The rectangular spiral
or the $n_1 \times n_2 \times \ldots \times n_k$ Points Problem

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Abstract: A generalization of Ripà's square spiral solution for the $n \times n \times \ldots \times n$ Points Upper Bound Problem. Additionally, we provide a non-trivial lower bound for the k-dimensional $n_1 \times n_2 \times \ldots \times n_k$ Points Problem. In this way, we can build a range in which, with certainty, all the best possible solutions to the problem we are considering will fall. Finally, we provide a few characteristic numerical examples in order to appreciate the fineness of the result arising from the particular approach we have chosen.
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1 Introduction

Nearly a century ago, the classic *Nine Dots Problem* appeared in Samuel Loyd's *Cyclopedia of Puzzles* [1–4]. The challenge was as follows: "…draw a continuous line through the center of all the eggs so as to mark them off in the fewest number of strokes" [3–5].

That puzzle can naturally be extended to an arbitrarily large number of distinct (zero-dimensional) points for each row / column [7]. This new problem asks to connect $n \times n$ points, arranged in a grid formed by n rows and n columns, using the fewest straight lines connected at their end points. Ripà and Remirez [6] showed that it is possible to do this for every $n \in \mathbb{N} - \{0, 1, 2\}$, using only $2 \cdot n - 2$ straight lines. For any $n \geq 5$, we can combine a given 8 line solution for the 5×5 problem and the square spiral frame [10]. In the same paper, they extended the $n \times n$ result to a three dimensional space [8] and finally to a generic k-dimensional space (for $k > 3$).

Starting from that outcome, we consider the same problem and rules by [6]. We can apply the "pure" spiral method to a $n_1 \times n_2$ rectangular grid (where $n_1 \leq n_2$). In this way, it is quite simple to discover that the minimum number of lines we need to connect every point (solving the problem inside the box, connecting points without crossing a line, and visiting any dot just once) is given by the Eq. 1 [9].

$$h = 2 \cdot n - 1 \quad \forall \, n \in \mathbb{N} - \{0, 1\} \tag{1}$$

Figure 1. The original problem from Samuel Loyd's *Cyclopedia of Puzzles*,
New York, 1914, p. 301.

2   The $n_1 \times n_2 \times \ldots \times n_k$ Problem upper bound

If we try to extend the result in Eq. 1 to a three dimensional space, where $n_1 \leq n_2 \leq n_3$, we need to modify a somewhat the standard strategy described in [6] in order to choose the best "plane by plane" approach that we can effect, even if there are a few exceptions (such as if $n_3 - n_2 \leq 1$, see Appendix). So, we need to identify the correct starting plane to lay the first straight line. Using basic mathematics, it is quite easy to prove that, in general, the best option is to start from the $[n_2; n_3]$ plane.

Hence, under the additional constraints that we must solve the problem inside the box only, connecting points without crossing a line, and visiting each dot just once, our strategy is as follows:

- Step 1) Take one of the external planes identified by $[n_2; n_3]$: here is the plane to lay our first line;
- Step 2) Starting from one point on an angle of this grid, draw the first straight line to connect n_3 points, until we have reached the last point in that row;
- Step 3) The next line is on the same plane as well. It lays on $[n_2; n_3]$, it is orthogonal to the previous one, and it links $n_2 - 1$ points;
- Step 4) Repeat the square (rectangular) spiral pattern until we connect every point belonging to this $n_2 \cdot n_3$ set to the others on the same surface;

- Step 5) Draw another line which is orthogonal to the $[n_2; n_3]$ plane we have considered before, doubling the same scheme (in reverse) with the opposite face of this three dimensional box with the shape of a (n_1, n_2, n_3) parallelogram. Repeat the same pattern for any $n_2 \times n_3$ grid, $n_1 - 2$ times more.

The rectangular spiral solution also gives us the shortest path we can find to connect every point: the total length of the line segments used to fit all the points is minimal.

Nota Bene: Just a couple of trivial considerations. Referring to the rectangular spiral pattern applied to a k-dimensional space ($k \geq 2$), we can return to the starting point using exactly one additional line (it works for any number of dimensions we can consider at or above 1). For any odd value of n_1, we can visit a maximum of $\left\lceil \frac{n_{k-1}}{2} \right\rceil - 1$ points twice, simply extending the line end (if we do not, we will not visit any dot more than once, otherwise we can visit $\left\lceil \frac{n_{k-1}}{2} \right\rceil - 1$ points, at most). Moreover, it is possible to visit up to $n_{k-1} - 2$ points twice if we move the second to last line too (crossing some more lines as well). Finally, considering $k \geq 2$, if we are free to extend the ending line until we are close to the next (already visited) point (i.e., let ε be the distance between the last line and the nearest point and let the distance between two adjacent points be unitary, we have that $0 < \varepsilon < 1$), it is possible to return to the starting point without visiting any point more than once.

The total number of lines we use to connect every point is always lower or equal to

$$h = 2 \cdot n_1 \cdot n_2 - 1 \tag{2}$$

In fact, $h = (2 \cdot n_2 - 1) \cdot n_1 + n_1 - 1$.

Nevertheless, $(2 \cdot n_2 - 1) \cdot n_1 + n_1 - 1 = 2 \cdot n_1 \cdot n_2 - n_1 + n_1 - 1 = 2 \cdot n_1 \cdot n_2 - 1 = 2 \cdot n_1 \cdot n_2 - n_2 + n_2 - 1 = (2 \cdot n_1 - 1) \cdot n_2 + n_2 - 1$ (Q.E.D.)

