NNTDM 12 (2006), 1, 13-20

SOME *q*-BINOMIAL COEFFICIENTS

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Abstract

This paper considers some q-extensions of binomial coefficients. Some of the results are applied to some generalized Fibonacci numbers, and others are included as ideas for further investigation, particularly into q-Bernoulli polynomials.

1. Introduction

Leonard Carlitz generalized many results by considering q-series as analogues of factorials, and from them he constructed a q-series analogue of the ordinary binomial coefficient. Variations of these had been studied earlier by Gauss and Cayley (see Macmahon [18] and Gordon and Houten [10,11]). More recently, Blumen [2] has generalized the q-binomial theorem with non-commuting quantities in quantum algebras and quantum superalgebras.

The form of these analogues suggests a similar analogue of the binomial coefficient formed from rising factorials. With the development of the properties of the rising factorial it is shown in this paper that the rising binomial coefficient is in fact a generalization of the ordinary binomial coefficient. Elsewhere the present author, with Richard Ollerton, has extended other ways of generalizing binomial coefficients [19].

2. *q*-series

Carlitz has used q-series in numerous papers; for example [3,4,5,6,8,9]. Recently, T. Kim and his colleagues have extended some elegant results in both analytic and elementary number theory with such series in a sequence of papers [15-17]. They are defined basically by

$$(q)_n = (1-q)(1-q^2)\ldots(1-q^n), \tag{2.1}$$

with $(q)_0 = 1$. Arising out of these are the so-called q-binomial coefficients which are analogous to ordinary binomial coefficients. Their simplest definition is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}} \tag{2.2}$$
$$= \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-k+1})}{(1-q)(1-q^2)\ldots(1-q^k)}.$$

Carlitz [7] and Horadam[13] have used them in the form that follows in their papers on generating functions for powers of elements of second order recursive sequences. If we let

$$q = \frac{\beta}{\alpha}$$

in the above definition, where α, β are the roots of

$$x^2 - px + q = 0,$$

then we find that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{\left(1-\left(\beta/\alpha\right)^n\right)\ldots\left(1-\left(\beta/\alpha\right)^{n-k+1}\right)}{\left(1-\left(\beta/\alpha\right)\right)\left(1-\left(\beta/\alpha\right)^2\right)\ldots\left(1-\left(\beta/\alpha\right)^k\right)}$$
$$= \alpha^{k(n-k)} \frac{U_n U_{n-1} \ldots U_{n-k+1}}{U_1 U_2 \ldots U_k}$$
$$= \alpha^{k(n-k)} \begin{Bmatrix} n \\ k \end{Bmatrix}$$
$$= U_n C_{n,k} \alpha^{k(n-k)}$$

in which the sequence $\{U_n\}$is generated by the homogeneous second order linear recurrence relation

$$U_n = pU_{n-1} - qU_{n-2},$$

and

$$C_{n,k} = \frac{U_{n-1} \ldots U_{n-k+1}}{U_1 U_2 \ldots U_k}. \tag{2.3}$$

The significance of the $C_{n,k}$ can be seen in some papers by Hoggatt [12] in which he developed properties for ordinary Fibonacci numbers where the $\begin{bmatrix} n \\ k \end{bmatrix}$ were called Fibonomial coefficients. The properties of these were extended in Jerbic's thesis [14] under the supervision of the late Verner Hoggatt. A preliminary result is

$$\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \frac{2-q^k-q^{n-k}}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}. \tag{2.4}$$

Proof:

$$\begin{aligned} \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} &= \frac{(1-q^{n-1})(1-q^{n-2})\ldots(1-q^{n-k})}{(1-q)(1-q^2)\ldots(1-q^k)} + \frac{(1-q^{n-1})\ldots(1-q^{n-k+1})}{(1-q)\ldots(1-q^{k-1})} \\ &= \frac{(1-q^{n-1})(1-q^{n-2})\ldots(1-q^{n-k+1})}{(1-q)(1-q^2)\ldots(1-q^{k-1})} \left\{ \frac{1-q^{n-k}}{1-q^k} + 1 \right\}. \end{aligned}$$

■

Equation (2.4) is a variation of the well-known identities

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

and (Andrews [1])

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Equation (2.4) gives rise to two new results for generalized Fibonacci numbers. The first of these is

$$\left(\frac{2-q^k-q^{n-k}}{1-q^n} \right) \alpha^n U_n = \alpha^{n-k} U_{n-k} + \alpha^k U_k. \tag{2.5}$$

Proof:

$$\begin{aligned} \begin{bmatrix} n-1 \\ k \end{bmatrix} &= \alpha^{k(n-k-1)} U_{n-k} \frac{U_{n-1} \ldots U_{n-k+1}}{U_1 U_2 \ldots U_k} \\ &= U_{n-k} C_{n,k} \alpha^{k(n-k-1)} \\ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} &= \alpha^{(k-1)(n-k)} U_k \frac{U_{n-1} \ldots U_{n-k+1}}{U_1 U_2 \ldots U_{k-1} U_k} \\ &= U_k C_{n,k} \alpha^{(k-1)(n-k)}, \end{aligned}$$

and the result follows after induction.

