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THE AUXILIARY EQUATION ASSOCIATED WITH THE PLASTIC NUMBER

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ABSTRACT

This paper looks at some of the properties of the auxiliary equation associated with the plastic number which, in turn, is related to the sequences of numbers $\{P_n\}$, $\{Q_n\}$ and $\{R_n\}$ respectively, defined by

$$\begin{aligned} P_n &= P_{n-2} + P_{n-3}, n > 3, \; P_1 = 1, P_2 = 1, P_3 = 1, \\ Q_n &= Q_{n-2} + Q_{n-3}, n > 3, \; Q_1 = 0, Q_2 = 2, Q_3 = 3, \\ R_n &= R_{n-2} + R_{n-3}, n > 3, \; R_1 = 1, R_2 = 0, R_3 = 1. \end{aligned}$$

The dominant root of the associated auxiliary equation is found by a contraction process related to Bernoulli's iteration and the Jacobi-Perron Algorithm. The latter is one way of generalizing the ordinary continued fraction algorithm and an alternative way is explored which also relates to the auxiliary equations of the sequences. Various methods for reduction of the order of the cubic auxiliary equation are also considered.

1. INTRODUCTION

It is the aim of this paper to explore some of the properties of the auxiliary equation (in Equation (2.1) below) and its dominant root, the plastic number, for the sequences of Cordonnier$\{P_n\}$, Perrin$\{Q_n\}$and van der Laan numbers $\{R_n\}$, respectively, defined by third-order homogeneous recurrence relations in (1.1), (1.2) and (1.3) respectively. The historical background of these sequences is outlined in [23] and will not be repeated here in detail. Suffice it to say that Equation (2.1) has distinct roots $\alpha(= p), \beta, \overline{\beta}$, in which $p \approx 1.324718$ (the so-called *plastic number*) is the unique real solution and dominant root of (1.1). Incidentally, the rather unattractive name "plastic number" (*plastische getal*) is due to van der Laan, not by allusion to synthetic materials, but in the continental sense of 'plastic' = 'three-dimensional, palpable'.

The plastic number corresponds to the *golden number* $\varphi \approx 1.618034$ associated with the equiangular spiral related to the conjoined squares in Fibonacci numbers, for example,

$$\lim_{n\to\infty} \tfrac{P_{n+1}}{P_n} = p\left(= \lim_{n\to\infty} \tfrac{Q_{n+1}}{Q_n}\right).$$

Some essential definitions are listed here for ease of reference. Numbers P_n and Q_n, along with the linking numbers R_n, are defined by recurrence relations or by generating functions in Equations (1.1) to (1.6). The first few numbers are displayed in Table 1.

Recurrence Relations

$$P_n = P_{n-2} + P_{n-3}, n > 3, \ \ P_1 = 1, P_2 = 1, P_3 = 1, \tag{1.1}$$

$$Q_n = Q_{n-2} + Q_{n-3}, n > 3, \ \ Q_1 = 0, Q_2 = 2, Q_3 = 3, \tag{1.2}$$

$$R_n = R_{n-2} + R_{n-3}, n > 3, \ \ R_n = 1, R_2 = 0, R_3 = 1. \tag{1.3}$$

Generating Functions

$$\sum_{n=1}^{\infty} P_n y^{n-1} = (1+y)\left[1-\left(y^2+y^3\right)\right]^{-1}, \tag{1.4}$$

$$\sum_{n=1}^{\infty} Q_{n+1} y^{n-1} = (2+3y)\left[1-\left(y^2+y^3\right)\right]^{-1}, \tag{1.5}$$

$$\sum_{n=1}^{\infty} R_n y^{n-1} = \left[1-\left(y^2+y^3\right)\right]^{-1}. \tag{1.6}$$

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P_n	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37
Q_n	0	2	3	2	5	5	7	10	12	17	22	29	39	51	68
R_n	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21

Table 1: First few values of $P_n\, Q_n$ and R_n

2. AUXILIARY EQUATIONS

The auxiliary equation for these numbers can be written as

$$x^3 = x + 1, \tag{2.1}$$

so that, in turn,

$$
\begin{aligned}
x^4 &= x^2 + x,\\
x^5 &= x^3 + x^2 = x^2 + x + 1,\\
x^6 &= x^3 + x^2 + x = x^2 + 2x + 1.
\end{aligned}
$$