The "savings", in terms of unused segments, are zero if (and only if)

$$n_1 < 2 \cdot (n_3 - n_2) + 3 \tag{3}$$

In general, (also if $n_1 \geq 2 \cdot (n_3 - n_2) + 3$), the Eq. 2 can be rewritten as:

$$h = 2 \cdot n_1 \cdot n_2 - c \tag{4}$$

Where $c = 1$ if the "savings" are zero, while $c \geq 2$ if not.

As an example, let us consider the following cases:

a) $n_1 = 5$; $n_2 = 6$; $n_3 = 9$.

b) $n_1 = 11$; $n_2 = 12$; $n_3 = 13$.

While in the first hypothesis $c = 1$ (in fact $5 < 2 \cdot (9 - 3) + 3$), $h = 2 \cdot 5 \cdot 6 - 1 = 59$, in case b) we have $c = 13$, $h = 2 \cdot 11 \cdot 12 - 13 = 251$. This is by virtue of the fact that the fifth and the sixth connecting lines allow us to "save" one line for every subsequent plane, whereas each plane "met" after the sixth can be solved using two fewer lines (if compared with the first four we have considered).
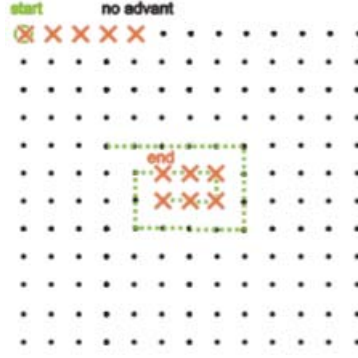
Figure 2. The rectangular spiral for the case of the example b): $n_1 = 11$, $n_2 = 12$, $n_3 = 13$.

If $n_1 \geq 2 \cdot (n_3 - n_2) + 4$, the (pure) rectangular spiral method, with specific regard to the three dimensional problem, can be summarized as follows:

$$h = n_1 - 1 + [2 \cdot (n_3 - n_2) + 2] \cdot (2 \cdot n_2 - 1) + 2 \cdot (2 \cdot n_2 - 2) + [2 \cdot (n_3 - n_2) + 4] \cdot (2 \cdot n_2 - 3)$$
$$+ 4 \cdot (2 \cdot n_2 - 4) + [2 \cdot (n_3 - n_2) + 6] \cdot (2 \cdot n_2 - 5) + 6 \cdot (2 \cdot n_2 - 6) + \ldots + d,$$

where d represents the product of the number of line segments used to solve the plane, which contains the fewest lines (the last plane we have considered, the plane which cuts roughly half-way through our imaginary *box*) and $n_1 - \{[2 \cdot (n_3 - n_2) + 2]\} + 2 + [2 \cdot (n_3 - n_2) + 4] + 4 + \ldots\}$".

Thus, we can synthesize the previous formula as

$$h = n_1 - 1 + \sum_{j=0}^{j_{max}} [(2 \cdot n_2 - 2 \cdot j - 1) \cdot (2 \cdot (n_3 - n_2) + 2 \cdot (j + 1))$$
$$+ 2 \cdot (j + 1) \cdot (2 \cdot n_2 - 2 \cdot (j + 1))] + b$$

Hence

$$h = -\frac{8 \cdot j_{max}^3}{3} + 6 \cdot j_{max}^2 \cdot n_2 - 2 \cdot j_{max}^2 \cdot n_3 - 11 \cdot j_{max}^2 - 4 \cdot j_{max} \cdot n_2^2$$

$$+ 4 \cdot j_{max} \cdot n_2 \cdot n_3 + 16 \cdot j_{max} \cdot n_2 - 4 \cdot j_{max} \cdot n_3 - \frac{43 \cdot j_{max}}{3} - 4 \cdot n_2^2$$

$$+ 4 \cdot n_2 \cdot n_3 + 10 \cdot n_2 - 2 \cdot n_3 - 7 + n_1 + b,$$

where j_{max} represents the <u>maximum</u> value of the upper bound of the summation, let us say $\tilde{j}$, such that

$$n_1 \geq \sum_{j=0}^{\tilde{j}} [2 \cdot (n_3 - n_2) + 2 \cdot (j + 1) + 2 \cdot (j + 1)] \rightarrow n_1 \geq 2 \cdot (\tilde{j} + 1) \cdot (n_3 - n_2 + \tilde{j} + 2),$$

while

$$b := \begin{cases} [n_1 - 2 \cdot (j_{max} + 1) \cdot (n_3 - n_2 + j_{max} + 2)] \cdot (2 \cdot n_2 - 2 \cdot j_{max} - 3) \\ \textbf{\textit{if}} \quad n_1 - 2 \cdot (j_{max} + 1) \cdot (n_3 - n_2 + j_{max} + 2) \leq 2 \cdot (n_3 - n_2) + 2 \cdot (j_{max} + 2) \\ \\ [2 \cdot (n_3 - n_2) + 2 \cdot (j_{max} + 2)] \cdot (2 \cdot n_2 - 2 \cdot j_{max} - 3) + \\ \{n_1 - 2 \cdot (j_{max} + 1) \cdot (n_3 - n_2 + j_{max} + 2) - [2 \cdot (n_3 - n_2) + 2 \cdot (j_{max} + 2)]\} \cdot (2 \cdot n_2 - 2 \cdot j_{max} - 4) \\ \textbf{\textit{if}} \quad n_1 - 2 \cdot (j_{max} + 1) \cdot (n_3 - n_2 + j_{max} + 2) > 2 \cdot (n_3 - n_2) + 2 \cdot (j_{max} + 2) \end{cases}$$

