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New modular relations for the Rogers–Ramanujan type functions of order fifteen

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Abstract: In this paper, we establish two modular relations for the Rogers–Ramanujan–Slater functions of order fifteen. These relations are analogues to Ramanujan's famous forty identities for the Rogers–Ramanujan functions.

Furthermore, we give interesting partition theoretic interpretations of these relations.
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1 Introduction

Throughout the paper, we assume $|q| < 1$ and for positive integer n, we use the standard notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

The well-known Rogers–Ramanujan functions are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}. \tag{1.1}$$

These functions satisfy the famous Rogers–Ramanujan identities

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \text{ and } H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.2}$$

In [20], Ramanujan remarks, I have now found an algebraic relation between $G(q)$ and $H(q)$, viz.:

$$H(q)\{G(q)\}^{11} - q^2 G(q)\{H(q)\}^{11} = 1 + 11q\{G(q)H(q)\}^6.$$

Another interesting formula is

$$H(q)G(q^{11}) - q^2 G(q)H(q^{11}) = 1.$$

These two identities are form a list of forty identities involving the Rogers–Ramanujan functions that Ramanujan had complied. In 1921, L. G. Rogers [23] established ten out of these forty identities. Later G. N. Watson [27] proved eight of the identities, but with two of them from the list that Rogers proved. Ramanujan's forty identities for $G(q)$ and $H(q)$ were first brought to the mathematical world by B. J. Birch [12] in 1975. After Watson's paper, D. Bressoud [13] proved fifteen from the list of the forty in 1977. A. J. F. Biagioli [11] used modular forms to prove seven of the remaining nine identities. Recently B. C. Berndt et al. [9] offered proofs of 35 of the 40 identities. Most likely these proofs might have given by Ramanujan himself. A number of mathematician tried to find new identities for the Rogers–Ramanujan functions similar to those which have been found by Ramanujan [21], including Berndt and H. Yesilyurt [10] and C. Gugg [15].

Two important analogues of the Rogers–Ramanujan functions are the Ramanujan–Göllnitz–Gordan functions. In addition to that, the Rogers–Ramanujan and Ramanujan–Göllnitz-Gordan functions share some remarkable properties. S.-S. Huang [18] has derived several modular relations analogues to Ramanujan's forty identities for the Rogers–Ramanujan functions. S.-L. Chen and Huang [14] also derived some modular relations for Ramanujan-Göllnitz-Gordan functions. N. D. Baruah, J. Bora and N. Saikia [8], offered new proofs of many of the identities of Chen and Huang [14], their methods yields further new relations as well. In [16, 17], H. Hahn has established several modular relations for the septic analogues of the Rogers–Ramanujan functions and also obtained several relations that are connected with the Rogers–Ramanujan and Göllnitz-Gordan functions. In [7], Baruah and Bora have established several modular relations for the nonic analogues of the Rogers–Ramanujan functions. They also established several other modular relations that are connected with the Rogers–Ramanujan functions, Göllnitz–Gordan functions and septic analogues of Rogers–Ramanujan type functions. In [6] Baruah and Bora have established several modular relations involving two functions analogues to the Rogers–Ramanujan functions.

In [4], C. Adiga, K. R. Vasuki and B. R. Srivatsa Kumar have established modular relations involving two functions of Rogers–Ramanujan type. In [26], Vasuki, G. Sharath and K. R. Rajanna have established modular relations for cubic functions and are shown to be connected to the Ramanujan cubic continued fraction. In 2012, Adiga, Vasuki and N. Bhaskar [3] have established modular relations for cubic functions. Vasuki and P. S. Guruprasad [25] have established certain modular relations for the Rogers–Ramanujan type functions of order twelve of which some of them are proved by Baruah and Bora [6] on employing different method. Recently, Adiga and N. A. S. Bulkhali [1, 2] have established several modular relations for the Rogers–Ramanujan type functions of order ten. Almost all of these functions which have been studied so far are due to Rogers [22] and L. G. Slater [24].

In [21, p. 33], Ramanujan stated the following identity:

$$\frac{f(aq^3,\ a^{-1}q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}\,(-a^{-1}q;\ q^2)_n\,(-aq;\ q^2)_n}{(q^2;\ q^2)_{2n}}. \tag{1.3}$$

