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THE INTEGER STRUCTURE OF THE DIFFERENCE OF TWO ODD INTEGERS RAISED TO AN EVEN POWER

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Abstract: Using the modular ring Z_4, it is shown that the row structures of $x^n - y^n$, x, y odd, $n = 2^m$, are incompatible with the row structures of z^n. Even though some structures are close, the right-end-digits (REDs) are quite distinct. The analysis shows how the effort to find counter-examples for such theorems may be drastically reduced.

Keywords: primes, composites, modular rings, right-end digits, integer structure.

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1 Introduction

We have recently shown that the integer structure of $x^n - y^n$, x, y, n odd is quite different from z^n (z even). This is to be expected from Fermat's Last Theorem. That is, the integer produced from $x^n - y^n$ can never fit the structure of z^n, and therefore can never equal z^n [3]. Here we examine the structure of $x^n - y^n$, x, y odd, n even.

If x^n, y^n are of the form $(x^m)^t$ with $mt = n$ and t odd, then the results will be the same as for $x^n - y^n$, n odd. For example,

$$\begin{aligned}(x^2)^3 - (y^2)^3 &= (x^2 - y^2)(x^4 + y^4 + x^2y^2) &(1.1)\\ &= (x-y)(x^5 + y^5 + xy(x^3 + y^3 + xy(x+y))) &(1.2)\end{aligned}$$

which is consistent with the general form:

$$x^n - y^n = (x-y)\left(x^{n-1} + y^{n-1} + \frac{xy}{x-y}\left(x^{n-2} - y^{n-2}\right)\right). \quad (1.3)$$

We therefore confine our analysis to $n = 2^m, m$ odd or even. In fact the analysis of $x^4 - y^4$ will suffice to indicate the structure of these types of systems.

2 Structure of $x^4 - y^4$, x, y odd

We have previously analysed a related system where x is odd and y even [1,2]. However, the focus there was on the structure of the factors and the production of primes. The general equation (1.3) yields [1]

$$\begin{aligned} x^4 - y^4 &= (x-y)\left(x^3 + y^3 + xy(x+y)\right) \\ &= (x-y)f(x,y). \end{aligned} \tag{2.1}$$

Table 1 illustrates possible class structure for x,y and the subsequent structures of the various components of Equation (2.1).

No.	x	y	x-y	x^3+y^3	x+y	**x**$\mathbf{x}$**y**	$f(x,y)$	x^4-y^4
1	$\bar{1}_4$	$\bar{1}_4$	$\bar{0}_4$	$\bar{2}_4$	$\bar{2}_4$	$\bar{1}_4$	$\bar{0}_4$	$\bar{0}_4(4r_o)(4R_0)$
2	$\bar{1}_4$	$\bar{3}_4$	$\bar{2}_4$	$\bar{0}_4$	$\bar{0}_4$	$\bar{3}_4$	$\bar{0}_4$	$\bar{0}_4(4r_2+2)(4R_0)$
3	$\bar{3}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{0}_4$	$\bar{0}_4$	$\bar{3}_4$	$\bar{0}_4$	$\bar{0}_4(4r_2+2)(4R_0)$
4	$\bar{3}_4$	$\bar{3}_4$	$\bar{0}_4$	$\bar{2}_4$	$\bar{2}_4$	$\bar{1}_4$	$\bar{0}_4$	$\bar{0}_4(4r_o)(4R_0)$

Table 1

For Numbers 1 and 4, the class $\bar{0}_4$ dominates. Consider Number 1. Here

$$x = 4r_1 + 1,$$

and

$$y = 4r_1' + 1,$$

so that

$$\begin{aligned} (x-y) &= 4(r_1 - r_1') \\ &= 4r_0 \end{aligned} \tag{2.2}$$

and

$$f(x) = 4\left(R_1' + R_1''\right) + 2 + \left(4\left(4r_1'r_1 + \left(r_1 + r_1'\right)\right) + 1\right)\left(4\left(r_1 + r_1'\right) + 2\right) \tag{2.3}$$

and with

$$\begin{aligned} R_1' + R_1'' &= r_2 \\ r_1 + r_1' &= r_2' \end{aligned}$$

and

$$4r_1'r_1 + \left(r_1 + r_1'\right) = R_1$$

so that

$$(x-y)f(x,y) = x^4 - y^4 = 4r_0 4\left(r_2 + 4R_1r_2' + r_2' + 2R_1 + 1\right). \tag{2.4}$$

Thus the row of $x^4 - y^4$ is

$$\text{row} = 4r_0\left(r_2 + 4R_1r_2' + r_2' + 2R_1 + 1\right) \tag{2.5}$$

which always falls in $\overline{0}_4$. The row of the row is

$$\text{row of row} \quad = r_0\left(r_2 + 4R_1 r_2^{|} + r_2^{|} + 2R_1 + 1\right) \tag{2.6}$$

which will depend on the parity of the elements.

