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# Calculating terms of associated polynomials of Perrin and Cordonnier numbers

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**Abstract:** In this paper, we calculate terms of associated polynomials of Perrin and Cordonnier numbers by using determinants and permanents of various Hessenberg matrices. Since these polynomials are general forms of Perrin and Cordonnier numbers, our results are valid for the Perrin and Cordonnier numbers.
**Keywords:** Perrin and Cordonnier numbers, Associated polynomial of Perrin and Cordonnier numbers, Hessenberg matrix, Determinant, Permanent.
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## 1 Introduction

Lucas [10] in 1876 introduced a sequence, called Perrin sequence after R. Perrin [13]. The well known Perrin and Cordonnier sequences are respectively

$$Q_n = Q_{n-2} + Q_{n-3} \text{ for } n > 3 \text{ and } Q_1 = 0,\ Q_2 = 2,\ Q_3 = 3$$

and

$$P_n = P_{n-2} + P_{n-3} \text{ for } n > 3 \text{ and } P_1 = P_2 = P_3 = 1.$$

The characteristic equation associated with the Perrin and Cordonnier sequences is

$$x^3 - x - 1 = 0$$

with real solution $\rho \approx 1,324718$, called plastic number. The plastic number corresponds to the golden number $\phi \approx 1,618034$ associated with the Fibonacci numbers, for example,

$$\lim_{n\to\infty}\frac{Q_{n+1}}{Q_n}=\lim_{n\to\infty}\frac{P_{n+1}}{P_n}=\rho.$$

In [14], authors defined associated polynomials of Perrin and Cordonnier sequences as

$$Q_n(x)=x^2Q_{n-2}(x)+Q_{n-3}(x)\ \text{ for } n>3 \ \text{ and } Q_1(x)=0,\ Q_2(x)=2,\ Q_3(x)=3x$$

and

$$P_n(x)=x^2P_{n-2}(x)+P_{n-3}(x)\ \text{ for } n>3 \ \text{ and } P_1(x)=1,\ P_2(x)=x,\ P_3(x)=x^2$$

respectively, and studied on these polynomials. In addition, Kaygısız and Bozkurt [2] defined $k$ sequences of generalized order-$k$ Perrrin numbers.

Many researchers studied on determinantal and permanental representations of number sequences. For example, Minc [11] defined a square matrix whose permanent is equal to the generalized order-$k$ Fibonacci numbers. Some of other such papers are [3, 4, 5, 6, 7, 8, 9, 12, 15].

In this paper we give some determinantal and permanental representations of associated polynomials of Perrin and Cordonnier numbers by using various Hessenberg matrices.

# 2 The determinantal representations

An $n\times n$ matrix $A_n=(a_{ij})$ is called lower Hessenberg matrix if $a_{ij}=0$ when $j-i>1$ i.e.,

$$A_n=\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0\\ a_{21} & a_{22} & a_{23} & \cdots & 0\\ a_{31} & a_{32} & a_{33} & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n}\\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n}\end{bmatrix}.$$

**Theorem 2.1** ([1]). *Let $A_n$ be the $n\times n$ lower Hessenberg matrix for all $n\geq 1$ and define $\det(A_0)=1$. Then $\det(A_1)=a_{11}$ and for $n\geq 2$*

$$\det(A_n)=a_{n,n}\det(A_{n-1})+\sum_{r=1}^{n-1}((-1)^{n-r}a_{n,r}\prod_{j=r}^{n-1}a_{j,j+1}\det(A_{r-1})).$$

**Theorem 2.2.** *Let $n\geq 1$ be an integer, $Q_n(x)$ be the associated polynomials of Perrin numbers and $C_n=(c_{rs})$ be an $n\times n$ Hessenberg matrix, where*

$$c_{rs} = \begin{cases} ix^2 & \textit{if } r-s=-1 \\ i & \textit{if } s\neq 1 \textit{ and } r-s=1 \\ \dfrac{-1}{x^4} & \textit{if } s\neq 1 \textit{ and } r-s=2 \\ \dfrac{2i}{x^2} & \textit{if } s=1 \textit{ and } r-s=1 \\ \dfrac{-3x}{x^4} & \textit{if } s=1 \textit{ and } r-s=2 \\ 0 & \textit{otherwise} \end{cases}$$

*i.e.,*

$$C_n = \begin{bmatrix} 0 & ix^2 & 0 & 0 & 0 & 0 \\ \dfrac{2i}{x^2} & 0 & ix^2 & 0 & 0 & 0 \\ \dfrac{-3x}{x^4} & i & 0 & ix^2 & 0 & 0 \\ 0 & \dfrac{-1}{x^4} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & i & 0 & ix^2 \\ 0 & 0 & 0 & \dfrac{-1}{x^4} & i & 0 \end{bmatrix}$$

*Then,* $\det(C_n) = Q_n(x)$, *where* $i = \sqrt{-1}$.