The preceding result of Ramanujan yields infinitely many identities of Rogers–Ramanujan–Slater type when a is set to $\pm q^r$ for $r \in \mathbb{Q}$. In [19], J. Mc Laughlin, A. V. Sills and P. Zimmer have listed the following Rogers–Ramanujan–Slater identities:

$$A(q) := \frac{f(-q^7,-q^8)}{f(-q^5)} = \sum_{n=0}^{\infty} \frac{q^{5n^2}\,(q^2;q^5)_n\,(q^3;q^5)_n}{(q^5;q^5)_{2n}}, \tag{1.4}$$

$$B(q) := \frac{f(-q^4,-q^{11})}{f(-q^5)} = 1 - \sum_{n=1}^{\infty} \frac{q^{5n^2-1}\,(q^4;q^5)_{n-1}\,(q;q^5)_{n+1}}{(q^5;q^5)_{2n}}, \tag{1.5}$$

$$C(q) := \frac{f(-q^2,-q^{13})}{f(-q^5)} = 1 - \sum_{n=1}^{\infty} \frac{q^{5n^2-3}\,(q^2;q^5)_{n-1}\,(q^3;q^5)_{n+1}}{(q^5;q^5)_{2n}}, \tag{1.6}$$

$$D(q) := \frac{f(-q,-q^{14})}{f(-q^5)} = 1 - \sum_{n=1}^{\infty} \frac{q^{5n^2-4}\,(q;q^5)_{n-1}\,(q^4;q^5)_{n+1}}{(q^5;q^5)_{2n}}. \tag{1.7}$$

The main purpose of this paper is to establish two modular relations involving $A(q)$, $B(q)$, $C(q)$ and $D(q)$, which are analogues of Ramanujan's forty identities and further we extract partition theoretic interpretations of these relations.

2 Definitions and preliminary results

In this section, we present some basic definitions and preliminary results on Ramanujan's theta functions. Ramanujan's general theta function is

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\,b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.1}$$

Then it is easy to verify that [5, Entry 18]

$$f(a,b) = f(b,a), \tag{2.2}$$

$$f(1,a) = 2f(a,a^3), \tag{2.3}$$

$$f(-1,a) = 0. \tag{2.4}$$

The well-known Jacobi triple product identity [5, Entry 19] is given by

$$f(a,b) = (-a;ab)_\infty\,(-b;ab)_\infty\,(ab;ab)_\infty. \tag{2.5}$$

The three most interesting special cases of (2.1) are [5, Entry 22]

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{2.6}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.7}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{2.8}$$

Throughout the paper, we shall use f_n to denote $f(-q^n)$.

3 Main results

In this section, we prove the following two modular relations involving $A(q)$, $B(q)$, $C(q)$ and $D(q)$. For simplicity, for a positive integer k, we set $A_k := A(q^k)$, $B_k := B(q^k)$, $C_k := C(q^k)$, $D_k := D(q^k)$.

$$A_1B_1 + qB_1C_1 - q^2C_1D_1 - qD_1A_1 = 1, \tag{3.1}$$

$$A_{11}D_1 - q^3B_{11}A_1 - q^{10}C_{11}B_1 + q^{15}D_{11}C_1 = \frac{f_3f_{33}}{f_5f_{55}} - q. \tag{3.2}$$

Proofs of our identities strongly depend on results of Rogers [23] and Bressoud [13]. We adopt Bressoud's notation, except that we use $q^{\frac{n}{24}}f(-q^n)$ instead of P_n, and the variable q instead of x. Let $g_\alpha^{(p,n)}$ and $\Phi_{\alpha,\beta,m,p}$ be defined as follows:

$$g_\alpha^{(p,n)} := g_\alpha^{(p,n)}(q) = q^{\alpha(\frac{12n^2-12n+3-p}{24p})} \prod_{r=0}^{\infty} \frac{(1-(q^\alpha)^{pr+\frac{p-2n+1}{2}})\,(1-(q^\alpha)^{pr+\frac{p+2n-1}{2}})}{\prod_{k=1}^{p-1}(1-(q^\alpha)^{pr+k})}, \tag{3.3}$$

for any positive odd integer p, integer n, and natural number α, and

$$\Phi_{\alpha,\beta,m,p} := \Phi_{\alpha,\beta,m,p}(q) = \sum_{n=1}^{p} \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{\frac{1}{2}\{p\alpha(r+m\frac{2n-1}{2p})^2 + p\beta(s+\frac{2n-1}{2p})^2\}}, \tag{3.4}$$

where α, β and p are natural numbers, and m is an odd positive integer. Then we have the following propositions.

Proposition 3.1. *[13, eqs. (2.12) and (2.13).] We have*

$$g_\alpha^{(5,1)} = q^{-\frac{\alpha}{60}} G(q^\alpha) \quad \text{and} \quad g_\alpha^{(5,2)} = q^{\frac{11\alpha}{60}} H(q^\alpha).$$

Proposition 3.2. *We have*

$$
\begin{aligned}
g_\alpha^{(15,1)} =& q^{-\frac{\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} A_\alpha, &(3.5)\\
g_\alpha^{(15,2)} =& q^{\frac{\alpha}{30}} \frac{f(-q^{3\alpha})}{f(-q^\alpha)} G(q^{3\alpha}), &(3.6)\\
g_\alpha^{(15,3)} =& q^{\frac{\alpha}{6}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)}, &(3.7)\\
g_\alpha^{(15,4)} =& q^{\frac{11\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} B_\alpha, &(3.8)\\
g_\alpha^{(15,5)} =& q^{\frac{19\alpha}{30}} \frac{f(-q^{3\alpha})}{f(-q^\alpha)} H(q^{3\alpha}), &(3.9)\\
g_\alpha^{(15,6)} =& q^{\frac{29\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} C_\alpha, &(3.10)\\
g_\alpha^{(15,7)} =& q^{\frac{41\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} D_\alpha. &(3.11)
\end{aligned}
$$