The second of the two results is

$$\begin{bmatrix} n \\ m \end{bmatrix} = U_{m+1}\alpha^{m}\begin{bmatrix} n-1 \\ m \end{bmatrix} - qU_{n-m+1}\alpha^{n-m}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$

Proof:
It can be readily shown that

$$\begin{aligned} U_n &= U_{m+1}U_{n-m} - qU_m U_{n-m-1} \\ U_n C_{n,k}\alpha^{m(n-m)} &= U_{m+1}\left(U_{n-m}C_{n,k}\alpha^{m(n-m-1)}\right)\alpha^{m} - qU_{n-m-1}\left(U_m C_{n,k}\alpha^{(m-1)(n-m)}\right)\alpha^{n-m} \end{aligned}$$

and the required result comes from the use of the definition of $C_{n,k}$. This result can be compared with Equation (F) of Hoggatt [12].

3. Rising Factorials

The falling factorial, an *r*-permutation of *n* distinct objects, is given by

$$n^{\underline{r}} = P(n,r)$$

and is such that

$$\begin{aligned} \nabla P(n,r) &= P(n,r) - P(n-1,r) \\ &= rP(n-1,r-1), \end{aligned}$$

(see, for example Riordan [20]). Similarly, we can show for the rising factorial of *n* that

$$\begin{aligned} \nabla n^{\overline{r}} &= n^{\overline{r}} - (n-1)^{\overline{r}} \\ &= rn^{\overline{r-1}}. \end{aligned} \tag{3.1}$$

Proof:

$$\begin{aligned} (n-1)^{\overline{r}} + rn^{\overline{r-1}} &= (n-1)n(n+1)...(n+r-2) + rn(n+1)...(n+r-2) \\ &= n(n+1)...(n+r-2)(n+r-1) \\ &= n^{\overline{r}}. \end{aligned}$$

■

This is a recurrence relation for $n^{\overline{r}}$, which is an *r* permutation of *n*+*r*-1 objects, and which is related to the Stirling numbers. Now, Carlitz [6] considers an equation of the form

$$f_n(x+1) - f_n(x) = nf_{n-1}(x), \quad (x \geq 1)$$

as an extension of the criterion for an Appell set. Thus (3.1) may be considered as extension of this criterion.

It is well known that

$$\begin{aligned} \binom{n}{r} &= \frac{n(n-1)...(n-r+1)}{r(r-1)...(r-r+1)} \\ &= \frac{n^{\underline{r}}}{r^{\underline{r}}} \end{aligned}$$

in which $n^{\underline{r}}$ is the falling r-factorial of n. Consider

$$C(n,r) = \frac{n^{\overline{r}}}{r^{\overline{r}}} \tag{3.2}$$

in which $n^{\overline{r}}$ is the rising r-factorial of n. Thus

$$C(n,r) = \frac{n(n+1)...(n+r-1)}{r(r+1)...(r+r-1)}, \tag{3.3}$$

which is also suggested by the Gauss-Cayley form of the generalized binomial coefficient. This leads to

$$nC(n-r,r) = (n-r)C(n-r+1,r). \tag{3.4}$$

Proof:

$$\begin{aligned} \frac{n}{n-r}C(n-r,r) &= \frac{n(n-r)(n-r+1)...(n-1)}{(n-r)r(r+1)...(2r-1)} \\ &= \frac{(n-r+1)...(n-1)n}{r(r+1)...(2r-1)} \\ &= C(n-r+1,r). \end{aligned}$$

■

This can also be put in the form

$$\begin{aligned} \nabla nC(n-r+1,r) &= nC(n-r+1,r) - nC(n-r,r) \\ &= nC(n-r+1,r) - (n-r)C(n-r+1,r) \\ &= rC(n-r+1,r). \end{aligned}$$

Another line of approach is to define

$$C(n, j; r) = \frac{n^{\overline{r}}}{j^{\overline{r}}}. \tag{3.5}$$

Then

$$C(n, r; r) = C(n, r)$$

and

$$C(-n, -r; r) = \binom{n}{r}.$$

Moreover,

$$\begin{aligned} \nabla C(n, r; r) &= C(n, r; r) - C(n-1, r; r) \\ &= C(n, r+1; r-1). \end{aligned} \tag{3.6}$$