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Making some calculations, we have that

$$b := \begin{cases} \begin{aligned} & 4 \cdot j_{max}^3 - 8 \cdot j_{max}^2 \cdot n_2 + 4 \cdot j_{max}^2 \cdot n_3 + 18 \cdot j_{max}^2 - 2 \cdot j_{max} \cdot n_1 + 4 \cdot j_{max} \cdot n_2^2 - 4 \cdot j_{max} \cdot n_2 \cdot n_3 - \\ & 22 \cdot j_{max} \cdot n_2 + 10 \cdot j_{max} \cdot n_3 + 26 \cdot j_{max} + 2 \cdot n_1 \cdot n_2 - 3 \cdot n_1 + 4 \cdot n_2^2 - 4 \cdot n_2 \cdot n_3 - 14 \cdot n_2 + 6 \cdot n_3 + 12 \\ & \quad\quad \mathbf{if} \quad\quad n_1 \leq 2 \cdot (j_{max} + 2) \cdot (j_{max} - n_2 + n_3 + 2) \end{aligned} \\[2em] \begin{aligned} & 4 \cdot j_{max}^3 - 8 \cdot j_{max}^2 \cdot n_2 + 4 \cdot j_{max}^2 \cdot n_3 + 20 \cdot j_{max}^2 - 2 \cdot j_{max} \cdot n_1 + 4 \cdot j_{max} \cdot n_2^2 - 4 \cdot j_{max} \cdot n_2 \cdot n_3 - \\ & 24 \cdot j_{max} \cdot n_2 + 12 \cdot j_{max} \cdot n_3 + 34 \cdot j_{max} + 2 \cdot n_1 \cdot n_2 - 4 \cdot n_1 + 4 \cdot n_2^2 - 4 \cdot n_2 \cdot n_3 - 18 \cdot n_2 + 10 \cdot n_3 + 20 \\ & \quad\quad \mathbf{if} \quad\quad n_1 > 2 \cdot (j_{max} + 2) \cdot (j_{max} - n_2 + n_3 + 2) \end{aligned} \end{cases}$$

Thus, the general solution is given by:

$$h = \begin{cases} \begin{aligned} & \frac{4 \cdot j_{max}^3}{3} - 2 \cdot j_{max}^2 \cdot n_2 + 2 \cdot j_{max}^2 \cdot n_3 + 7 \cdot j_{max}^2 - 2 \cdot j_{max} \cdot n_1 - 6 \cdot j_{max} \cdot n_2 + \\ & \quad 6 \cdot j_{max} \cdot n_3 + \frac{35 \cdot j_{max}}{3} + 2 \cdot n_1 \cdot n_2 - 2 \cdot n_1 - 4 \cdot n_2 + 4 \cdot n_3 + 5 \\ & \mathbf{if} \quad n_1 \leq 2 \cdot (j_{max}^2 - j_{max} \cdot n_2 + j_{max} \cdot n_3 + 4 \cdot j_{max} - 2 \cdot n_2 + 2 \cdot n_3 + 4) \end{aligned} \\[2em] \begin{aligned} & \frac{4 \cdot j_{max}^3}{3} - 2 \cdot j_{max}^2 \cdot n_2 + 2 \cdot j_{max}^2 \cdot n_3 + 9 \cdot j_{max}^2 - 2 \cdot j_{max} \cdot n_1 - 8 \cdot j_{max} \cdot n_2 + \\ & \quad 8 \cdot j_{max} \cdot n_3 + \frac{59 \cdot j_{max}}{3} + 2 \cdot n_1 \cdot n_2 - 3 \cdot n_1 - 8 \cdot n_2 + 8 \cdot n_3 + 13 \\ & \mathbf{if} \quad n_1 > 2 \cdot (j_{max}^2 - j_{max} \cdot n_2 + j_{max} \cdot n_3 + 4 \cdot j_{max} - 2 \cdot n_2 + 2 \cdot n_3 + 4) \end{aligned} \end{cases} \tag{5}$$

where j_{max} is the <u>maximum</u> value $j \in \mathbb{N}_0$ such that

$$n_1 \geq 2 \cdot [j^2 + (n_3 - n_2 + 3) \cdot j + n_3 - n_2 + 2]$$

$$\rightarrow j_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{n_3^2 + n_2^2 - 2 \cdot n_2 \cdot n_3 + 2 \cdot n_3 - 2 \cdot n_2 + 2 \cdot n_1 + 1} + n_2 - n_3 - 3 \right) \right\rfloor .$$