Proof. Putting $p = 15$, and $n = 1$ in (3.3), we obtain

$$
\begin{aligned}
g_\alpha^{(15,1)} =& q^{\alpha(-\frac{1}{30})} \prod_{r=0}^{\infty} \frac{(1-(q^\alpha)^{15r+7})(1-(q^\alpha)^{15r+8})}{\prod_{k=1}^{14}(1-(q^\alpha)^{15r+k})} \\
=& \frac{q^{(-\alpha/30)}}{(q^\alpha, q^{2\alpha}, q^{3\alpha}, q^{4\alpha}, q^{5\alpha}, q^{6\alpha}, q^{9\alpha}, q^{10\alpha}, q^{11\alpha}, q^{12\alpha}, q^{13\alpha}, q^{14\alpha}; q^{15\alpha})_\infty} \\
=& q^{-\frac{\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} A_\alpha,
\end{aligned}
$$

which gives (3.5). Similarly, we can prove (3.6) - (3.11). □

Lemma 3.3. *[13, Proposition 5.1.] We have*

$$
g_\alpha^{(p,n)} = g_\alpha^{(p,-n+1)}, \quad g_\alpha^{(p,n)} = g_\alpha^{(p,n-2p)}, \quad g_\alpha^{(p,n)} = g_\alpha^{(p,2p-n+1)},
$$

$$
g_\alpha^{(p,n)} = -g_\alpha^{(p,n-p)}, \quad g_\alpha^{(p,n)} = -g_\alpha^{(p,p-n+1)}, \quad g_\alpha^{(p,(p+1)/2)} = 0.
$$

Theorem 3.4. *[13, Proposition 5.4.] For odd $p > 1$,*

$$
\Phi_{\alpha,\beta,m,p} = 2q^{\frac{\alpha+\beta}{24}} f(-q^\alpha) f(-q^\beta) \left(\sum_{n=1}^{(p-1)/2} g_\beta^{(p,n)} g_\alpha^{(p,(2mn-m+1)/2)} \right).
$$

Putting $p = 5$ and $p = 15$ in Theorem 3.4 and then using Lemma 3.3, in the resulting identities, we obtain the following useful lemmas.

Lemma 3.5. *[13, Corollary 5.7.] We have*

$$
\begin{aligned}
\Phi_{\alpha,\beta,1,5} =& 2q^{\frac{\alpha+\beta}{40}} f(-q^\alpha) f(-q^\beta) \left(G(q^\beta)G(q^\alpha) + q^{\frac{\alpha+\beta}{5}} H(q^\beta)H(q^\alpha) \right), &(3.12)\\
\Phi_{\alpha,\beta,3,5} =& 2q^{\frac{9\alpha+\beta}{40}} f(-q^\alpha) f(-q^\beta) \left(G(q^\beta)H(q^\alpha) - q^{\frac{-\alpha+\beta}{5}} H(q^\beta)G(q^\alpha) \right). &(3.13)
\end{aligned}
$$

Lemma 3.6. *We have*

$$\begin{aligned}\Phi_{\alpha,\beta,1,15} =&2q^{\frac{\alpha+\beta}{120}}f(-q^{5\alpha})f(-q^{5\beta})\\ &\{A_\beta A_\alpha + q^{\frac{2(\alpha+\beta)}{5}}B_\beta B_\alpha + q^{\alpha+\beta}C_\beta C_\alpha + q^{\frac{7(\alpha+\beta)}{5}}D_\beta D_\alpha + q^{\frac{\alpha+\beta}{5}}\\ &+ q^{\frac{\alpha+\beta}{15}}\frac{f(-q^{3\alpha})f(-q^{3\beta})}{f(-q^{5\alpha})f(-q^{5\beta})}[G(q^{3\beta})G(q^{3\alpha}) + q^{\frac{3(\alpha+\beta)}{5}}H(q^{3\beta})H(q^{3\alpha})]\},\end{aligned} \tag{3.14}$$

$$\begin{aligned}\Phi_{\alpha,\beta,7,15} =&2q^{\frac{49\alpha+\beta}{120}}f(-q^{5\alpha})f(-q^{5\beta})\\ &\{A_\beta B_\alpha + q^{\frac{(12\beta+18\alpha)}{30}}B_\beta C_\alpha - q^{\beta+\alpha}C_\beta D_\alpha - q^{\frac{(42\beta-12\alpha)}{30}}D_\beta A_\alpha - q^{\frac{(\beta-\alpha)}{5}}\\ &- q^{\frac{(2\beta+8\alpha)}{30}}\frac{f(-q^{3\alpha})f(-q^{3\beta})}{f(-q^{5\alpha})f(-q^{5\beta})}[G(q^{3\beta})H(q^{3\alpha}) - q^{\frac{3(\beta-\alpha)}{5}}H(q^{3\beta})G(q^{3\alpha})]\},\end{aligned} \tag{3.15}$$

