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Conductors for sets of large integer squares

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Abstract: We calculate the Frobenius conductor for the infinite set $\{n^2, (n + 1)^2, \ldots\}$ through $n = 200$, demonstrate that the conductor's growth rate as function of n is $o(n^{2+\epsilon})$ for any positive ϵ, and calculate specific numerical bounds for several $\epsilon > 0.0145$.

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It is an elementary result that every positive integer can be expressed as a sum of squares, either by virtue of the Four Square Theorem of Lagrange (see [2] or [3]) or more trivially by recognizing that $1 = 1^2$ can be used as a repeated summand with any multiplicity. But what if we restrict the size of the square summands below by n^2 for some fixed value of n? It is evident that no integer less than n^2 can be expressed as a sum of these large squares, and that the first location where two consecutive integers could possibly both be expressed as sums of these squares must be at least $2n^2$. However, since n^2 and $(n + 1)^2$ are relatively prime, Sylvester's solution of the Frobenius Coin Problem for two coins [5] shows that every integer greater than or equal to $(n^2 - 1)[(n + 1)^2 - 1]$ can be written as a non-negative linear combination of this restricted set of squares.

In this paper, we investigate the location of the minimum threshold value beyond which every integer is expressible as a sum of these large squares, specifically

$$V(n) = \min\{M : m \geq M \Rightarrow m \text{ is expressible as a sum of } n^2, (n + 1)^2, (n + 2)^2, \ldots\}$$

We provide pseudo code for a simple sieve solution for finding this threshold value, and demonstrate that its growth as a function of n is slower than any power of n greater than 2.

Theorem: $V(n) = o(n^{2+\epsilon})$ *for any* $\epsilon > 0$.

For our presentation we will use the "conductor" notation of [4]. For any set S of mutually relatively prime positive integers the conductor $\chi(S)$ is defined by

$$\chi(S) = \min\{M: m \geq M \Rightarrow m \text{ is an element of the semigroup generated by S}\}$$

so that $V(n) = \chi(\{n^2, (n + 1)^2, (n + 2)^2, \ldots\})$. Note that the conductor definition just adds 1 to the traditional "Frobenius number" for the set S considered as a collection of coins, but it makes many important results much less cumbersome to state. First, we prove an extension of a result of Brauer and Shockley [1].

Lemma: *Suppose that the set S of mutually prime positive integers is comprised of two types of positive integer, labeled* $a_1, a_2, \ldots$ *and* $b_1, b_2 \ldots$, *and suppose that the positive integer t divides each of the integers* $a_1, a_2, \ldots$. *Then,*

$$\chi(\{a_1, a_2, \ldots, b_1, b_2, \ldots\}) \le t\,\chi(\{a_1/t, a_2/t, \ldots, b_1, b_2, \ldots\}) + \chi(\{t, b_1, b_2, \ldots\}).$$

Proof: Suppose m is an integer satisfying $m \ge t\,\chi(\{a_1/t, a_2/t, \ldots, b_1, b_2, \ldots\}) + \chi(\{t, b_1, b_2, \ldots\})$, and set $w = m - t\,\chi(\{a_1/t, a_2/t, \ldots, b_1, b_2, \ldots\})$. Then, $w \ge \chi(\{t, b_1, b_2, \ldots\})$, so that we can write w as a finite sum $w = ct + d_1b_1 + d_2b_2 + \ldots$ with $c \ge 0$ and each $d_i \ge 0$. Now, we define x to be the number $x = ct + t\,\chi(\{a_1/t, a_2/t, \ldots, b_1, b_2, \ldots\})$. Since $x/t \ge \chi(\{a_1/t, a_2/t, \ldots, b_1, b_2, \ldots\})$, we can write x/t as a finite sum $x/t = e_1(a_1/t) + e_2(a_2/t) + \ldots + f_1b_1 + f_2b_2 + \ldots$ with each $e_i \ge 0$ and each $f_i \ge 0$. Then, we have

$$m = (w - ct) + t(x/t) = d_1b_1 + d_2b_2 + \ldots + t\,[e_1(a_1/t) + e_2(a_2/t) + \ldots + f_1b_1 + f_2b_2 + \ldots]$$

$$= e_1a_1 + e_2a_2 + \ldots + (d_1 + tf_1)b_1 + (d_2 + tf_2)b_2 + \ldots,$$

and this is a sum of elements of S. □

In particular, if we apply this to a set of large squares with the $a_1, a_2, \ldots$ being the even squares, $b_1, b_2, \ldots$ being the odd squares and $t = 4$, then we obtain the following result.