The Eq. 5 can be rewritten more elegantly as

$$h = \begin{cases} \begin{aligned} & \frac{4}{3} \cdot j_{max}^3 + [2 \cdot (n_3 - n_2) + 7] \cdot j_{max}^2 + \left[6 \cdot (n_3 - n_2) - 2 \cdot n_1 + \frac{35}{3} \right] \cdot j_{max} + 4 \cdot (n_3 - n_2) + 2 \cdot n_1 \cdot (n_2 - 1) + 5 \\ & \quad\quad if \quad\quad n_1 \leq 2 \cdot [j_{max}^2 + (n_3 - n_2 + 4) \cdot j_{max} + 2 \cdot (n_3 - n_2) + 4] \end{aligned} \\[2em] \begin{aligned} & \frac{4}{3} \cdot j_{max}^3 + [2 \cdot (n_3 - n_2) + 9] \cdot j_{max}^2 + \left[8 \cdot (n_3 - n_2) - 2 \cdot n_1 + \frac{59}{3} \right] \cdot j_{max} + 8 \cdot (n_3 - n_2) + n_1 \cdot (2 \cdot n_2 - 3) + 13 \\ & \quad\quad if \quad\quad n_1 > 2 \cdot [j_{max}^2 + (n_3 - n_2 + 4) \cdot j_{max} + 2 \cdot (n_3 - n_2) + 4] \end{aligned} \end{cases} \tag{6}$$

where $j_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{n_3^2 + n_2^2 - 2 \cdot n_2 \cdot n_3 + 2 \cdot (n_3 - n_2 + n_1) + 1} + n_2 - n_3 - 3 \right) \right\rfloor .$

Nota Bene. For obvious reasons, the Eq. 6 is always applicable, on condition that $n_1 \geq 2 \cdot (n_3 - n_2) + 4$. Otherwise, the solution follows immediately from Eq. 4, since c can assume only two distinct values: 1 or 2 ($c = 1$ if the condition (3) is verified, $c = 2$ if the (3) is not satisfied, but the Eq. 6 cannot be used – therefore, this is the case $n_1 = 2 \cdot (n_3 - n_2) + 3$.
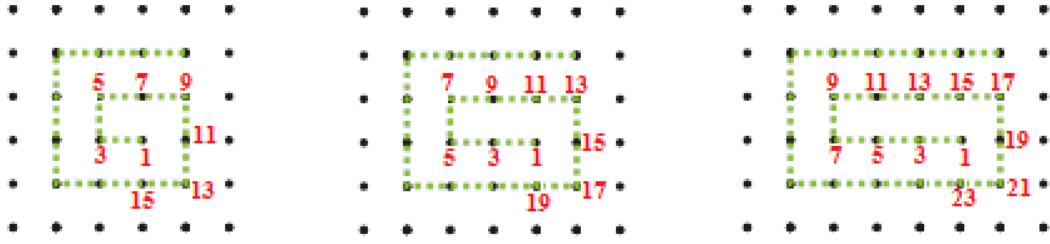
Figure 3. The rectangular spiral and its development [2]
for the cases of (from left to right) $n_3 - n_2 = 0$, $n_3 - n_2 = 1$ and $n_3 - n_2 = 2$.

Therefore, it is possible to extend the aforementioned result we have previously shown in the k-dimensional case: $n_1 \times n_2 \times \ldots \times n_k$. The method to determine an acceptable upper limit for the optimal solution remains the same as in the case $n_1 = n_2 = \ldots = n_k$:

$$h = (t + 1) \cdot \prod_{j=1}^{k-3} n_j - 1 \tag{7}$$

Where t, the lowest upper limit available for the $n_{k-2} \times n_{k-1} \times n_k$ problem, is given by Eq. 4 (with the exception of the very particular cases we introduced at the beginning of the paper [6]) and it is made explicit by (2)–(6).

Specifically, we will start considering an external grid defined by $[n_{k-1}; n_k]$, and we will connect the corresponding $n_k \cdot n_{k-1}$ points using $2 \cdot n_{k-1} - 1$ lines (following the rectangular spiral pattern), then, from the ending point of that external grid, we will draw the line segment which is orthogonal to any $[n_{k-1}; n_k]$ plane (along the n_{k-2} points direction), and so on.

3　The $n_1 \times n_2 \times \ldots \times n_k$ Problem bounded from below

In this section, we provide a non-trivial lower bound for the k-dimensional $n_1 \times n_2 \times \ldots \times n_k$ Points Problem. In this way, we can build a range in which all the best possible solutions to the problem we are considering (for any natural number n_i and number of dimensions k) will certainly fall. In conclusion, we provide a few characteristic numerical examples in order to appreciate the quality of the result arising from the particular approach we have chosen.

For $k = 3$ ($n_1 \leq n_2 \leq n_3$), let us examine first the structure of the grid: it is not possible to intersect more than $(n_3 - 1) + (n_2 - 1) = n_3 + n_2 - 2$ points using two consecutive lines; however, there is one exception (which, for simplicity, we may assume as in the case of the first two lines drawn). In this circumstance, it is possible to fit n_3 points with the first line and $n_2 - 1$ points using the second one, just as in the case of the pure rectangular spiral solution that we have already considered.

Let us observe now that, lying (by definition) each segment on a unique plan, it will be necessary to provide $n_1 - 1$ lines to connect the various plans that are addressed in succession (of any type): there is no way to avoid using fewer than $n_1 - 1$ lines to connect (at most) $n_1 - 1$ points at a time (under the constraint previously explained above to connect $n_3 + n_2 - 1$ points with the first two line segments). Each of these lines could be interposed between as many rectilinear line segments capable of connecting $n_k - 1$ points at any one time.

Following the same pattern, we notice that the previous result, in the k-dimensions case ($k \geq 3$), does not substantially change.