$$\begin{aligned}\Phi_{\alpha,\beta,11,15} =&2q^{\frac{121\alpha+\beta}{120}}f(-q^{5\alpha})f(-q^{5\beta})\\ &\{A_\beta C_\alpha - q^{\frac{2(\beta+\alpha)}{5}}B_\beta D_\alpha + q^{\beta-\alpha}C_\beta A_\alpha - q^{\frac{(7\beta-3\alpha)}{5}}D_\beta B_\alpha + q^{\frac{(\beta-4\alpha)}{5}}\\ &- q^{\frac{(\beta-14\alpha)}{15}}\frac{f(-q^{3\alpha})f(-q^{3\beta})}{f(-q^{5\alpha})f(-q^{5\beta})}[G(q^{3\beta})G(q^{3\alpha}) + q^{\frac{3(\beta+\alpha)}{5}}H(q^{3\beta})H(q^{3\alpha})]\},\end{aligned} \tag{3.16}$$

$$\begin{aligned}\Phi_{\alpha,\beta,13,15} =&2q^{\frac{169\alpha+\beta}{120}}f(-q^{5\alpha})f(-q^{5\beta})\\ &\{A_\beta D_\alpha - q^{\frac{(2\beta-7\alpha)}{5}}B_\beta A_\alpha - q^{\beta-\alpha}C_\beta B_\alpha + q^{\frac{(7\beta-2\alpha)}{5}}D_\beta C_\alpha + q^{\frac{(\beta-6\alpha)}{5}}\\ &- q^{\frac{(\beta-11\alpha)}{15}}\frac{f(-q^{3\alpha})f(-q^{3\beta})}{f(-q^{5\alpha})f(-q^{5\beta})}[G(q^{3\beta})H(q^{3\alpha}) - q^{\frac{3(\beta-\alpha)}{5}}H(q^{3\beta})G(q^{3\alpha})]\}.\end{aligned} \tag{3.17}$$

Proof. Applying Theorem 3.4 with $m = 1$ and $p = 15$, we have

$$\begin{aligned}\Phi_{\alpha,\beta,1,15} =&2q^{\frac{\alpha+\beta}{24}}f(-q^{\alpha})f(-q^{\beta})\{g_\beta^{(15,1)}g_\alpha^{(15,1)} + g_\beta^{(15,2)}g_\alpha^{(15,2)} + g_\beta^{(15,3)}g_\alpha^{(15,3)}\\ &+ g_\beta^{(15,4)}g_\alpha^{(15,4)} + g_\beta^{(15,5)}g_\alpha^{(15,5)} + g_\beta^{(15,6)}g_\alpha^{(15,6)} + g_\beta^{(15,7)}g_\alpha^{(15,7)}\}.\end{aligned} \tag{3.18}$$

Using (3.5) - (3.11) in (3.18) and then simplifying, we obtain (3.14). The identities (3.15) - (3.17) can be proved in a similar way by setting $m = 7, 11, 13$, respectively, and $p = 15$ in Theorem 3.4. □

Corollary 3.7. *[13, Corollary 5.5.] If $\Phi_{\alpha,\beta,m,p}$ is defined by (3.4), then*

$$\Phi_{\alpha,\beta,m,1} = 0. \tag{3.19}$$

Theorem 3.8. *[13, Corollary 7.3.] Let $\alpha_i,\ \beta_i,\ m_i,\ p_i$ where $i = 1, 2$, be positive integers with m_1 and m_2 both odd. If $\lambda_1 := (\alpha_1 m_1^2 + \beta_1)/p_1$ and $\lambda_2 := (\alpha_2 m_2^2 + \beta_2)/p_2$, and the conditions*

$$\lambda_1 = \lambda_2, \tag{3.20}$$

$$\alpha_1\beta_1 = \alpha_2\beta_2, \tag{3.21}$$

$$\alpha_1 m_1 \equiv \pm\alpha_2 m_2 (mod \lambda_1) \tag{3.22}$$

hold, then

$$\Phi_{\alpha_1,\beta_1,m_1,p_1} = \Phi_{\alpha_2,\beta_2,m_2,p_2}.$$

Theorem 3.9. *The identity (3.1) holds.*

Proof. Let N denote the set of positive integers. Setting

$$\alpha_1 = 3u,\ \beta_1 = 3u,\ m_1 = 7,\ p_1 = 15u,$$

$$\alpha_2 = u,\ \beta_2 = 9u,\ m_2 = 1,\ p_2 = u,$$

in Theorem 3.8, we obtain

$$\Phi_{3u,3u,7,15u} = \Phi_{u,9u,1,u}\ , \quad u \in N. \tag{3.23}$$

In particular, by taking $u = 1$ in (3.23) and then using (3.15) and (3.19), we have