Corollary: $V(n) \le 4V(\text{ceiling}(n/2)) + 3[(n + 1)^2 - 1]$.

Proof: The numbers $a_1/4, a_2/4, \ldots$ are consecutive squares, and $a_1/4 \le b_1$. So by writing sqrt(a_1)/2 as ceiling(n/2), we have $t\,\chi(\{a_1/t, a_2/t, \ldots, b_1, b_2, \ldots\}) = 4V(\text{ceiling}(n/2))$. The remaining term is found by writing $\chi(\{t, b_1, b_2, \ldots\}) \le \chi(\{t, b_1\}) = (t - 1)(b_1 - 1)$ with b_1 equal to either n^2 or $(n + 1)^2$. □

We can now prove the main result.

Proof of Theorem: Define $\beta = 2 + \epsilon/2$ and choose $N \ge \text{ceiling}[1/(2^{(1-2/\beta)} - 1)]$. Then for $n \ge N$ we have the following chain of consequences

$$1/n \le 2^{(1-2/\beta)} - 1$$

$$(1 + 1/n)/2 \le (1/4)^{1/\beta}$$

$$[(n + 1)/2]^\beta/n^\beta \le 1/4$$

$$n^\beta - 4[(n + 1)/2]^\beta \ge 0$$

Now define $f(n) = 3(2n)^2/[n^\beta - 4[(n+1)/2]^\beta]$. For $n \ge N$ this is a positive and decreasing function of n, since $f'(n) = [12n^2(2 - \beta)[n^{\beta-1} - 2[(n + 1)/2]^{\beta-1}] - 48n[(n + 1)/2]^{\beta-1}]/[n^\beta - 4[(n + 1)/2]^\beta]^2$ with $2 - \beta < 0$, $n^{\beta-1} - 2[(n + 1)/2]^{\beta-1} > 0$, and the denominator $[n^\beta - 4[(n + 1)/2]^\beta]^2 > 0$. Then define $k_1 = f(N)$, $k_2 = \max\{V(n)/n^\beta : n = 1, \ldots, N\}$, and $k = \max\{k_1, k_2\}$. Then, we automatically have $V(n) \le kn^\beta$ for all $n \le N$. We will now establish this bound for all n by induction. Suppose that $n > N$ and that we already have $V(\text{ceiling}(n/2)) \le k(\text{ceiling}(n/2))^\beta$. Then, by the corollary,

$$\begin{aligned} V(n) &\le 4k(\text{ceiling}(n/2))^{\beta} + 3[(n+1)^2 - 1] \\ &\le 4k[(n+1)/2]^{\beta} + 3(n+1)^2 \\ &= kn^{\beta}[2^{2-\beta}(1+1/n)^{\beta} + 3n^{-\beta}(n+1)^2/k] \\ &\le kn^{\beta}[2^{2-\beta}(1+1/n)^{\beta} + 3n^{-\beta}(2n)^2/f(n)] \\ &= kn^{\beta} \end{aligned}$$

This shows that $V(n) = O(n^{\beta}) = o(n^{2+\epsilon})$. ☐

This proof actually provides an explicit construction of a universal bound $V(n) \le kn^{\beta}$ whenever we have already calculated $V(n)$ for all $n = 1, \ldots, N$ with $N \ge \text{ceiling}[1/(2^{(1-2/\beta)} - 1)]$. The required N goes up steeply with decreasing β, as shown below (Table 1).

β	Minimum value of *N*
3	4
2.1	30
2.01	290
2.001	2887
2.0001	28855
2.00001	288541

Table 1.

Using a value of N higher than the listed minimum can dramatically improve the tightness of the bound you can compute, since $k_1 = f(N)$ for the decreasing function f. For the best results you should increase N until the values of k_1 and k_2 are nearly equal, if possible.

