the following

$$\mathbf{Q}_4: \quad \begin{array}{ccccccccccc} & & & & & \mathbf{1} & & & & & \\ & & & & 0 & \mathbf{1} & 0 & & & & \\ & & & 0 & 0 & \mathbf{1} & 0 & 0 & & & \\ & & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ & & & \vdots & & \vdots & & \vdots & & & \end{array}$$

$$\mathbf{Q}_5: \quad \begin{array}{ccccccccccccc} & & & & & & \mathbf{1} & & & & & & \\ & & & & & 1 & \mathbf{0} & 1 & & & & & \\ & & & & 2 & -1 & \mathbf{1} & -1 & 2 & & & & \\ & & & 3 & -1 & 0 & \mathbf{1} & 0 & -1 & 3 & & & \\ & & 5 & -2 & 1 & -1 & \mathbf{2} & -1 & 1 & -2 & 5 & & \\ & 8 & -3 & 1 & 0 & -1 & \mathbf{3} & -1 & 0 & 1 & -3 & 8 & \\ 13 & -5 & 2 & -1 & 1 & -2 & \mathbf{5} & -2 & 1 & -1 & 2 & -5 & 13 \\ & & & & \vdots & & \vdots & & \vdots & & & & \end{array}$$

Now, to construct the second scheme, let us use the same infinite triangles below

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & a_{2,2} & a_{2,3} & & \\ & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\ & \vdots & & \vdots & & \vdots & \end{array}$$

but here, for every natural number i and

1. for every natural number j for which $2 \leq j \leq i$ it will be valid:

$$a_{i,j} = a_{i,j-1} - a_{i-1,j-1};$$

2. for every natural number j for which $i \le j \le 2i-1$ it will be valid:

$$a_{i,j} = a_{i,j+1} - a_{i-1,j-1}.$$

In the second construction, the above examples obtain the following forms.

$$\mathbf{Q}_6 : \quad \begin{array}{c}
\mathbf{1} \\
1 \quad \mathbf{0} \quad 1 \\
1 \quad 0 \quad \mathbf{0} \quad 0 \quad 1 \\
1 \quad 0 \quad 0 \quad \mathbf{0} \quad 0 \quad 0 \quad 1 \\
1 \quad 0 \quad 0 \quad 0 \quad \mathbf{0} \quad 0 \quad 0 \quad 0 \quad 1 \\
1 \quad 0 \quad 0 \quad 0 \quad 0 \quad \mathbf{0} \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \\
1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \mathbf{0} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

Therefore, $\mathbf{Q}_6$ and $\mathbf{Q}_1$ coincide.

$$\mathbf{Q}_7 : \quad \begin{array}{c}
\mathbf{1} \\
2 \quad \mathbf{1} \quad 2 \\
4 \quad 2 \quad \mathbf{1} \quad 2 \quad 4 \\
8 \quad 4 \quad 2 \quad \mathbf{1} \quad 2 \quad 4 \quad 8 \\
16 \quad 8 \quad 4 \quad 2 \quad \mathbf{1} \quad 2 \quad 4 \quad 8 \quad 16 \\
32 \quad 16 \quad 8 \quad 4 \quad 2 \quad \mathbf{1} \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \\
64 \quad 32 \quad 16 \quad 8 \quad 4 \quad 2 \quad \mathbf{1} \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \quad 64 \\
128 \quad 64 \quad 32 \quad 16 \quad 8 \quad 4 \quad 2 \quad \mathbf{1} \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \quad 64 \quad 128 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

$$\mathbf{Q}_8 : \quad \begin{array}{c}
\mathbf{1} \\
3 \quad \mathbf{2} \quad 3 \\
9 \quad 6 \quad \mathbf{4} \quad 6 \quad 9 \\
27 \quad 18 \quad 12 \quad \mathbf{8} \quad 12 \quad 18 \quad 27 \\
81 \quad 54 \quad 36 \quad 24 \quad \mathbf{16} \quad 24 \quad 36 \quad 54 \quad 81 \\
243 \quad 162 \quad 108 \quad 72 \quad 48 \quad \mathbf{32} \quad 48 \quad 72 \quad 108 \quad 162 \quad 243 \\
729 \quad 486 \quad 324 \quad 216 \quad 144 \quad 96 \quad \mathbf{64} \quad 96 \quad 144 \quad 216 \quad 324 \quad 486 \quad 729 \\
2187 \quad 1458 \quad 972 \quad 648 \quad 432 \quad 288 \quad 192 \quad \mathbf{128} \quad 192 \quad 288 \quad 432 \quad 648 \quad 972 \quad 1458 \quad 2187 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