$$2q^{5/4} f^2(-q^{15})\{A_3B_3 + q^3B_3C_3 - q^6C_3D_3 - q^3D_3A_3 - 1\} = 0.$$

We first observe that identity (3.1) holds for $q = 0$. For $q \neq 0$, dividing both sides of the above equation by $2q^{5/4}f^2(-q^{15})$ and replacing q^3 by q, we deduce that

$$A_1B_1 + qB_1C_1 - q^2C_1D_1 - qD_1A_1 = 1.$$

□

Theorem 3.10. *The identity (3.2) holds.*

Proof. Let N denote the set of positive integers. Setting

$$\alpha_1 = u,\ \beta_1 = 11u,\ m_1 = 13,\ p_1 = 15u,$$

$$\alpha_2 = 11u,\ \beta_2 = u,\ m_2 = 1,\ p_2 = u,$$

in Theorem 3.8, we obtain

$$\Phi_{u,11u,13,15u} = \Phi_{11u,u,1,u}\ , u \in N. \tag{3.24}$$

In particular, by taking $u = 1$ in (3.24) and then by using (3.17) and (3.19), we have

$$\begin{aligned} 2q^{3/2} f(-q^5)f(-q^{55})\{&A_{11}D_1 - q^3B_{11}A_1 - q^{10}C_{11}B_1 + q^{15}D_{11}C_1 \\ &+ q - \frac{f(-q^{33})f(-q^3)}{f(-q^{55})f(-q^5)}[G(q^{33})H(q^3) - q^6H(q^{33})G(q^3)]\} = 0. \end{aligned} \tag{3.25}$$

The first published proof of the following identity was given by Rogers [23]. Watson [27] also gave a proof.

$$G(q^{11})H(q) - q^2H(q^{11})G(q) = 1. \tag{3.26}$$

Employing (3.26) with q replaced by q^3 in (3.25), we get the required result. □

4 Applications to the theory of partitions

In this section, we present partition theoretic interpretations of (3.1) and (3.2). For simplicity, we adopt the standard notation

$$(a_1, a_2, \ldots a_n; q)_\infty := \prod_{j=1}^{n} (a_j; q)_\infty$$

and define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integers and $r < s$.

We also need the notation of colored partitions. A positive integer n has k color if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integer into parts with colors are called "colored partitions".

For example, if 1 is allowed to have two colors, say r (red), and g (green), then all colored partitions of 3 are 3, $2 + 1_r$, $2 + 1_g$, $1_r + 1_r + 1_r$, $1_r + 1_r + 1_g$, $1_r + 1_g + 1_g$, and $1_g + 1_g + 1_g$.

An important fact is that

$$\frac{1}{(q^u;\ q^v)_\infty^k}$$

is the generating function for the number of partitions of n, where all the parts are congruent to $u\ (mod\ v)$ and have k colors.

Theorem 4.1. *Let $P_1(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, $\pm 5\ (mod\ 15)$, with $\pm 5\ (mod\ 15)$ having two colors.*

Let $P_2(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 5, $\pm 7\ (mod\ 15)$, with $\pm 5\ (mod\ 15)$ having two colors.

Let $P_3(n)$ denote the number of partitions of n into parts congruent to ± 4, ± 5, $\pm 7\ (mod\ 15)$, with $\pm 5\ (mod\ 15)$ having two colors.

Let $P_4(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 4, $\pm 5\ (mod\ 15)$, with $\pm 5\ (mod\ 15)$ having two colors.

Let $P_5(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, ± 4, $\pm 7\ (mod\ 15)$.

Then, for any positive integer $n \geq 2$, we have

$$P_1(n) + P_2(n-1) - P_3(n-2) - P_4(n-1) = P_5(n).$$

Proof. Using (1.4) – (1.7) and (2.5) in (3.1) and simplifying we obtain

$$\frac{1}{(q^{1\pm}, q^{2\pm}, q^{5\pm}, q^{5\pm}; q^{15})_\infty} + \frac{q}{(q^{1\pm}, q^{5\pm}, q^{5\pm}, q^{7\pm}; q^{15})_\infty} - \frac{q^2}{(q^{4\pm}, q^{5\pm}, q^{5\pm}, q^{7\pm}; q^{15})_\infty}$$
$$- \frac{q}{(q^{2\pm}, q^{4\pm}, q^{5\pm}, q^{5\pm}; q^{15})_\infty} = \frac{1}{(q^{1\pm}, q^{2\pm}, q^{4\pm}, q^{7\pm}; q^{15})_\infty}.$$