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Let h_l be the number of line segments of our lower bound, for any $k \geq 3$, so that we have

$$\prod_{i=1}^{k} n_i :\leq n_k + \sum_{j=1}^{k-2}(n_j - 1)^2 + (n_k - 1) \cdot \sum_{j=1}^{k-2}(n_j - 1) + \left[h_l - 2 \cdot \sum_{j=1}^{k-2}(n_j - 1) - 1\right] \cdot \left\lfloor \frac{n_k + n_{k-1}}{2} - 1\right\rfloor \quad (8)$$

Taking into account the fact that, $\forall\, n_k, n_{k-1}$, $\left\lfloor \frac{n_k + n_{k-1}}{2} - 1\right\rfloor \leq \left\lfloor \frac{n_k + n_{k-1} - 1}{2}\right\rfloor$, doing some basic calculations, we get the following result:

$$\begin{cases} h_l \geq \left\lceil \left| \frac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} n_j{}^2 + (3 - n_k) \cdot \sum_{j=1}^{k-2} n_j + n_k \cdot (k - 3) - 2 \cdot k + 4 + (n_k + n_{k-1} - 2) \cdot \sum_{j=1}^{k-2}(n_j - 1)\right]\right|\right\rceil + 1 \\ \qquad\qquad\qquad\qquad if \quad \frac{n_k + n_{k-1}}{2} \in \mathbb{N}\backslash\{0,1\} \\[2em] h_l \geq \left\lceil \left| \frac{2}{n_k + n_{k-1} - 1} \cdot \left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} n_j{}^2 + (3 - n_k) \cdot \sum_{j=1}^{k-2} n_j + n_k \cdot (k - 3) - 2 \cdot k + 4 + (n_k + n_{k-1} - 1) \cdot \sum_{j=1}^{k-2}(n_j - 1)\right]\right|\right\rceil + 1 \\ \qquad\qquad\qquad\qquad if \quad \frac{n_k + n_{k-1} + 1}{2} \in \mathbb{N}\backslash\{0,1\} \end{cases}$$

Hence

$$h_l \geq \begin{cases} \left\lceil \left| \frac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} n_j{}^2 + \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2\right)\right]\right|\right\rceil + 1 \\ \qquad\qquad\qquad if \quad \frac{n_k + n_{k-1}}{2} \in \mathbb{N}\backslash\{0,1\} \\[2em] \left\lceil \left| \frac{2}{n_k + n_{k-1} - 1} \cdot \left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} n_j{}^2 + 2 \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2\right) - k + 2\right]\right|\right\rceil + 1 \\ \qquad\qquad\qquad if \quad \frac{n_k + n_{k-1} + 1}{2} \in \mathbb{N}\backslash\{0,1\} \end{cases}$$

$$(9)$$

Notice now how we can improve the result by the (9) whereas the linking lines between the various plans cannot actually join $n_i - 1$ points each time: to connect all the points of every plane belonging to the dimension/s distinguished by the fewest points aligned (the values of the n_i characterized by the lowest subscript) it is possible to connect $n_i - 1$ points with the first line segment, $n_i - 2$ using the second line segment, $n_i - 3$ points with the next one, and so on.

Therefore, with reference to the three-dimensional case, these $n_1 - 1$ linking lines intersect $\sum_{j=1}^{n_1-1}(n_1 - j) = \frac{n_1 \cdot (n_1 - 1)}{2}$ new (unvisited) points. As noted above, we can assume that, at most, each one of them will precede and follow as many line segments that intersect $n_k - 1$ points.

Thus

$$\prod_{i=1}^{k} n_i :\leq n_k + \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j \cdot (n_j - 1) + (n_k - 1) \cdot \sum_{j=1}^{k-2}(n_j - 1) + \left[h_l - 2 \cdot \sum_{j=1}^{k-2}(n_j - 1) - 1\right] \cdot \left|\frac{n_k + n_{k-1}}{2} - 1\right|$$

$$(10)$$

Hence

$$h_l \geq \begin{cases} \left\lceil \left| \frac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j^2 - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) + k - 2 \right] \right| \right\rceil + 1 \\ \qquad\qquad \textbf{if} \quad \frac{n_k + n_{k-1}}{2} \in \mathbb{N} \backslash \{0,1\} \\[2em] \left\lceil \left| \frac{2}{n_k + n_{k-1} - 1} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j^2 + \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) \right] \right| \right\rceil + 1 \\ \qquad\qquad \textbf{if} \quad \frac{n_k + n_{k-1} + 1}{2} \in \mathbb{N} \backslash \{0,1\} \end{cases} \tag{11}$$

In detail (looking at the (11)), if $k = 3$, it follows that

$$h_l \geq \begin{cases} \left\lceil \frac{2 \cdot n_1 \cdot n_2 \cdot n_3 - n_1^2 + 2 \cdot n_1 \cdot n_2 - n_1 - 2 \cdot n_2 - 2 \cdot n_3 + 2}{n_3 + n_2 - 2} \right\rceil + 1 \\ \qquad \textbf{if} \quad \frac{n_3 + n_2}{2} \in \mathbb{N} \backslash \{0,1\} \\[2em] \left\lceil \frac{2 \cdot n_1 \cdot n_2 \cdot n_3 - n_1^2 + 2 \cdot n_1 \cdot n_2 + n_1 - 2 \cdot n_2 - 2 \cdot n_3}{n_3 + n_2 - 1} \right\rceil + 1 \\ \qquad \textbf{if} \quad \frac{n_3 + n_2 + 1}{2} \in \mathbb{N} \backslash \{0,1\} \end{cases} \tag{12}$$

Now, let us consider that, for every $n_k \geq n_{k-1} \geq \ldots \geq n_2 \geq n_1 \geq 2$ ($\forall n_i \in \mathbb{N} \backslash \{0,1\}$),

$$\frac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j^2 - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) + k - 2 \right] \geq$$
$$\frac{2}{n_k + n_{k-1} - 1} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j^2 + \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) \right].$$

Thus, considering the fact that we can arbitrarily change the value of n_k (i.e., we can take $\widetilde{n_k} := n_k + 1$ if we like) without varying the number of line segments we need to connect every point, we can assume, without loss of generality, that

$$h_l \geq \left\lceil \frac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j^2 - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 3 \right) + k - 4 \right] \right\rceil - 1 \tag{13}$$

for any $[n_k, n_{k-1}, \ldots, n_2, n_1]$.