Hence, it is valid

Lemma 2: For every natural number n:

$$\begin{array}{c}
\mathbf{1} \\
n \qquad \mathbf{n-1} \qquad n \\
n^2 \qquad n^2-n \qquad \mathbf{(n-1)^2} \qquad n^2-n \qquad n \\
n^3 \quad n^3-n^2 \quad n^3-2n^2+n \quad \mathbf{(n-1)^3} \quad n^3-2n^2+n \quad n^3-n^2 \quad n^3 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

The analogous of the above described two other examples are the following

$$
\mathbf{Q}_9: \quad
\begin{array}{c}
\mathbf{1}\\
0 \;\; \mathbf{-1} \;\; 0\\
0 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 0\\
0 \;\; 0 \;\; 0 \;\; \mathbf{-1} \;\; 0 \;\; 0 \;\; 0\\
0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 0 \;\; 0 \;\; 0\\
0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{-1} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\\
0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\\
0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{-1} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\\
\vdots \qquad \vdots \qquad \vdots
\end{array}
$$

$$
\mathbf{Q}_{10}: \quad
\begin{array}{c}
\mathbf{1}\\
1 \;\; \mathbf{0} \;\; 1\\
2 \;\; 1 \;\; \mathbf{1} \;\; 1 \;\; 2\\
3 \;\; 1 \;\; 0 \;\; \mathbf{-1} \;\; 0 \;\; 1 \;\; 3\\
5 \;\; 2 \;\; 1 \;\; 1 \;\; \mathbf{2} \;\; 1 \;\; 1 \;\; 2 \;\; 5\\
8 \;\; 3 \;\; 1 \;\; 0 \;\; -1 \;\; \mathbf{-3} \;\; 1 \;\; 0 \;\; 1 \;\; 3 \;\; 8\\
13 \;\; 5 \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; \mathbf{5} \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; 5 \;\; 13\\
21 \;\; 8 \;\; 3 \;\; 1 \;\; 0 \;\; -1 \;\; -3 \;\; \mathbf{8} \;\; -3 \;\; -1 \;\; 0 \;\; 1 \;\; 3 \;\; 8 \;\; 21\\
\vdots \qquad \vdots \qquad \vdots
\end{array}
$$

We can modify $\mathbf{Q}_{10}$ to the form

$$
\mathbf{Q}_{11}: \quad
\begin{array}{c}
\mathbf{0}\\
1 \;\; \mathbf{1} \;\; 1\\
1 \;\; 0 \;\; \mathbf{-1} \;\; 0 \;\; 1\\
2 \;\; 1 \;\; 1 \;\; \mathbf{2} \;\; 1 \;\; 1 \;\; 2\\
3 \;\; 1 \;\; 0 \;\; -1 \;\; \mathbf{-3} \;\; -1 \;\; 0 \;\; 1 \;\; 3\\
5 \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; \mathbf{5} \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; 5\\
8 \;\; 3 \;\; 1 \;\; 0 \;\; -1 \;\; -3 \;\; \mathbf{-8} \;\; -3 \;\; 1 \;\; 0 \;\; 1 \;\; 3 \;\; 8\\
13 \;\; 5 \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; 5 \;\; \mathbf{13} \;\; 5 \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; 5 \;\; 13\\
21 \;\; 8 \;\; 3 \;\; 1 \;\; 0 \;\; -1 \;\; -3 \;\; -8 \;\; \mathbf{-21} \;\; -8 \;\; -3 \;\; -1 \;\; 0 \;\; 1 \;\; 3 \;\; 8 \;\; 21\\
\vdots \qquad \vdots \qquad \vdots
\end{array}
$$

Therefore, if the generating sequence contains the members of Fibonacci sequence, i.e., they are elements if sequence $\{F_i\}_{i\geq 0}$, then the generated sequence is $\{(-1)^{i+1}F_i\}_{i\geq 0}$. If we like to change the places of the generating and generated seuences, then we obtain the following Pascal's like triangle.