Proof:

$$\begin{aligned} C(n-1, r; r) + C(n, r+1; r-1) &= \frac{(n-1)^{\overline{r}}}{r^{\overline{r}}} + \frac{n^{\overline{r-1}}}{(r+1)^{\overline{r-1}}} \\ &= \frac{(n-1)n...(n+r-2)}{r(r+1)...(2r-1)} + \frac{n(n+1)...(n+r-2)}{(r+1)(r+2)...(2r-1)} \\ &= \frac{n(n+1)...(n+r-2)(n-1+r)}{r(r+1)...(2r-1)} \\ &= \frac{n^{\overline{r}}}{r^{\overline{r}}}. \end{aligned}$$

■

We note in passing that Riordan [20] has stated that the falling factorial occupies a central position in the finite difference calculus because

$$\nabla x^{\underline{n}} = n x^{\underline{n-1}}$$

which is analogous to the result which we have already proved in (3.1), namely

$$\nabla x^{\overline{n}} = n x^{\overline{n-1}}.$$

4. Concluding Comments

We can also define a rising factorial analogue of the exponential function as follows: Let

$$e(n,r,x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^{\bar{r}}}. \tag{4.1}$$

Then

$$\begin{aligned} e(n,r,x) &= \sum_{n=0}^{\infty} x^n \prod_{s=1}^{r} (n+s)^{-1} \\ &= \prod_{s=1}^{r} \sum_{n=0}^{\infty} \frac{x^n}{n+s}. \end{aligned}$$

In particular,

$$\begin{aligned} e(n,r,1) &= \prod_{s=1}^{r} \sum_{n=0}^{\infty} (s+n)^{-1} \\ &= \prod_{s=1}^{r} \zeta(1,s) \end{aligned}$$

where $\zeta(j,s)$ represents the generalized Zeta function defined by

$$\zeta(j,s) = \sum_{n=0}^{\infty} (s+n)^{-j}.$$

These and the results in Section 3 can be used to develop q-Bernoulli polynomials.

References

1. G.E. Andrews. A Polynomial Identity Which Implies the Rogers-Ramanujan Identities. *Scripta Mathematica*. **28** (1970): 297-305.

2. Sacha C. Blumen, Two Generalizations of the Binomial Theorem, *The Australian Mathematical Society Gazette*, **33** (2006): 39-43.

3. L. Carlitz. q-Bernoulli Numbers and Polynomials. *Duke Mathematical Journal*. **15** (1948): 987-1000.

4. L. Carlitz. Expansions of q-Bernoulli Numbers. *Duke Mathematical Journal*. **25** (1958): 355-364.

5. L. Carlitz. Some Integral Equations satisfied by the Complete Elliptic Integrals of the First and Second Kind. *Bolletino della Unione Matematica Italiana*. (**3**) 16 (1961): 264-268.

6. L. Carlitz. Characterization of Certain Sequences of Orthogonal Polynomials. *Portugaliae Mathematica*. **20** (1961): 43-46.

7. L. Carlitz. Generating Functions for Powers of Certain Sequences of Numbers. *Duke Mathematical Journal*. **29** (1962): 521-537.

8. L. Carlitz. A q-identity. *Monatshefte für Mathematik*. **67** (1963): 305-310.
9. L. Carlitz. A Note on Products of Sequences. *Bolletino della Unione Matematica Italiana*. (**4**) 1 (1968): 362-365.
10. B. Gordon and L. Houten. Notes on Plane Partitions. I. *Journal of Combinatorial Theory*. **4** (1968): 72-80.
11. B. Gordon and L. Houten. Notes on Plane Partitions. II. *Journal of Combinatorial Theory*. **4** (1968): 81-99.
12. V.E. Hoggatt, Jr. Fibonacci Numbers and Generalized Binomial Coefficients. *The Fibonacci Quarterly*. **5** (1967): 383-400.
13. A.F. Horadam. Generating Functions for Powers of a Certain Generalized Sequence of Numbers. *Duke Mathematical Journal*. **32** (1965): 437-446.
14. S.K. Jerbic. *Fibonomial Coefficients – A Few Summation Properties.* Master's Thesis, San José State College, California, 1968.
15. T. Kim, Some Formulae for the q-Bernoulli and Euler Polynomials. *Journal of Mathematical Analysis and Applications*. **273** (2002): 236-242.
16. T. Kim. Analytic Continuation of Multiple q-zeta Functions and their Values at Negative Integers. *Russian Journal of Mathematics & Physics*. **11** (2004): 71-76.
17. T. Kim and S.H. Rim. On Changed q-Euler Numbers and Polynomials. *Advanced Studies in Contemporary Mathematics*. **9** (2004): 81-86.
18. P.A. Macmahon. *Combinatory Analysis.* Cambridge: Cambridge University Press, 1916.
19. R.L. Ollerton and A.G. Shannon. Extensions of Generalized Binomial Coefficients. In F. Howard (ed.), *Applications of Fibonacci Numbers, Volume 9.* Dordrecht: Kluwer, 2004, pp.187-199.
20. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: Wiley, 1958, p.3.

AMS Classification Numbers: 11B65, 11B39, 05A30