To speed up the process, Gnanadoss [7] suggested a contraction process as follows:

$$x^6 = \left(x^3\right)^2 = x^2 + 2x + 1,$$

and so on, until

$$x^{48} = 170625x^2 + 226030x + 128801$$

and

$$x^{49} = 226030x^2 + 299426x + 170625,$$

so that the dominant root of the auxiliary equation is

$$
\begin{aligned}
\alpha &= \frac{x^{49}}{x^{48}}\\
&\approx \begin{cases} 1.324717957 & when \quad x = 1,\\ 1.324717955 & when \quad x = 2,\\ 1.324717973 & when \quad x = 1.3. \end{cases}
\end{aligned}
$$

which Gnanadoss compared with Hildebrand [10] for the same polynomial.
More generally, we can further speed up the contraction with the following result.
If

$$x^3 = x + 1,$$

then

$$x^{n+3} = \sum_{j=0}^{2} Q_{n,j} x^j, \tag{2.2}$$

where

$$\begin{bmatrix} Q_{n,1} \\ Q_{n,2} \end{bmatrix} = \begin{bmatrix} Q_{n-1,0} \\ Q_{n-1,1} \end{bmatrix} + \begin{bmatrix} A_2 \\ A_1 \end{bmatrix} Q_{n-1,2} \tag{2.3}$$

with

$$
\begin{aligned}
A_2 &= 1, & A_1 &= 0,\\
Q_{n,0} &= 1, & Q_{0,m} &= A_{3-m},\\
Q_{n,m} &= 0 \quad for & m &\geq 3, m, n < 0.
\end{aligned}
$$

The result is similar to one in Henrici [9] and it can be generalized to any order [24]. For notational convenience, let

$$x^3 = \sum_{j=0}^{2} A_{3-j} x^j, \qquad A_3 = 1.$$

Proof of (2.2): We use induction on *n*. Now, for *n*=0,

$$\begin{aligned}\sum_{j=0}^{2} Q_{0,j} x^j &= \sum_{j=0}^{2} A_{3-j} x^j \\ &= A_3 + A_2 x + 0 \\ &= 1 + x \\ &= x^3,\end{aligned}$$

in which we have used the given values of A_1, A_2, A_3. Therefore, (2.2) is true for *n*=0. Assume the result is true for *n*=1,2,3,…,*s*. That is,

$$x^{s+3} = \sum_{j=0}^{2} Q_{s,j} x^j.$$

Then

$$\begin{aligned}x^{s+4} &= \sum_{j=0}^{2} Q_{s,j} x^{j+1} \\ &= Q_{s,2} x^3 + \sum_{j=0}^{1} Q_{s,j} x^{j+1} \\ &= Q_{s,2} x^3 + \sum_{j=0}^{2} Q_{s,j-1} x^j \\ &= Q_{s,2} x^3 + \sum_{j=0}^{2} Q_{s+1,j} x^j - \sum_{j=0}^{2} A_{3-j} Q_{s,2} x^j \quad by\,(6.3) \\ &= Q_{s,2} x^3 - Q_{s,2} \sum_{j=0}^{2} A_{3-j} x^j + \sum_{j=0}^{2} Q_{s+1,j} x^j \\ &= \sum_{j=0}^{2} Q_{s+1,j} x^j \qquad .\end{aligned}$$

(inductive assumption).

This establishes (2.2). The general form of (2.2) is related to the Jacobi-Perron Algorithm (JPA) [2]. The contraction is a form of Bernoulli's iteration (Householder [14]).

At this stage one might note that Goldstern *et al* [8] have determined the asymptotic distribution function of the ratios of the terms of a linear recurrence. In doing so they have studied the characteristic polynomials. De Pillis [18] has highlighted fascinating and surprising features of Newton's formula for finding a root of a non-linear function when applied to cubic polynomials and has speculated on the generalization of his observations.