$$
\mathbf{Q}_{12}: \quad
\begin{array}{cccccccccccc}
 & & & & & & & & & \mathbf{0} & & \\
 & & & & & & & & 1 & \mathbf{1} & 1 & \\
 & & & & & & & 3 & 2 & \mathbf{1} & 2 & \ldots \\
 & & & & & & 8 & 5 & 3 & \mathbf{2} & 3 & \ldots \\
 & & & & & 21 & 13 & 8 & 5 & \mathbf{3} & 5 & \ldots \\
 & & & & 55 & 34 & 21 & 13 & 8 & \mathbf{5} & 8 & \ldots \\
 & & & 144 & 89 & 55 & 34 & 21 & 13 & \mathbf{8} & 13 & \ldots \\
 & & 377 & 233 & 144 & 89 & 55 & 34 & 21 & \mathbf{13} & 21 & \ldots \\
 & 987 & 610 & 377 & 233 & 144 & 89 & 55 & 34 & \mathbf{21} & 34 & \ldots \\
2584 & 1597 & 987 & 610 & 377 & 233 & 144 & 89 & 55 & \mathbf{34} & 55 & \ldots \\
 & & & & & & & \vdots & & \vdots & & \vdots
\end{array}
$$

The other spacial sequences alos generate interesting PLTs.
The Lucas sequence (as generating one) gives us possibility to construct the PLT

$$
\mathbf{Q}_{13}: \quad
\begin{array}{cccccccccccccccc}
 & & & & & & & & & \mathbf{2} & & & & & & \\
 & & & & & & & & 1 & \mathbf{-1} & 1 & & & & & \\
 & & & & & & & 3 & 2 & \mathbf{3} & 2 & 3 & & & & \\
 & & & & & & 4 & 1 & -1 & \mathbf{-4} & -1 & 1 & 4 & & & \\
 & & & & & 7 & 3 & 2 & 3 & \mathbf{7} & 3 & 2 & 3 & 7 & & \\
 & & & & 11 & 4 & 1 & -1 & -4 & \mathbf{11} & -4 & -1 & 1 & 4 & 11 & \\
 & & & 18 & 7 & 3 & 2 & 3 & 7 & \mathbf{18} & 7 & 3 & 2 & 3 & 7 & \ldots \\
 & & 29 & 11 & 4 & 1 & -1 & -4 & -11 & \mathbf{-29} & -11 & -4 & -1 & 1 & 4 & \ldots \\
 & 47 & 18 & 7 & 3 & 2 & 3 & 7 & 18 & \mathbf{47} & 18 & 7 & 3 & 2 & 3 & \ldots \\
76 & 29 & 11 & 4 & 1 & -1 & -4 & -11 & -29 & \mathbf{76} & -29 & -11 & -4 & -1 & 1 & \ldots \\
 & & & & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}
$$

Jacobsthal's sequence (as generating one) generates the PLT

$$
\mathbf{Q}_{14}: \quad
\begin{array}{ccccccccccccccccc}
 & & & & & & & & & \mathbf{0} & & & & & & & \\
 & & & & & & & & 1 & \mathbf{1} & 1 & & & & & & \\
 & & & & & & & 1 & 0 & \mathbf{-1} & 0 & 1 & & & & & \\
 & & & & & & 3 & 2 & 2 & \mathbf{3} & 2 & 2 & 3 & & & & \\
 & & & & & 5 & 2 & 0 & -2 & \mathbf{-5} & -2 & 0 & 2 & 5 & & & \\
 & & & & 11 & 6 & 4 & 4 & 6 & \mathbf{11} & 6 & 4 & 4 & 6 & 11 & & \\
 & & & 21 & 10 & 4 & 0 & -4 & -10 & \mathbf{-21} & -10 & -4 & 0 & 4 & 10 & 21 & \\
 & & 43 & 22 & 12 & 8 & 8 & 12 & 22 & \mathbf{43} & 22 & 12 & 8 & 8 & 12 & 22 & \ldots \\
 & 85 & 42 & 20 & 8 & 0 & -8 & -20 & -42 & \mathbf{-85} & -42 & -20 & -8 & 0 & 8 & 20 & \ldots \\
171 & 86 & 44 & 24 & 16 & 16 & 24 & 44 & 86 & \mathbf{171} & 86 & 44 & 24 & 16 & 16 & 24 & \ldots \\
 & & & & & & & \vdots & & \vdots & & \vdots & & & & &
\end{array}
$$