Consequently, if $k = 3$,

$$h_l \geq \left\lceil \frac{n_1 \cdot (2 \cdot n_2 \cdot n_3 - n_1 + 2 \cdot n_2 - 1) - 2}{n_3 + n_2 - 2} \right\rceil - 1 \tag{14}$$

On specifics, for $2 \leq n_1 = n_2 = n_3 := n$,

$$h_l \geq \left\lceil n^2 + \frac{3 \cdot n}{2} \right\rceil \tag{15}$$

4 Conclusion

Given $k = 3$, by combining Eq. 14 with the (2)–(6), we get the intervals in which the best possible solutions of the problem will certainly fall.

How wide this range is (and therefore how interesting this outcome may be considered) also depends on the particular values of n_1, n_2 and n_3.

Example 1: $n_1 = 10$, $n_2 = 13$, $n_3 = 15$.

$$155 \leq h \leq 253$$

Example 2: $n_1 = 10$, $n_2 = 21$, $n_3 = 174$.

$$380 \leq h \leq 419$$

If $k > 3$, the interval is given by

$$\left\lceil \frac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \left(\sum_{j=1}^{k-2} n_j{}^2 + \sum_{j=1}^{k-2} n_j \right) + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 3 \right) + k - 4 \right] \right\rceil - 1 \leq h \leq (t+1) \cdot \prod_{j=1}^{k-3} n_j - 1$$

Where t, the minimal upper limit for the $n_{k-2} \times n_{k-1} \times n_k$ Points Problem, is the result obtained from the (4)–(6) or, if $n_{k-1} \leq n_k - 1$, from the (16)–(17) (see Appendix).

In this case, how great the interval is also depends on the particular value of k (in general, the lager the k, the wider is the interval).

Example 3: $k = 4$, $n_1 = 10$, $n_2 = 16$, $n_3 = 18$, $n_4 = 48$ (thus $t = 575$).

$$4328 \leq h \leq 5759$$

If I had to gamble, setting $k := 3$, I would put money on any betting odds higher than $1 + 10^{-80} : 1$ (there are roughly 10^{80} atoms in the visible universe) that "h_{best}" (the number of straight line segments associated with the best possible solution) is significantly closer to the upper bound I defined and can be small compared to its counterpart – mathematically, I would be willing to bet on the fact that, for the vast majority of the possible combinations $[D, n_3]$,

$$\frac{h_u - h_{best}}{h_{best} - h_l} < 1.$$

Finally, it is interesting to note that, for some particular combinations, the upper bound and the lower bound coincide, thus allowing us to obtain a complete and definitive resolution of the given problem.

E.g., for $k = 3$, $n_1 = n_2 = 3$, $n_3 = 61$, it follows that $h_l = h_u = h_{best} = 17$. Ditto if $k = 3$, $n_1 = 3$, $n_2 = 4$, $n_3 = 57$. In fact, $h_l = h_u = h_{best} = 23$. While, if $k = 4$, $n_1 = n_2 = n_3 = 2$, $n_4 = 46$, $h_l = h_u = h_{best} = 15$.