On the other hand though, if

$$z \quad = 4R_0,$$

then

$$z^4 \quad = 4^4 R_0^4,$$

so that the row structure will be of the form $\overline{0}_4\ \overline{0}_4\ \overline{0}_4\ \overline{0}_4 \ldots$. Thus, for $x^4 - y^4$ to match the row structure of z^4, $r_0 r_2, r_2^{|}$ and R_1 need to be very restricted.

With

$$z \quad = 4R_2 + 2,$$

then

$$z^4 \quad = 4\left(4^3 R_2^4 + 2x4^3 R_2{}^3 + 6x4^2 R_2^2 + 32R_2 + 4\right), \tag{2.7}$$

so that

$$\text{row} \quad = 4\left(4^2 R_2^4 + 2x4^2 R_2{}^3 + 6x4R_2^2 + 8R_2 + 1\right),$$

and the row of the row is in class $\overline{1}_4$, but the row of the row of the row is in class $\overline{0}_4$ again so that the row structure will be $\overline{0}_4\ \overline{1}_4\ \overline{0}_4 \ldots$. Table 2 illustrates the typical row structure for $x^4 - y^4$. This structure can then be compared with that of z^4 in Table 3. Obviously, only a limited number of $x^4 - y^4$ row structures resemble those of z^4. As expected, z^4 generally has a row structure $\overline{0}_4\ \overline{0}_4\ \overline{0}_4 \ldots$ or $\overline{0}_4\ \overline{1}_4\ \overline{0}_4$.

The row structures of $x^4 - y^4$ of the form $\overline{0}_4\ \overline{2}_4\ \ldots$ or $\overline{0}_4\ \overline{3}_4\ \ldots$ or $\overline{0}_4\ \overline{1}_4\ \overline{1}_4\ \ldots$ or $\overline{0}_4\ \overline{1}_4\ \overline{2}_4\ \ldots$ or $\overline{0}_4\ \overline{1}_4\ \overline{3}_4\ \ldots$ are incompatible with the structure of z^4, and so the *x*,*y* values which produce these structures can be eliminated. *x*,*y* which yield $\overline{0}_4\ \overline{0}_4\ \overline{1}_4 \ldots$ may also be eliminated as only 12^4 gives this structure.

001001	(6)	01013001	(8)	0200322	(7)	0302121	(7)
00111332	(8)	01013331	(8)	0201321	(7)	030311	(6)
00121011	(8)	011121011	(9)	021213	(6)	03110321	(8)
0020002	(7)	0113003	(7)	02131332	(8)	0312	(4)
00213021	(8)	0113130	(7)	0222233	(7)	03130231	(8)
00223322	(8)	0120302	(7)	02222331	(8)	0322113	(7)
00302	(5)	012331	(6)	023023	(6)	03230331	(8)
00303331	(8)	0123321	(7)	02303201	(8)	0323132	(7)
0031103	(7)	0132132	(7)	02303301	(8)	03310302	(8)
003213	(6)	01330011	(8)	023300031	(9)	03311011	(8)
0100322031	(10)	01013001	(8)	030123	(6)	0331111	(7)
01012	(5)	02002022	(8)	0302111	(7)	03321	(5)

Table 2: Typical row structure for $x^4 - y^4$: bracketed no's correspond with the number of rows

000000000001	(12)	000100121011	(12)	0100111312301	(13)	01020110101	(11)
0000000001	(10)	00010013032222	(14)	0100122011	(10)	010202310322	(12)
0000000001011	(13)	00010031103	(11)	010020210201	(12)	0102112	(7)
00000001	(8)	000101210011	(12)	01002311023	(11)	01022121	(8)
00000001011	(11)	0001012332	(10)	010031103	(9)	010230301302	(12)
000000010312	(12)	0001022121	(10)	0100321333111	(13)	010231133111	(12)
000001	(6)	0001023113311	(13)	0101013333	(10)	01030123	(8)
000001011	(9)	0001030123	(10)	010102012233	(12)	010312	(6)
00000102112	(11)	00010312	(8)	01011	(5)	010332220013	(12)
00001030123	(12)	000103231332	(12)	010123321	(9)	0103231332	(10)
0000010312	(10)	00023222	(8)	01013112321	(11)	010330212031	(12)
0001	(4)	001011	(6)	01013221322	(11)	01033321002	(11)

Table 3: Typical row structure for z^4

The few similar row structures for $x^4 - y^4$ and z^4 are shown in Table 4 together with the corresponding right-end-digit (RED) structures. The length of the row pattern must be the same so there is no real match, but it is of interest that the RED structure is sometimes close. For example, No.4 of $x^4 - y^4$ compared with No.3 of z^4. Of course, we did not expect any matches, but before this had been established much time and effort was spent on the search for counter-examples. Such effort could have been greatly reduced with the use of integer structure theory.

Number	Rows of $x^4 - y^4$	REDs of $x^4 - y^4$	Rows of z^4	REDs of z^4
1	0100322031	0564584871	0100321333111	4100589717151
2	01012	0592	010123321	410561761
3	01013001	05697641	010131123211	056971561251
4	01013331	05697171	01013221322	41051203302

Table 4: Some row structures for $x^4 - y^4$ and z^4

References

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