while Jacobsthal-Lucas's sequence (as generating one) generates the PLT

$$
\mathbf{Q}_{15}:\quad
\begin{array}{rrrrrrrrrrrrrr}
 & & & & & & & & & \mathbf{2} & & & & \\
 & & & & & & & & 1 & \mathbf{-1} & 1 & & & \\
 & & & & & & & 5 & 4 & \mathbf{5} & 4 & 5 & & \\
 & & & & & & 7 & 2 & -2 & \mathbf{-7} & -2 & 2 & 7 & \\
 & & & & & 17 & 10 & 8 & 10 & \mathbf{17} & 10 & 8 & 10 & \ldots \\
 & & & & 31 & 14 & 4 & -4 & -14 & \mathbf{-31} & -14 & -4 & 4 & \ldots \\
 & & & 65 & 34 & 20 & 16 & 20 & 34 & \mathbf{65} & 34 & 20 & 16 & \ldots \\
 & & 127 & 62 & 28 & 8 & -8 & -28 & -62 & \mathbf{-127} & -62 & -28 & -8 & \ldots \\
 & 257 & 130 & 68 & 40 & 32 & 40 & 68 & 130 & \mathbf{257} & 130 & 68 & 40 & \ldots \\
511 & 254 & 124 & 56 & 16 & -16 & -56 & -124 & -254 & \mathbf{-511} & -254 & -124 & -56 & \ldots \\
 & & & & & & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

If the generating sequence is $\{n\}_{n\geq 1}$, then we obtain the PLT

$$
\mathbf{Q}_{16}:\quad
\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 2 & \mathbf{1} & 2 & & & & \\
 & & & 3 & 1 & \mathbf{0} & 1 & 3 & & & \\
 & & 4 & 1 & 0 & \mathbf{0} & 0 & 1 & 4 & & \\
 & 5 & 1 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & 5 & \\
6 & 1 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 1 & 6 \\
 & & & \vdots & & \vdots & & \vdots & & &
\end{array}
$$

while, if we like this sequence to be a generated one, then the PLT has the form

$$
\mathbf{Q}_{17}:\quad
\begin{array}{rrrrrrrrrrrrrrrr}
 & & & & & & & & & \mathbf{1} & & & & & & \\
 & & & & & & & & 3 & \mathbf{2} & 3 & & & & & \\
 & & & & & & & 8 & 5 & \mathbf{3} & 5 & 8 & & & & \\
 & & & & & & 20 & 12 & 7 & \mathbf{4} & 7 & 12 & 20 & & & \\
 & & & & & 48 & 28 & 16 & 9 & \mathbf{5} & 9 & 16 & 28 & 48 & & \\
 & & & & 112 & 64 & 36 & 20 & 11 & \mathbf{6} & 11 & 20 & 36 & 64 & 112 & \\
 & & & 256 & 144 & 80 & 44 & 24 & 13 & \mathbf{7} & 13 & 24 & 44 & 80 & 144 & \ldots \\
 & & 576 & 320 & 176 & 96 & 52 & 28 & 15 & \mathbf{8} & 15 & 28 & 52 & 96 & 176 & \ldots \\
 & 1280 & 704 & 384 & 208 & 112 & 60 & 32 & 17 & \mathbf{9} & 17 & 32 & 60 & 112 & 208 & \ldots \\
2816 & 1536 & 832 & 448 & 240 & 128 & 68 & 36 & 19 & \mathbf{10} & 19 & 36 & 68 & 128 & 240 & \ldots \\
 & & & & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}
$$

We see that for $n \geq 1$, the elements of the generating sequence $\{a_{n,1}\}_{n\geq 1}$ satisfy the equality

$$a_{n,1} = 3a_{n-1,1} - 4a_{n-3,1}.$$

By analogy with the research from [3], here we construct the $(0,1)$-PLTs, that correspond to the above ones PLTs.

$$\mathbf{N}_1: \quad \begin{array}{c}
\mathbf{1} \\
1 \;\; \mathbf{0} \;\; 1 \\
1 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 1 \\
1 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 1 \\
1 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; 1 \\
1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

Therefore, $\mathbf{N}_1$ and $\mathbf{O}_1$ coincide. Let us denote the $(0,1)$-analogous of the PLT $\mathbf{O}_i$ by $\mathbf{N}_i$ for $1 \leq i \leq 17$.