*Proof.* Proof is by mathematical induction on $n$. The result is true for $n = 1$ by hypothesis. Assume that it is true for all positive integers less than or equal to $n$, that is $\det(C_n) = Q_n(x)$. Using Theorem 2.1 we have

$$\begin{aligned} \det(C_{n+1}) &= c_{n+1,n+1}\det(C_n) + \sum_{r=1}^{n}\left((-1)^{n+1-r}c_{n+1,r}\prod_{j=r}^{n}c_{j,j+1}\det(C_{r-1})\right) \\ &= \sum_{r=1}^{n-2}\left((-1)^{n+1-r}c_{n+1,r}\prod_{j=r}^{n}c_{j,j+1}\det(C_{r-1})\right) \\ &+ \sum_{r=n-1}^{n}\left((-1)^{n+1-r}c_{n+1,r}\prod_{j=r}^{n}c_{j,j+1}\det(C_{r-1})\right) \\ &= \sum_{r=n-1}^{n}\left((-1)^{n+1-r}c_{n+1,r}\prod_{j=r}^{n}c_{j,j+1}\det(C_{r-1})\right) \\ &= (-1)^{n+1-n+1}c_{n+1,n-1}\prod_{j=n-1}^{n}c_{s,s+1}\det(C_{n-2}) \\ &+ (-1)^{n+1-n}c_{n+1,n}c_{n,n+1}\det(C_{n-1}) \end{aligned}$$

$$
\begin{aligned}
&= (\frac{-1}{x^4}) ix^2 ix^2 \det(C_{n-2}) + (-1)i.ix^2 \det(C_{n-1}) \\
&= \det(C_{n-2}) + x^2 \det(C_{n-1}).
\end{aligned}
$$

From the hypothesis and the definition of associated polynomials of Perrin numbers, we obtain

$$\det(C_{n+1}) = x^2 Q_{n-1}(x) + Q_{n-2}(x) = Q_{n+1}(x).$$

Therefore, the result is true for all possitive integers. $\square$

**Example 2.3.** *We obtain $Q_5(x)$, by using Theorem 2.2.*

$$\det(C_5) = \det \begin{bmatrix} 0 & ix^2 & 0 & 0 & 0 \\ \frac{2i}{x^2} & 0 & ix^2 & 0 & 0 \\ \frac{-3x}{x^4} & i & 0 & ix^2 & 0 \\ 0 & \frac{-1}{x^4} & i & 0 & ix^2 \\ 0 & 0 & \frac{-1}{x^4} & i & 0 \end{bmatrix} = 3x^3 + 2.$$

**Theorem 2.4.** *Let $n \geq 2$ be an integer, $P_n(x)$ be the associated polynomials of Cordonnier numbers and $P_n = (p_{rs})$ be an $n \times n$ Hessenberg matrix, where*

$$p_{rs} = \begin{cases} ix^2 & \text{if } r - s = -1 \\ i & \text{if } s \neq 1 \text{ and } r - s = 1 \\ \frac{-1}{x^4} & \text{if } s \neq 1 \text{ and } r - s = 2 \\ \frac{i}{x^2} & \text{if } s = 1 \text{ and } r - s = 1 \\ \frac{-1}{x^3} & \text{if } s = 1 \text{ and } r - s = 2 \\ 0 & \text{otherwise.} \end{cases}$$

*Then*

$$\det(P_n) = P_{n-1}(x)$$

*where $i = \sqrt{-1}$.*

*Proof.* Proof is similar to the proof of Theorem 2.2 using Theorem 2.1. $\square$

**Theorem 2.5.** *Let $n \geq 1$ be an integer, $Q_n(x)$ be the associated polynomials of Perrin numbers* *and $B_n = (b_{ij})$ be an $n \times n$ Hessenberg matrix, where*

$$b_{ij} = \begin{cases} -x^2 & \text{if } i-j=-1 \\ 1 & \text{if } j \neq 1 \text{ and } i-j=1 \\ \dfrac{1}{x^4} & \text{if } j \neq 1 \text{ and } i-j=2 \\ \dfrac{2}{x^2} & \text{if } j=1 \text{ and } i-j=1 \\ \dfrac{3x}{x^4} & \text{if } j=1 \text{ and } i-j=2 \\ 0 & \text{otherwise} \end{cases}$$