For the homogeneous linear recurrence relation of order N represented by

$$\sum_{k=0}^{N} a_N u_{N+k} = 0$$

with distinct roots α_i, $i=1,2,\ldots,N,$ the general solution is

$$u_k = \sum_{i=1}^{N} A_i \alpha_i^k .$$

Bernoulli's method is an application of the equivalence between recurrence relations and their characteristic polynomials to find the dominant root α. Then if $A_1 \neq 0$,

$$q_k = \tfrac{u_{k+1}}{u_k} \to \alpha \ as\ k \to \infty.$$

which has been investigated computationally by Turner [27]. Ferguson [5] also established the limit of the ratio of adjacent terms as the dominant root of the cubic auxiliary polynomial. In particular, for the sequence $\{G_n\}$, where $\{G_n\}$ is defined by the third order recurrence relation

$$G_n = G_{n-1} + G_{n-3}, \quad n \geq 1, \tag{2.4}$$

with initial terms $G_1 = G_2 = G_3 = 1,$ so that the first few terms are given by

$$\{G_n\} = \{1,1,1,2,3,4,6,9,13,\ldots\}.$$

$$\lim_{n\to\infty} \tfrac{G_n}{G_{n-1}} \approx 1.465571231.$$

Benjamin *et al* [1] provide a combinatorial proof that

$$G_{3n+2} = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} \binom{n-j-k}{i}\binom{n-i-k}{j}\binom{n-i-j}{k}.$$

Ferguson also showed that this lies as a sum of binomial coefficients lying upon lines of slope 2 through Pascal's triangle; for example,

$$
\begin{aligned}
6 &= 1+4+1 \\
9 &= 1+5+3 \\
13 &= 1+6+6 \\
19 &= 1+7+1,
\end{aligned}
$$

and so on. Ferguson's results relate to those of Sofo [26] who, *inter alia*, used the theory of difference-delay equations to prove that for $\{G_n\}$:

$$
\begin{aligned}
F(z) &= \frac{z^3}{z^3 - z^2 - 1} \\
&= \sum_{r=0}^{\infty} \frac{z^{1-2r}}{(z-1)^{1+r}},
\end{aligned}
$$

and

$$
G_n = \sum_{r=0}^{\lfloor n/3 \rfloor} \binom{n-2r}{r}.
$$

3. REDUCTION OF THE ORDER

An alternative to developing new properties for these sequences of order 3 is to reduce them to generalized Fibonacci and Lucas sequences of order 2 since there are so many results in the literature for these sequences. We shall consider three approaches to this idea, each of which sheds slightly different light on the underlying structures.

We can take advantage of the fact that the characteristic equation (2.1) is in Cardano form and utilize some of its properties. Among these is

$$
x = a + b
$$

where

$$
a = \left[\frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{27}}\right)\right]^{1/3}
$$

and

$$
b = \left[\frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{27}}\right)\right]^{1/3}
$$

to reduce it by a method of Williams [28] to

$$
x^2 - p_1 x + q_1 = 0 \tag{3.1}
$$

in which $p_1 = (a+b)$ and $q_1 = ab$. Equation (3.1) is then the characteristic equation of $\{w_n\}$, the second order sequence which satisfies

$$w_n = p_1 w_{n-1} - q_1 w_{n-2}, \;\; n \geq 2 \tag{3.2}$$

which has been explored extensively by Horadam [12,13].

Another approach is to denote by E the endomorphism of the R-module R^N of all sequences over an arbitrary commutative ring with unity which sends the sequence $\{P_n\}$ to the sequence $\{P_{n+1}\}$ [19]; that is,

$$EP_n = P_{n+1}, \tag{3.3}$$

so that the third order recurrence relation (1.1) becomes

$$\begin{aligned} 0 &= \left(E^3 - E - 1\right)P_n \\ &= \left(E - \alpha\right)\left(E^2 + \alpha E + \tfrac{1}{\alpha}\right)P_n \\ &= \left(E^2 - \alpha E + \tfrac{1}{\alpha}\right)u_n \end{aligned}$$

since

$$\alpha^2 = 1 + \tfrac{1}{\alpha} \tag{3.4}$$

and where

$$u_n = \left(E - \alpha\right)P_n = P_{n+1} - \alpha P_n, \;\; n > 0,$$

and

$$u_n = -\alpha u_{n-1} - \tfrac{1}{\alpha} u_{n-2}, \;\; n > 2, \tag{3.5}$$

so that

$$\{u_n\} \equiv \left\{u_n\left(1, 1 - \alpha; -\alpha, \tfrac{1}{\alpha}\right)\right\}$$

is a (non-integral) second order sequence in the notation of Horadam. Similarly,

$$\{v_n\} \equiv \left\{v_n\left(3\alpha, 2; -\alpha, \tfrac{1}{\alpha}\right)\right\}$$

is analogous to the Perrin sequence $\{Q_n\}$. Shannon and Horadam [25] have used this approach to develop a generating function for powers of elements of third order sequence in order to generalize second order results of Carlitz [4].