We check directly that $\mathbf{N}_1$, $\mathbf{N}_3$, $\mathbf{N}_6$, $\mathbf{N}_8$ and $\mathbf{N}_9$ coincide; $\mathbf{N}_2$, $\mathbf{N}_4$ and $\mathbf{N}_7$ coincide; $\mathbf{N}_5$ and $\mathbf{N}_{10}$ coincide; and $\mathbf{N}_{11}$, $\mathbf{N}_{12}$, $\mathbf{N}_{13}$, $\mathbf{N}_{14}$, and $\mathbf{N}_{15}$ also coincide, where

$$\mathbf{N}_2: \quad \begin{array}{c}
\mathbf{1} \\
0 \;\; \mathbf{1} \;\; 0 \\
0 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 0 \\
0 \;\; 0 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 0 \;\; 0 \\
0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\
0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\
0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\
0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

$$\mathbf{N}_5: \quad \begin{array}{c}
\mathbf{1} \\
1 \;\; \mathbf{0} \;\; 1 \\
0 \;\; 1 \;\; \mathbf{1} \;\; 1 \;\; 0 \\
1 \;\; 1 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 1 \;\; 1 \\
1 \;\; 0 \;\; 1 \;\; 1 \;\; \mathbf{0} \;\; 1 \;\; 1 \;\; 0 \;\; 1 \\
0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; \mathbf{1} \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \\
1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

$$\mathbf{N}_{12}: \quad \begin{array}{c}
\mathbf{0} \\
1 \;\; \mathbf{1} \;\; 1 \\
1 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 1 \\
0 \;\; 1 \;\; 1 \;\; \mathbf{0} \;\; 1 \;\; 1 \;\; 0 \\
1 \;\; 1 \;\; 0 \;\; 1 \;\; \mathbf{1} \;\; 1 \;\; 0 \;\; 1 \;\; 1 \\
1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \\
0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; \mathbf{0} \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \\
1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; \mathbf{1} \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \\
1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; \mathbf{1} \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

The $(0,1)$-analogous of the PLT $\mathbf{O}_{16}$ and $\mathbf{O}_{17}$ are the following

$$\mathbf{N}_{16}: \quad \begin{array}{c} \mathbf{1} \\ 0 \;\; \mathbf{1} \;\; 0 \\ 1 \;\; 1 \;\; \mathbf{0} \;\; 1 \;\; 1 \\ 0 \;\; 1 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 1 \;\; 0 \\ 1 \;\; 1 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 1 \;\; 1 \\ 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \\ 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 1 \\ 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \\ \vdots \qquad \vdots \qquad \vdots \end{array}$$

$$\mathbf{N}_{17}: \quad \begin{array}{c} \mathbf{1} \\ 1 \;\; \mathbf{0} \;\; 1 \\ 0 \;\; 1 \;\; \mathbf{1} \;\; 1 \;\; 0 \\ 0 \;\; 0 \;\; 1 \;\; \mathbf{0} \;\; 1 \;\; 0 \;\; 0 \\ 0 \;\; 0 \;\; 0 \;\; 1 \;\; \mathbf{1} \;\; 1 \;\; 0 \;\; 0 \;\; 0 \\ 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; \mathbf{0} \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\ 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; \mathbf{1} \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\ 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; \mathbf{0} \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\ 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; \mathbf{1} \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\ 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; \mathbf{0} \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\ \vdots \qquad \vdots \qquad \vdots \end{array}$$

Now, we discuss some examples with parametric sequences.