Note that the five quotients of the above represent the generating functions for $P_1(n)$, $P_2(n)$, $P_3(n)$, $P_4(n)$ and $P_5(n)$ respectively. Hence, it is equivalent to

$$\sum_{n=0}^{\infty} P_1(n)q^n + q\sum_{n=0}^{\infty} P_2(n)q^n - q^2\sum_{n=0}^{\infty} P_3(n)q^n - q\sum_{n=0}^{\infty} P_4(n)q^n = \sum_{n=0}^{\infty} P_5(n)q^n,$$

where we set $P_1(0) = P_2(0) = P_3(0) = P_4(0) = P_5(0) = 1$. Equating coefficients of q^n $(n \geq 2)$ on both sides yields the desired result. □

Example 4.2. *The following table illustrates the case $n = 5$ in the Theorem 4.1.*

$P_1(5) = 5$	$5_r,\ 5_g,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$
$P_2(4) = 1$	$1+1+1+1$
$P_3(3) = 0$	$-$
$P_4(4) = 2$	$4,\ 2+2$
$P_5(5) = 4$	$4+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$

Theorem 4.3. *Let $P_1(n)$ denote the number of partitions of n into parts not congruent to* ±1, ±14, ±15, ±16, ±29, ±30, ±31, ±44, ±45, ±46, ±59, ±60, ±61, ±74, ±75, ±76, ±77 $(mod\ 165)$, *and parts congruent to* ±33, ±55, ±66 $(mod\ 165)$ *with two colors.*

Let $P_2(n)$ denote the number of partitions of n into parts not congruent to ±7, ±8, ±15, ±22, ±23, ±30, ±37, ±38, ±44, ±45, ±52, ±53, ±60, ±67, ±68, ±75, ±82 $(mod\ 165)$, *and parts congruent to* ±33, ±55, ±66 $(mod\ 165)$ *with two colors.*

Let $P_3(n)$ denote the number of partitions of n into parts not congruent to ±4, ±11, ±15, ±19, ±22, ±26, ±30, ±34, ±41, ±45, ±49, ±56, ±60, ±64, ±71, ±75, ±76 $(mod\ 165)$, *and parts congruent to* ±33, ±55, ±66 $(mod\ 165)$ *with two colors.*

Let $P_4(n)$ denote the number of partitions of n into parts not congruent to ±2, ±11, ±13, ±15, ±17, ±28, ±30, ±32, ±43, ±45, ±47, ±58, ±60, ±62, ±73, ±75, ±77 $(mod\ 165)$, *and parts congruent to* ±33, ±55, ±66 $(mod\ 165)$ *with two colors.*

Let $P_5(n)$ denote the number of partitions of n into parts not congruent to ±3, ±6, ±9, ±12, ±15, ±18, ±21, ±24, ±27, ±30, ±33, ±36, ±39, ±42, ±45, ±48, ±51, ±54, ±57, ±60, ±63, ±66, ±69, ±72, ±75, ±78, ±81 $(mod\ 165)$, *and parts congruent to* ±55 $(mod\ 165)$ *with two colors.*

Let $P_6(n)$ denote the number of partitions of n into parts not congruent to ±5, ±10, ±15, ±20, ±25, ±30, ±35, ±40, ±45, ±50, ±55, ±60, ±65, ±70, ±75 $(mod\ 165)$, *and parts congruent to* ±33, ±66 $(mod\ 165)$ *with two colors.*