*i.e.,*

$$B_n = \begin{bmatrix} 0 & -x^2 & 0 & 0 & 0 & 0 \\ \dfrac{2}{x^2} & 0 & -x^2 & 0 & 0 & 0 \\ \dfrac{3x}{x^4} & 1 & 0 & -x^2 & 0 & 0 \\ 0 & \dfrac{1}{x^4} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 0 & -x^2 \\ 0 & 0 & 0 & \dfrac{1}{x^4} & 1 & 0 \end{bmatrix}.$$

*Then*

$$\det(B_n) = Q_n(x).$$

*Proof.* Proof is similar to the proof of Theorem 2.2 using Theorem 2.1. $\square$

**Example 2.6.** *We obtain $Q_6(x)$, by using Theorem 2.5.*

$$\det B_6 = \det \begin{bmatrix} 0 & -x^2 & 0 & 0 & 0 & 0 \\ \dfrac{2}{x^2} & 0 & -x^2 & 0 & 0 & 0 \\ \dfrac{3x}{x^4} & 1 & 0 & -x^2 & 0 & 0 \\ 0 & \dfrac{1}{x^4} & 1 & 0 & -x^2 & 0 \\ 0 & 0 & \dfrac{1}{x^4} & 1 & 0 & -x^2 \\ 0 & 0 & 0 & \dfrac{1}{x^4} & 1 & 0 \end{bmatrix} = 3x + 2x^4.$$

**Theorem 2.7.** *Let $n \geq 2$ be an integer, $P_n(x)$ be the associated polynomials of Cordonnier* *numbers and $S_n = (s_{ij})$ be an $n \times n$ Hessenberg matrix, where*

$$s_{ij} = \begin{cases} -x^2 & \text{if } i-j=-1 \\ 1 & \text{if } j \neq 1 \text{ and } i-j=1 \\ \dfrac{1}{x^4} & \text{if } j \neq 1 \text{ and } i-j=2 \\ \dfrac{1}{x^2} & \text{if } j=1 \text{ and } i-j=1 \\ \dfrac{1}{x^3} & \text{if } j=1 \text{ and } i-j=2 \\ 0 & \text{otherwise.} \end{cases}$$

*Then*

$$\det(S_n) = P_{n-1}(x).$$

*Proof.* Proof is similar to the proof of Theorem 2.2 using Theorem 2.1. □

# 3 The permanent representations

Let $A = (a_{i,j})$ be an $n \times n$ square matrix over a ring R. The permanent of $A$ is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where $S_n$ denotes the symmetric group on $n$ letters.

**Theorem 3.1** ([12])**.** *Let $A_n$ be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and define $per(A_0) = 1$. Then $per(A_1) = a_{11}$ and for $n \geq 2$*

$$per(A_n) = a_{n,n} per(A_{n-1}) + \sum_{r=1}^{n-1} (a_{n,r} \prod_{j=r}^{n-1} a_{j,j+1} per(A_{r-1})).$$

**Theorem 3.2.** *Let $n \geq 2$ be an integer, $Q_n(x)$ be the associated polynomials of Perrin numbers and $H_n = (h_{rs})$ be an $n \times n$ Hessenberg matrix, where*

$$h_{rs} = \begin{cases} -ix^2 & \text{if } r-s=-1 \\ i & \text{if } s \neq 1 \text{ and } r-s=1 \\ \dfrac{-1}{x^4} & \text{if } s \neq 1 \text{ and } r-s=2 \\ \dfrac{2i}{x^2} & \text{if } s=1 \text{ and } r-s=1 \\ \dfrac{-3x}{x^4} & \text{if } s=1 \text{ and } r-s=2 \\ 0 & \text{otherwise} \end{cases}$$

*i.e.,*

$$H_n = \begin{bmatrix} 0 & -ix^2 & 0 & 0 & 0 & 0 \\ \frac{2i}{x^2} & 0 & -ix^2 & 0 & 0 & 0 \\ \frac{-3x}{x^4} & i & 0 & -ix^2 & 0 & 0 \\ 0 & \frac{-1}{x^4} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & i & 0 & -ix^2 \\ 0 & 0 & 0 & \frac{-1}{x^4} & i & 0 \end{bmatrix}.$$

*Then*

$$per(H_n) = Q_n(x)$$

*where* $i = \sqrt{-1}$.