Let a and b are two real (complex) numbers. If we use the first algorithm for subtraction, we can construct, e.g., the following PLTs.

$$\mathbf{Q}_{18}: \quad \begin{array}{c} \mathbf{a} \\ b \quad \mathbf{a-b} \quad b \\ a \quad b-a \quad \mathbf{2a-2b} \quad b-a \quad a \\ b \quad a-b \quad 2b-2a \quad \mathbf{4a-4b} \quad 2b-2a \quad a-b \quad b \\ a \quad b-a \quad 2a-2b \quad 4b-4a \quad \mathbf{8a-8b} \quad 4b-4a \quad 2a-2b \quad b-a \quad a \\ b \quad a-b \quad 2b-2a \quad 4a-4b \quad 8b-8a \quad \mathbf{16a-16b} \quad 8b-8a \quad 4a-4b \quad 2b-2a \quad \ldots \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{array}$$

When $b = a$ the above triangle obtains the form

$$\mathbf{Q}_{19}: \quad \begin{array}{c}
\mathbf{a} \\
a \;\; \mathbf{0} \;\; a \\
a \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; a \\
a \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; a \\
a \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; a \\
a \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; a \\
a \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; a \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

If we use the second algorithm for subtraction, the last two above PLTs, we can obtain

$$\mathbf{Q}_{20}: \quad \begin{array}{c}
\mathbf{a} \\
b \quad \mathbf{b-a} \quad b \\
a \quad a-b \quad \mathbf{2a-2b} \quad a-b \quad a \\
b \quad b-a \quad 2b-2a \quad \mathbf{4b-4a} \quad 2b-2a \quad b-a \quad b \\
a \quad a-b \quad 2a-2b \quad 4a-4b \quad \mathbf{8a-8b} \quad 4a-4b \quad 2a-2b \quad a-b \quad a \\
b \quad b-a \quad 2b-2a \quad 4b-4a \quad 8b-8a \quad \mathbf{16b-16a} \quad 8b-8a \quad 4b-4a \quad 2b-2a \quad \ldots \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{array}$$

When $b = a$ the above PLT obtains the form $\mathbf{Q}_{19}$.

$$\mathbf{Q}_{21}: \quad \begin{array}{c}
\mathbf{a} \\
a \;\; \mathbf{0} \;\; a \\
a \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; a \\
a \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; a \\
a \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; a \\
a \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; a \\
a \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{0} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; a \\
\vdots \qquad \vdots \qquad \vdots
\end{array}$$

More general, let $\{a_n\}_{n\geq 1}$ be a (infinite) sequence of real numbers. Let they are sequential members of the generated sequence and let us use the first algorithm for subtraction. Then we obtain the PLT

$$\mathbf{Q}_{22}: \quad \begin{array}{cccccc}
 & & & & \mathbf{a_1} & \\
 & & & a_2 & \mathbf{a_1 - a_2} & \ldots \\
 & & a_3 & a_2 - a_3 & \mathbf{a_1 - 2a_2 + a_3} & \ldots \\
 & a_4 & a_3 - a_4 & a_2 - 2a_3 + a_4 & \mathbf{a_1 - 3a_2 + 3a_3 - a_4} & \ldots \\
a_5 & a_4 - a_5 & a_3 - 2a_4 + a_5 & a_2 - 3a_3 + 3a_4 - a_5 & \mathbf{a_1 - 4a_2 + 6a_3 - 4a_4 + a_5} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Therefore, it is valid

Lemma 3: The generating sequence is $\{a_n\}_{n\geq 1}$, if and only if the generated sequence is

$$\left\{\sum_{i=1}^{n}(-1)^{i-1}\left(\begin{array}{c} n-1 \\ i-1 \end{array}\right)a_i\right\}_{n\geq 1}.$$

For the opposite case, when $\{a_n\}_{n\geq 1}$ is the generating sequence, we obtain

$$\begin{array}{lrrrrcl}
\mathbf{Q}_{23}: & & & & & \mathbf{a_1} & \\
 & & & & a_1-a_2 & \mathbf{a_2} & \ldots \\
 & & & a_1-2a_2+a_3 & a_2-a_3 & \mathbf{a_3} & \ldots \\
 & & a_1-3a_2+3a_3-a_4 & a_2-2a_3+a_4 & a_3-a_4 & \mathbf{a_4} & \ldots \\
 & a_1-4a_2+6a_3-4a_4+a_5 & a_2-3a_3+3a_4-a_5 & a_3-2a_4+a_5 & a_4-a_5 & \mathbf{a_5} & \ldots \\
 & & & \vdots & & \vdots & \vdots
\end{array}$$

Hence, it is valid

Lemma 4: The generating sequence is $\left\{\sum_{i=1}^{n}(-1)^{i-1}\left(\begin{array}{c} n-1 \\ i-1 \end{array}\right)a_i\right\}_{n\geq 1}$, if and only if the generated sequence is $\{a_n\}_{n\geq 1}$.