5 Appendix

If we do not take into account all the additional constraints (solving the problem "inside the box" only, no intersections between lines, and so on) we could improve our "plane by plane" upper bound. For example, we could use the basic pattern below (Fig. 4), for any $n \geq 4$. This kind of solutions can be applied to the $n \times n \times \ldots \times n$ Points Problem and to the $n_1 \times n_2 \times \ldots \times n_k$ Points Problem as well (e.g., $n_k - n_{k-1} = 1$, see Fig. 5):
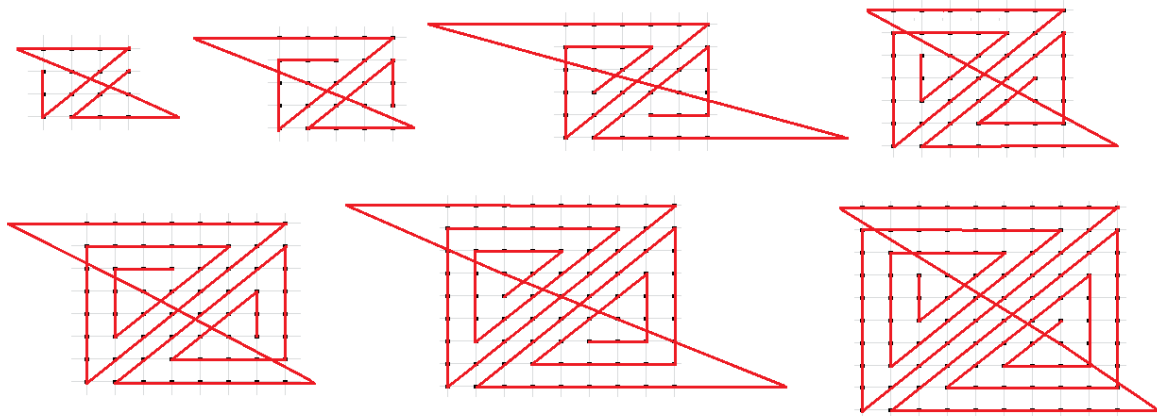
Three alternative patterns
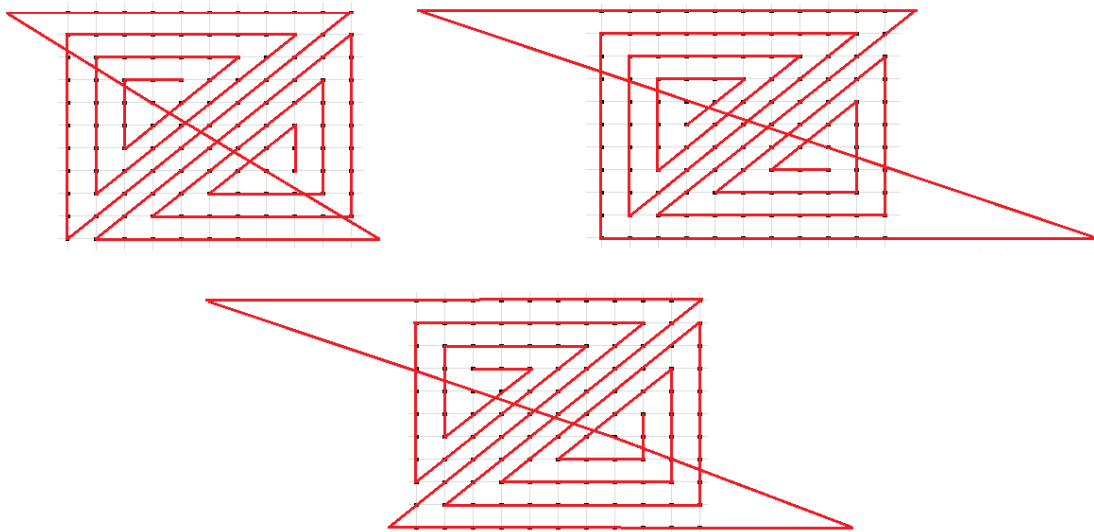
Figure 4. The "double spiral" pattern for $n_k = n_{k-1}(2 \cdot n_{k-1} - 2$ lines).
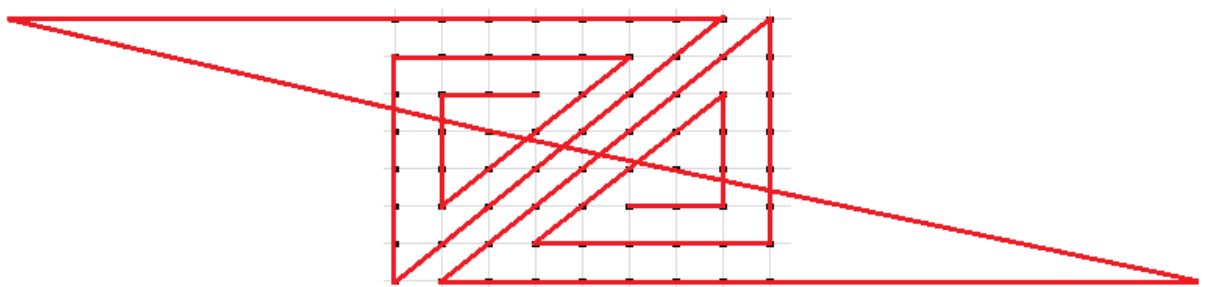
Figure 5. The "double spiral" pattern if $n_k - n_{k-1} = 1$ ($2 \cdot n_{k-1} - 1$ lines).

Looking at the pattern of Fig. 5, we can easily discover that we can use it to reduce the 3D upper bounds by the rectangular spiral: e.g., for $n_1 = n_2 = 22$, $n_3 = 23$ it follows that $h_u = 902$, which is far better than 917, the rectangular spiral one.

Therefore, if $n_1 = n_2 = n_3 := n$, the best "thinking outside the box" upper bounds are as follows.

Table 1: $n \times n \times n$ Points Puzzle upper bounds
following the "double spiral pattern" by Fig. 4.

n	Best Upper Bound Currently Discovered	n	Best Upper Bound Currently Discovered	n	Best Upper Bound Currently Discovered
1	/	18	**587**	35	**2258**
2	**7**	19	**655**	36	**2391**
3	**14**	20	**726**	37	**2528**
4	**26**	21	**801**	38	**2669**
5	**42**	22	**880**	39	**2814**
6	**62**	23	**963**	40	**2963**
7	**85**	24	**1050**	41	**3115**
8	**112**	25	**1141**	42	**3270**
9	**143**	26	**1236**	43	**3429**
10	**178**	27	**1335**	44	**3592**
11	**216**	28	**1438**	45	**3759**
12	**257**	29	**1544**	46	**3930**
13	**302**	30	**1653**	47	**4105**
14	**351**	31	**1766**	48	**4284**
15	**404**	32	**1883**	49	**4467**
16	**461**	33	**2004**	50	**4654**
17	**522**	34	**2129**	51	**4845**

Nota Bene. The upper bounds for $n = 4$ and $n = 5$ are only two particular cases. They are based on a combination of a few, different, two-dimensional patterns. A personal conjecture is that it is possible to do the same for any $n \geq 4$; i.e., we would be able to solve every $n \times n \times n$ ($n \geq 4$) puzzle with a plane by plane approach using at least one line less than the "pure" double spiral solution.