Then, for any positive integer $n \geq 15$, we have

$$P_1(n) - P_2(n-3) - P_3(n-10) + P_4(n-15) = P_5(n) - P_6(n-1).$$

Proof. Using (1.4) - (1.7) and (2.5) in (3.2) and simplifying we obtain

$$\frac{1}{(q^{2\pm},q^{3\pm},q^{4\pm},q^{5\pm},q^{6\pm},q^{7\pm},q^{8\pm},q^{9\pm},q^{10\pm},q^{11\pm},q^{12\pm},q^{13\pm},q^{17\pm},q^{18\pm},q^{19\pm},q^{20\pm};q^{165})_\infty}$$
$$\times\frac{1}{(q^{21\pm},q^{22\pm},q^{23\pm},q^{24\pm},q^{25\pm},q^{26\pm},q^{27\pm},q^{28\pm},q^{32\pm},q^{33\pm},q^{33\pm},q^{34\pm},q^{35\pm},q^{36\pm};q^{165})_\infty}$$
$$\times\frac{1}{(q^{37\pm},q^{38\pm},q^{39\pm},q^{40\pm},q^{41\pm},q^{42\pm},q^{43\pm},q^{47\pm},q^{48\pm},q^{49\pm},q^{50\pm},q^{51\pm},q^{52\pm},q^{53\pm};\ q^{165})_\infty}$$
$$\times\frac{1}{(q^{54\pm},q^{55\pm},q^{55\pm},q^{56\pm},q^{57\pm},q^{58\pm},q^{62\pm},q^{63\pm},q^{64\pm},q^{65\pm},q^{66\pm},q^{66\pm},q^{67\pm},q^{68\pm};\ q^{165})_\infty}$$
$$\times\frac{1}{(q^{69\pm},q^{70\pm},q^{71\pm},q^{72\pm},q^{73\pm},q^{78\pm},q^{79\pm},q^{80\pm},q^{81\pm},q^{82\pm};\ q^{165})_\infty}$$
$$-\frac{q^3}{(q^{1\pm},q^{2\pm},q^{3\pm},q^{4\pm},q^{5\pm},q^{6\pm},q^{9\pm},q^{10\pm},q^{11\pm},q^{12\pm},q^{13\pm},q^{14\pm},q^{16\pm},q^{17\pm},q^{18\pm};q^{165})_\infty}$$
$$\times\frac{1}{(q^{19\pm},q^{20\pm},q^{21\pm},q^{24\pm},q^{25\pm},q^{26\pm},q^{27\pm},q^{28\pm},q^{29\pm},q^{31\pm},q^{32\pm},q^{33\pm},q^{33\pm},q^{34\pm};q^{165})_\infty}$$
$$\times\frac{1}{(q^{35\pm},q^{36\pm},q^{39\pm},q^{40\pm},q^{41\pm},q^{42\pm},q^{43\pm},q^{46\pm},q^{47\pm},q^{48\pm},q^{49\pm},q^{50\pm},q^{51\pm},q^{54\pm};\ q^{165})_\infty}$$
$$\times\frac{1}{(q^{55\pm},q^{55\pm},q^{56\pm},q^{57\pm},q^{58\pm},q^{59\pm},q^{61\pm},q^{62\pm},q^{63\pm},q^{64\pm},q^{65\pm},q^{66\pm},q^{66\pm},q^{69\pm};\ q^{165})_\infty}$$
$$\times\frac{1}{(q^{70\pm},q^{71\pm},q^{72\pm},q^{73\pm},q^{74\pm},q^{76\pm},q^{77\pm},q^{78\pm},q^{79\pm},q^{80\pm},q^{81\pm};\ q^{165})_\infty}$$
$$-\frac{q^{10}}{(q^{1\pm},q^{2\pm},q^{3\pm},q^{5\pm},q^{6\pm},q^{7\pm},q^{8\pm},q^{9\pm},q^{10\pm},q^{12\pm},q^{13\pm},q^{14\pm},q^{16\pm},q^{17\pm},q^{18\pm},q^{20\pm};q^{165})_\infty}$$
$$\times\frac{1}{(q^{21\pm},q^{23\pm},q^{24\pm},q^{25\pm},q^{27\pm},q^{28\pm},q^{29\pm},q^{31\pm},q^{32\pm},q^{33\pm},q^{33\pm},q^{35\pm},q^{36\pm},q^{37\pm};q^{165})_\infty}$$
$$\times\frac{1}{(q^{38\pm},q^{39\pm},q^{40\pm},q^{42\pm},q^{43\pm},q^{44\pm},q^{46\pm},q^{47\pm},q^{48\pm},q^{50\pm},q^{51\pm},q^{52\pm},q^{53\pm},q^{54\pm};\ q^{165})_\infty}$$
$$\times\frac{1}{(q^{55\pm},q^{55\pm},q^{57\pm},q^{58\pm},q^{59\pm},q^{61\pm},q^{62\pm},q^{63\pm},q^{65\pm},q^{66\pm},q^{66\pm},q^{67\pm},q^{68\pm},q^{69\pm};\ q^{165})_\infty}$$
$$\times\frac{1}{(q^{70\pm},q^{72\pm},q^{73\pm},q^{74\pm},q^{76\pm},q^{77\pm},q^{78\pm},q^{80\pm},q^{81\pm},q^{82\pm};\ q^{165})_\infty}$$
$$+\frac{q^{15}}{(q^{1\pm},q^{3\pm},q^{4\pm},q^{5\pm},q^{6\pm},q^{7\pm},q^{8\pm},q^{9\pm},q^{10\pm},q^{12\pm},q^{14\pm},q^{16\pm},q^{18\pm},q^{19\pm},q^{20\pm},q^{21\pm};q^{165})_\infty}$$
$$\times\frac{1}{(q^{22\pm},q^{23\pm},q^{24\pm},q^{25\pm},q^{26\pm},q^{27\pm},q^{29\pm},q^{31\pm},q^{33\pm},q^{33\pm},q^{34\pm},q^{35\pm},q^{36\pm},q^{37\pm};q^{165})_\infty}$$
$$\times\frac{1}{(q^{38\pm},q^{39\pm},q^{40\pm},q^{41\pm},q^{42\pm},q^{44\pm},q^{46\pm},q^{48\pm},q^{49\pm},q^{50\pm},q^{51\pm},q^{52\pm},q^{53\pm},q^{54\pm};\ q^{165})_\infty}$$
$$\times\frac{1}{(q^{55\pm},q^{55\pm},q^{56\pm},q^{57\pm},q^{59\pm},q^{61\pm},q^{63\pm},q^{64\pm},q^{65\pm},q^{66\pm},q^{66\pm},q^{67\pm},q^{68\pm},q^{69\pm};\ q^{165})_\infty}$$
$$\times\frac{1}{(q^{70\pm},q^{71\pm},q^{72\pm},q^{74\pm},q^{76\pm},q^{78\pm},q^{79\pm},q^{80\pm},q^{81\pm},q^{82\pm};\ q^{165})_\infty}$$