*Proof.* Proof is similar to the proof of Theorem 2.2 by using Theorem 3.1. □

**Example 3.3.** *We obtain* $Q_6(x)$, *by using Theorem 3.2.*

$$perH_6 = per \begin{bmatrix} 0 & -ix^2 & 0 & 0 & 0 & 0 \\ \frac{2i}{x^2} & 0 & -ix^2 & 0 & 0 & 0 \\ \frac{-3x}{x^4} & i & 0 & -ix^2 & 0 & 0 \\ 0 & \frac{-1}{x^4} & i & 0 & -ix^2 & 0 \\ 0 & 0 & \frac{-1}{x^4} & i & 0 & -ix^2 \\ 0 & 0 & 0 & \frac{-1}{x^4} & i & 0 \end{bmatrix} = 3x + 2x^4.$$

**Theorem 3.4.** *Let* $n \geq 2$ *be an integer,* $P_n(x)$ *be the associated polynomials of Cordonnier numbers and* $T_n = (t_{rs})$ *be an* $n \times n$ *Hessenberg matrix, where*

$$t_{rs} = \begin{cases} -ix^2 & \text{if } r-s=-1 \\ i & \text{if } s \neq 1 \text{ and } r-s=1 \\ \frac{-1}{x^4} & \text{if } s \neq 1 \text{ and } r-s=2 \\ \frac{i}{x^2} & \text{if } s=1 \text{ and } r-s=1 \\ \frac{-1}{x^3} & \text{if } s=1 \text{ and } r-s=2 \\ 0 & \text{otherwise.} \end{cases}$$

*Then*

$$per(T_n) = P_{n-1}(x)$$

*where* $i = \sqrt{-1}$.

*Proof.* Proof is similar to the proof of Theorem 2.2 by using Theorem 3.1. $\square$

**Theorem 3.5.** *Let $n \geq 2$ be an integer, $Q_n(x)$ be the associated polynomials of Perrin numbers and $L_n = (l_{ij})$ be an $n \times n$ Hessenberg matrix, where*

$$l_{ij} = \begin{cases} x^2 & \text{if } i - j = -1 \\ 1 & \text{if } j \neq 1 \text{ and } i - j = 1 \\ \dfrac{1}{x^4} & \text{if } j \neq 1 \text{ and } i - j = 2 \\ \dfrac{2}{x^2} & \text{if } j = 1 \text{ and } i - j = 1 \\ \dfrac{3x}{x^4} & \text{if } j = 1 \text{ and } i - j = 2 \\ 0 & \text{otherwise.} \end{cases}$$

*Then*

$$per(L_n) = Q_n(x).$$

*Proof.* Proof of the theorem is similar to the proof of Theorem 2.2 using Theorem 3.1. $\square$

**Theorem 3.6.** *Let $n \geq 2$ be an integer, $P_n(x)$ be the associated polynomials of Cordonnier numbers and $U_n = (u_{ij})$ be an $n \times n$ Hessenberg matrix, where*

$$u_{ij} = \begin{cases} x^2 & \text{if } i - j = -1 \\ 1 & \text{if } j \neq 1 \text{ and } i - j = 1 \\ \dfrac{1}{x^4} & \text{if } j \neq 1 \text{ and } i - j = 2 \\ \dfrac{1}{x^2} & \text{if } j = 1 \text{ and } i - j = 1 \\ \dfrac{1}{x^3} & \text{if } j = 1 \text{ and } i - j = 2 \\ 0 & \text{otherwise.} \end{cases}$$

*Then*

$$per(U_n) = P_{n-1}(x).$$

*Proof.* Proof of the theorem is similar to the proof of Theorem 2.2 using Theorem 3.1. $\square$

**Corollary 3.7.** *If we rewrite Theorem 2.2, Theorem 2.5, Theorem 3.2 and Theorem 3.5 for $x = 1$, we have*

$$\det(C_n) = \det(B_n) = per(H_n) = per(L_n) = Q_n.$$

**Corollary 3.8.** *If we rewrite Theorem 2.4, Theorem 2.7, Theorem 3.4 and Theorem 3.6 for $x = 1$, we have*

$$\det(P_n) = \det(S_n) = per(T_n) = per(U_n) = P_{n-1}.$$

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