In turn, this can be compared with the result of Kennedy [15] which yields the proposition that α is a root of the cubic equation (4.4), if and only if, α is a root of the quadratic equation

$$\alpha x^2 - x - 1 = 0, \tag{4.6}$$

which can be confirmed by substitution. Then, from Galvin and Výborný [6], we have the proposition that α is a root of the quadratic equation (4.6), if and only if, α is a root of the linear equation

$$\alpha x - 1 = 0. \tag{4.7}$$

4. SOME RESULTS OF LEON BERNSTEIN

Observe, in passing, that the characteristic polynomial, x^3-x-1, is also of interest because it represents a special case of polynomials of degree>2, in so far as its only real zero w turns out to be the fundamental unit of $Q(\alpha)$. From before we have that

$$\alpha \approx 1.325,$$

with conjugate roots

$$\beta \approx -0.662 + 0.562i$$

and

$$\overline{\beta} \approx -0.662 - 0.562i$$

with

$$|\beta| = |\overline{\beta}| \approx 0.868.$$

Bernstein [3] has proved that $(1,\alpha,\alpha^2)$ is a minimal basis of $Q(\alpha)$. He used a recursive approach to prove that

$$\alpha^n = r_n + s_n\alpha + t_n\alpha^2$$

and so

$$\begin{aligned}\alpha^{n+1} &= r_n\alpha + s_n\alpha^2 + t_n(1+\alpha)\\ &= r_{n+1} + s_{n+1}\alpha + t_{n+1}\alpha^2.\end{aligned}$$

Hence, by comparing coefficients of α,

$$\begin{bmatrix} s_{n+1} \\ t_{n+1} \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_n \\ t_n \\ u_n \end{bmatrix} \tag{5.1}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_{n-1} \\ s_n \\ s_{n+1} \end{bmatrix}. \tag{5.2}$$

Following Bernstein, write the matrix

$$T_n = [s_{n,} \, t_n, u_n]^T,$$

and

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then

$$|A| = 1,$$

and

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0,$$

so that the characteristic equation of A is the same as that of the sequences. Furthermore, we can prove by induction on m that

$$T_{n+m} = A^m T_n, \tag{5.3}$$

so that $\{T_n\}$ satisfies the recurrence relation (2.1) with initial conditions

$$T_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, T_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.4}$$

Let

$$R(x) = \sum_{n=0}^{\infty} r_n x^n .$$

Then, using the recurrence relation for $\{T_n\}$, we get

$$R(x) = \frac{1-x^2}{1-x^2-x^3}$$

and

$$r_n = \sum_{k=0}^{\infty} \binom{m-k-1}{2k-1+2\mu}$$

in which

$$m = \lfloor \tfrac{n}{2} \rfloor$$

and

$$\mu = \tfrac{1}{2} n - m.$$

Thus with some re-writing, we have x^3-x-1 as the recursion function for

$$f(n,2) = \sum_{j=0} (-1)^j \binom{n-2j}{j}.$$

In this way, Bernstein showed with methods similar to those considered here, but from a different point of view, that the question of the zeros of $f(n,2)$ is a combinatorial one. He observed further that the study of $f(n,2)$ for real values of n and of

$$f(n,k) = \sum_{j=0} (-1)^j \binom{n-kj}{j}$$

are open. The case $f(n,1)$ yields the Fibonacci numbers, as is well-known, and they too are amenable to a combinatorial explanation [11]. More recently, Rieger [21] has applied Newton approximation to the Golden Section. His consideration of the continued fraction convergents in this context has also been developed by Moore [16,17] who has also considered the asymptotic behaviour of golden numbers, as has Prodinger [20].

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