$$
\begin{aligned}
&= \frac{1}{(q^{1\pm}, q^{2\pm}, q^{4\pm}, q^{5\pm}, q^{7\pm}, q^{8\pm}, q^{10\pm}, q^{11\pm}, q^{13\pm}, q^{14\pm}, q^{16\pm}, q^{17\pm}, q^{19\pm}, q^{20\pm}, q^{22\pm}; q^{165})_\infty} \\
&\quad \times \frac{1}{(q^{23\pm}, q^{25\pm}, q^{26\pm}, q^{28\pm}, q^{29\pm}, q^{31\pm}, q^{32\pm}, q^{34\pm}, q^{35\pm}, q^{37\pm}, q^{38\pm}, q^{40\pm}, q^{41\pm}, q^{43\pm}; q^{165})_\infty} \\
&\quad \times \frac{1}{(q^{44\pm}, q^{46\pm}, q^{47\pm}, q^{49\pm}, q^{50\pm}, q^{52\pm}, q^{53\pm}, q^{55\pm}, q^{55\pm}, q^{56\pm}, q^{58\pm}, q^{59\pm}, q^{61\pm}, q^{62\pm};\ q^{165})_\infty} \\
&\quad \times \frac{1}{(q^{64\pm}, q^{65\pm}, q^{67\pm}, q^{68\pm}, q^{70\pm}, q^{71\pm}, q^{73\pm}, q^{74\pm}, q^{76\pm}, q^{77\pm}, q^{79\pm}, q^{80\pm}, q^{82\pm};\ q^{165})_\infty} \\
&\quad - \frac{q}{(q^{1\pm}, q^{2\pm}, q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{7\pm}, q^{8\pm}, q^{9\pm}, q^{11\pm}, q^{12\pm}, q^{13\pm}, q^{14\pm}, q^{16\pm}, q^{17\pm}, q^{18\pm}; q^{165})_\infty} \\
&\quad \times \frac{1}{(q^{19\pm}, q^{21\pm}, q^{22\pm}, q^{23\pm}, q^{24\pm}, q^{26\pm}, q^{27\pm}, q^{28\pm}, q^{29\pm}, q^{31\pm}, q^{32\pm}, q^{33\pm}, q^{33\pm}, q^{34\pm}; q^{165})_\infty} \\
&\quad \times \frac{1}{(q^{36\pm}, q^{37\pm}, q^{38\pm}, q^{39\pm}, q^{41\pm}, q^{42\pm}, q^{43\pm}, q^{44\pm}, q^{46\pm}, q^{47\pm}, q^{48\pm}, q^{49\pm}, q^{51\pm}, q^{52\pm};\ q^{165})_\infty} \\
&\quad \times \frac{1}{(q^{53\pm}, q^{54\pm}, q^{56\pm}, q^{57\pm}, q^{58\pm}, q^{59\pm}, q^{61\pm}, q^{62\pm}, q^{63\pm}, q^{64\pm}, q^{66\pm}, q^{66\pm}, q^{67\pm}, q^{68\pm};\ q^{165})_\infty} \\
&\quad \times \frac{1}{(q^{69\pm}, q^{71\pm}, q^{72\pm}, q^{73\pm}, q^{74\pm}, q^{76\pm}, q^{77\pm}, q^{78\pm}, q^{79\pm}, q^{81\pm}, q^{82\pm};\ q^{165})_\infty}.
\end{aligned}
$$

Note that the six quotients of the above represent the generating functions for $P_1(n)$, $P_2(n)$, $P_3(n)$, $P_4(n)$, $P_5(n)$ and $P_6(n)$ respectively. Hence, it is equivalent to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} P_1(n)q^n - q^3 \sum_{n=0}^{\infty} P_2(n)q^n - q^{10} \sum_{n=0}^{\infty} P_3(n)q^n + q^{15} \sum_{n=0}^{\infty} P_4(n)q^n \\
&= \sum_{n=0}^{\infty} P_5(n)q^n - q \sum_{n=0}^{\infty} P_6(n)q^n,
\end{aligned}
$$

where we set $P_1(0) = P_2(0) = P_3(0) = P_4(0) = P_5(0) = P_6(0) = 1$. Equating coefficients of q^n $(n \geq 15)$ on both sides yields the desired result. $\square$

Example 4.4. *The following table illustrate the case* $n = 15$ *in Theorem 4.3.*

$P_1(15) = 40$
$P_2(12) = 65$
$P_3(5) = 6$
$P_4(0) = 1$
$P_5(15) = 70$
$P_6(14) = 100$

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