Thus, $\forall\, n \geq 4$,

$$
h_u\,(n_1{=}n_2{=}n_3{:=}n) = \begin{cases}
9 \cdot (n - 1) + \sum_{i=1}^{i_{max}}(2 \cdot n - 2 \cdot i - 1) + \sum_{i=1}^{i_{max}}(2 \cdot i + 3)(2 \cdot n - 2 \cdot i - 2) + \\
\quad (n - i_{max}^{\,2} - 5 \cdot i_{max} - 4) \cdot (2 \cdot n - 2 \cdot i_{max} - 3) \\
\qquad\qquad \boldsymbol{if} \quad n - i_{max}^{\,2} - 5 \cdot i_{max} \leq 5 \\
\\
\\
9 \cdot (n - 1) + \sum_{i=1}^{i_{max}}(2 \cdot n - 2 \cdot i - 1) + \sum_{i=1}^{i_{max}}(2 \cdot i + 3)(2 \cdot n - 2 \cdot i - 2) + \\
\quad (2 \cdot n - 2 \cdot i_{max} - 3) + (n - i_{max}^{\,2} - 5 \cdot i_{max} - 5) \cdot (2 \cdot n - 2 \cdot i_{max} - 4) \\
\qquad\qquad \boldsymbol{if} \quad n - i_{max}^{\,2} - 5 \cdot i_{max} > 5
\end{cases}
$$

Hence

$$h_u\,(n_1{=}n_2{=}n_3{:=}n) = \begin{cases} \frac{2}{3}\cdot i_{max}{}^3 + 5\cdot i_{max}{}^2 - 2\cdot\left(n - \frac{14}{3}\right)\cdot i_{max} + 2\cdot n^2 - 2\cdot n + 3 \\ \qquad\qquad \textbf{\textit{if}} \quad n - i_{max}{}^2 - 5\cdot i_{max} \le 5 \\[2ex] \frac{2}{3}\cdot i_{max}{}^3 + 6\cdot i_{max}{}^2 - \left(2\cdot n - \frac{43}{3}\right)\cdot i_{max} + 2\cdot n^2 - 3\cdot n + 8 \\ \qquad\qquad \textbf{\textit{if}} \quad n - i_{max}{}^2 - 5\cdot i_{max} > 5 \end{cases} \tag{16}$$

where i_{max} is the <u>maximum</u> value $i \in \mathbb{N}_0$ such that $n \ge i^2 + 5\cdot i + 4$

$\rightarrow i_{max} = \left\lfloor \frac{1}{2}\cdot\left(\sqrt{4\cdot n + 9} - 5\right)\right\rfloor.$

While, if $n_1 = n_2 = n_3 - 1$, the best "thinking outside the box" upper bounds are given by Table 2.

Table 2: $n_1 \times n_2 \times n_3$ Points Puzzle upper bounds for $n_1 = n_2 = n_3 - 1$
following the "double spiral pattern" by Fig. 5.

$n_1 = n_2$ $= n_3 - 1$	Best Upper Bound Currently Discovered	$n_1 = n_2$ $= n_3 - 1$	Best Upper Bound Currently Discovered	$n_1 = n_2$ $= n_3 - 1$	Best Upper Bound Currently Discovered
1	**1**	18	**605**	35	**2293**
2	**7**	19	**674**	36	**2427**
3	**17**	20	**746**	37	**2565**
4	**31**	21	**822**	38	**2707**
5	**48**	22	**902**	39	**2853**
6	**68**	23	**986**	40	**3003**
7	**92**	24	**1074**	41	**3156**
8	**120**	25	**1166**	42	**3312**
9	**152**	26	**1262**	43	**3472**
10	**188**	27	**1362**	44	**3636**
11	**227**	28	**1466**	45	**3804**
12	**269**	29	**1573**	46	**3976**
13	**315**	30	**1683**	47	**4152**
14	**365**	31	**1797**	48	**4332**
15	**419**	32	**1915**	49	**4516**
16	**477**	33	**2037**	50	**4704**
17	**539**	34	**2163**	51	**4896**

Therefore, for any $[n_1 \geq 4, n_2 = n_3 - 1]$, it follows that

$$
h_u\left(n_2 = n_3 - 1\right) =
\begin{cases}
\frac{2}{3} \cdot i_{max}{}^3 + 5 \cdot i_{max}{}^2 + \left(\frac{28}{3} - 2 \cdot n_1\right) \cdot i_{max} + 2 \cdot n_1 \cdot n_2 - n_1 + 3 \\
\qquad \boldsymbol{if} \quad n_1 - i_{max}{}^2 - 5 \cdot i_{max} \leq 5 \\
\\
\frac{2}{3} \cdot i_{max}{}^3 + 6 \cdot i_{max}{}^2 + \left(\frac{43}{3} - 2 \cdot n_1\right) \cdot i_{max} + 2 \cdot n_1 \cdot n_2 - 2 \cdot n_1 + 8 \\
\qquad \boldsymbol{if} \quad n_1 - i_{max}{}^2 - 5 \cdot i_{max} > 5
\end{cases}
\tag{17}
$$

where i_{max} is the <u>maximum</u> value $i \in \mathbb{N}_0$ such that $n_1 \geq i^2 + 5 \cdot i + 4$
$\rightarrow i_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{4 \cdot n_1 + 9} - 5\right) \right\rfloor$.

Acknowledgement

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