Now, let us use the second algorithm for subtraction. Then, for the generating sequence $\{a_n\}_{n\geq 1}$, we obtain the PLT

$$\begin{array}{lccccc}
\mathbf{Q}_{24}: & & & & \mathbf{a_1} & \\
 & & & a_2 & \mathbf{-a_1+a_2} & \ldots \\
 & & a_3 & -a_2+a_3 & \mathbf{a_1-2a_2+a_3} & \ldots \\
 & a_4 & -a_3+a_4 & a_2-2a_3+a_4 & \mathbf{-a_1+3a_2-3a_3+a_4} & \ldots \\
a_5 & -a_4+a_5 & a_3-2a_4+a_5 & -a_2+3a_3-3a_4+a_5 & \mathbf{a_1-4a_2+6a_3-4a_4+a_5} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

It is valid

Lemma 5: The generating sequence is $\{a_n\}_{n\geq 1}$, if and only if the generated sequence is

$$\left\{\sum_{i=1}^{n}(-1)^{n+i}\left(\begin{array}{c} n-1 \\ i-1 \end{array}\right)a_i\right\}_{n\geq 1}.$$

Finally, let us use the second algorithm for subtraction. Then, for the generated sequence $\{a_n\}_{n\geq 1}$, we obtain the PLT

$$\begin{array}{lrrrrcl}
\mathbf{Q}_{23}: & & & & & \mathbf{a_1} & \\
 & & & & a_1+a_2 & \mathbf{a_2} & \ldots \\
 & & & a_1+2a_2+a_3 & a_2+a_3 & \mathbf{a_3} & \ldots \\
 & & a_1+3a_2+3a_3+a_4 & a_2+2a_3+a_4 & a_3+a_4 & \mathbf{a_4} & \ldots \\
 & a_1+4a_2+6a_3+4a_4+a_5 & a_2+3a_3+3a_4+a_5 & a_3+2a_4+a_5 & a_4+a_5 & \mathbf{a_5} & \ldots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

It is valid

Lemma 6: The generated sequence is $\{a_n\}_{n\geq 1}$, if and only if the generating sequence is

$$\left\{\sum_{i=1}^{n}\begin{pmatrix} n-1 \\ i-1 \end{pmatrix} a_i\right\}_{n\geq 1}.$$

References

[1] Atanassov, K. (2007) On some Pascal's like triangles. Part 1. *Notes on Number Theory and Discrete Mathematics*, 13(1), 31–36.

[2] Atanassov, K. (2007) On some Pascal's like triangles. Part 2. *Notes on Number Theory and Discrete Mathematics*, 13(2), 10–14.

[3] Atanassov, K. (2007) On some Pascal's like triangles. Part 3. *Notes on Number Theory and Discrete Mathematics*, 13(3), 20–25.

[4] Atanassov, K. (2007) On some Pascal's like triangles. Part 4. *Notes on Number Theory and Discrete Mathematics*, 13(4), 11–20.

[5] Atanassov, K. (2011) On some Pascal's like triangles. Part 5. *Advanced Studies in Contemporary Mathematics*, 21(3), 291–299.

[6] Atanassov, K. (2014) On some Pascal's like triangles. Part 6. *Notes on Number Theory and Discrete Mathematics*, 20(4), 40–46.

[7] Atanassov, K. (2014) On some Pascal's like triangles. Part 7. *Notes on Number Theory and Discrete Mathematics*, 20(5), 58–63.

[8] Atanassov, K. (2015) On some Pascal's like triangles. Part 8. *Notes on Number Theory and Discrete Mathematics*, 21(1), 42–50.

[9] Atanassov, K. (2015) On some Pascal's like triangles. Part 9. *Notes on Number Theory and Discrete Mathematics*, 21(2), 15–22.