We have calculated $V(n)$ for $n = 1, \ldots, 200$, using a simple sieve program based on the pseudo code

```
max_coin_root=10
max_index=max_coin_root^2
FOR n = 3 TO 200
   flag = TRUE
   WHILE flag DO
      FOR index = 1 TO max_index
         a[index] = FALSE
      NEXT (index)
      FOR coin_root = n TO max_coin_root
         a[coin_root^2] = TRUE
         FOR index = coin_root^2+1 TO max_index
            a[index]=a[index]||a[index – coin_root^2]
         NEXT (index)
      NEXT (coin_root)
      FOR index = max_index TO 1 STEP -1
         IF not(a[index]) THEN DO
            IF index+n^2 > max_index THEN DO
               max_index = 2*max_index
               max_coin_root = FLOOR(max_index^.5)
```

```
                  BREAK (back to WHILE)
               ELSE DO
                  PRINT n, index+1
                  max_index = CEILING(1.1*(1+1/n)^2*index+(n+1)^2)
                  max_coin_root=FLOOR(max_index^.5)
                  flag = FALSE
                  BREAK (back to WHILE)
               ENDIF
            ENDIF
         NEXT (index)
      ENDWHILE
   NEXT (n)
```

Implementing this program on Mathematica provided the first 50 values in 5 minutes, the first 100 values in about an hour, and all 200 values overnight. The main idea is to dynamically allocate only enough space to complete the required computations for each "n" as we get to it. For each "n" all the needed positions of the vector "a" are initially set to the Boolean value FALSE to show that no value of "index" has yet been expressed as a coin sum. The expression "coin_root^2" denotes the value of a coin being used. This coin guarantees that "index" has a coin sum expression when "index" equals the value "coin_root^2", and also whenever "index-coin_root^2" already has a coin sum expression. When we have used all of our coins, we count backwards from "max_index" until we find a location where "a[index]" is still FALSE. Hopefully, this value of "index" will be the largest integer that cannot be expressed as a coin sum. But we need to perform a check before we report "index+1" as the conductor for the set. We need to verify that there is a string of n^2 consecutive TRUE values following the FALSE, so that there cannot be a higher, out-of-range value of "index" which also lacks a coin sum expression. (Since the n^2 coin is in our set, the presence of n^2 consecutive TRUE values will guarantee that all the out-of-range values must also be TRUE.) If we do not find enough TRUE values, then we double the value of "max_index" and repeat the analysis for that value of "n".

If we find the required quantity of TRUE values, then we report "index+1" as the value of $V(n)$, and allocate space for the analysis of $V(n + 1)$. The formula

```
max_index = CEILING(1.1*(1+1/n)^2*index+(n+1)^2)
```

is based on the fact that experimentally the fraction $V(n)/n^2$ has only been observed to increase by a multiplicative factor of 1.1 or less when "n" is incremented by 1, except for the two starting cases where the new "n" value is 2 or 3. The "(n+1)^2" on the end is there to provide room for the required TRUE values. However, because the incremental growth of $V(n)/n^2$ could at some point change in an unanticipated manner, it seems prudent to keep the test for the needed set of consecutive TRUE values in the program. Notice that whenever the value of "max_index" is changed, the value of "max_coin_root" is also changed so that all coins that could possibly have impact are considered in the next analysis.

Here is an abbreviated table of $V(n)$ values, with an extra column which supports our unproven conjecture that $V(n) = O(n^2)$.

n	V(n)	V(n)/n^2
1	1	1
2	24	6
3	88	9.77778
4	120	7.5
5	202	8.08
6	313	8.69444
7	377	7.69388
8	456	7.125
9	617	7.61728
10	761	7.61
20	2765	6.9125
30	5524	6.13778
40	9857	6.16063
50	15233	6.0932
60	21409	5.94694
70	28193	5.75367
80	36449	5.69516
90	49049	6.05543
100	57409	5.7409
125	84993	5.43956
150	122881	5.46138
175	164865	5.38335
200	215041	5.37603

Table 2.

Using these tabulated values, we are now ready to express specific formulas for bounds on the growth rate of $V(n)$. For each β we use $N = 200$ to calculate the values of k_1 and k_2 from the proof of the theorem. The bound is $V(n) \le kn^{\beta}$ with $k = \max(k_1, k_2)$. For the rows where $k_1 > k_2$ these bounds might be substantially tightened by tabulating more values $V(n)$ in order to use a larger value for N.

β	k_1	k_2
2.9	0.2234	3.6377
2.8	0.4146	4.0602
2.7	0.7819	4.5316
2.6	1.5063	5.0579
2.5	2.9876	5.6452
2.4	6.1856	6.3007
2.3	13.725	7.0324
2.2	34.702	7.8490
2.1	123.63	8.7605
2.05	381.46	9.2552
2.04	558.00	9.3574
2.03	964.52	9.4608
2.02	2854.7	9.5653
2.015	31917	9.6180
2.0145	3438587	9.6233

Table 3.

References

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