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On the Structure of Certain Counting Polynomials

Robert J. Valenza
Claremont McKenna College

ABSTRACT. We consider a natural generalization $\zeta^{(k)}(s) = \sum \alpha_n / n^s$ $(k \geq 2)$ of the Riemann zeta function that arises from a modification of its classical Euler product expansion, for the most part here concentrating on the case $k = 2$. The associated coefficients α_n correspond to a counting problem that may be addressed via a family of multivariable generating functions. Examples computed via symbolic manipulation suggest a recursive structure for these functions, which we prove. With this result in hand, the calculation of the α_n may be facilitated by a more efficient, doubly modular algorithm, as worked out in a detailed example. We conclude with some observations and questions for the case $k > 2$.

KEYWORDS. Riemann zeta function, Euler product, Multivariable generating functions, Symbolic manipulation algorithms.

Introduction

This paper links a variation on the Riemann zeta function with an elementary counting problem, and this in turn with a family of multivariable generating functions having a surprising and beautiful recursive structure, a structure that one might at least metaphorically describe as fractal. To begin, among the most fundamental and celebrated properties of the Riemann zeta function is Euler product expansion:

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

where $s = \sigma + it$ is a complex variable, the summation taken over all positive integers n, and the product is taken over all positive (rational) prime integers p. (See [I], [J], or [RV] for technical and historical background at various levels of detail.) Formally speaking, this equality is exactly the fundamental theorem of arithmetic:

$$\prod_p \frac{1}{1-p^{-s}} = \prod_p (\sum_k p^{-ks}) = \sum (\prod_j p_j^{r_j})^{-s} \ .$$

Here the sum on the right is taken over all sequences $\{r_j\}$ of nonnegative integers that converge to 0, and p_j denotes the jth positive prime. Thus every positive integer n occurs exactly once.

To introduce the variation on $\zeta(s)$ that we have in mind, for each positive integer k define a subset Φ_k of the positive integers $\mathbf{N}_+$ containing precisely those n such that the number of primes (including multiplicity) occurring in the prime factorization of n is divisible by k. Hence, for example,

$$\begin{aligned}
&\Phi_1 = \mathbf{N}_+\\
&\Phi_2 = \{1, 4, 6, 9, 10, 14, 15, 16,\ldots\}\\
&\Phi_3 = \{1, 8, 12, 18, 27,\ldots\}\\
&\Phi_4 = \{1, 16, 24, 36, 54,\ldots\}
\end{aligned}$$

and, of course, this definition is contravariant in the sense that if $k \mid m$, then $\Phi_m \subseteq \Phi_k$. Accordingly, for each k we shall construct a function $\zeta^{(k)}(s)$ of the following form:

$$\zeta^{(k)}(s) = \sum_{n\in\Phi_k} \frac{\alpha_n}{n^s}$$

where the coefficients α_n (which implicitly depend on k as well as n) will be defined shortly. Our primary interest here is the case $k = 2$, and indeed the main object of our analysis will be the coefficients α_n in this instance. (We consider $k > 2$ in the final section below.)

To define $\zeta^{(2)}(s)$ consider the following modification of the Euler product:

$$\prod_{p\geq q} \frac{1}{1-(pq)^{-s}} .$$

Where absolutely convergent, the product is taken over all pairs of positive prime integers (p, q) where $p \geq q$, and indeed we can make an elementary calculation to establish such convergence in a half plane. Working logarithmically with absolute values, we have

$$\sum_p \sum_{p\geq q} \ln(\frac{1}{1-(pq)^{-\sigma}}) \leq \sum_p \sum_{p\geq q} \ln(\frac{1}{1-p^{-\sigma}}) \leq \sum_p p\ln(\frac{1}{1-p^{-\sigma}})$$

and this converges for $\sigma > 2$ by comparison with $-x\ln(1-x^{-\sigma})$. The point is that at least in this half plane we may rearrange the product *ad libitum* to obtain

$$\prod_{p\geq q} \frac{1}{1-(pq)^{-s}} = \sum_{n\in\Phi_2} \frac{\alpha_n}{n^s}$$

for α_n given combinatorially as follows. Let n have prime factorization

$$n = q_1^{r_1} \cdots q_l^{r_l} .$$

with $r = r_1 + \cdots + r_l$ even. Then α_n is the number of ways of pairing up r objects of which r_1 are of type 1, r_2 are of type 2, etc. Note that what matters here are not the q_j but the r_j ; that is, α depends not so much on n as on the structure of n's prime factorization.

EXAMPLE 1. Let $n = 7448 = 2^3 \cdot 7^2 \cdot 19$. For present purposes, we might say that n has prime factorization structure given by the unordered triple (3, 2, 1), and evidently the number of pairs that may be drawn from a collection of 6 objects having the corresponding type structure (3 of type 1, 2 of type 2, 1 of type 3) is 3. Hence $\alpha_n = 3$, as reflected in the following exhaustive list of “double prime factorizations” of n:

$$(2\cdot 2)(2\cdot 7)(7\cdot 19)$$
$$(2\cdot 2)(7\cdot 7)(2\cdot 19)$$
$$(2\cdot 7)(2\cdot 7)(2\cdot 19) .$$

Moreover, $n = 50{,}168{,}173 = 11^2 \cdot 17 \cdot 29^3$ yields the same value for α_n .

The Generating Function for Counting Pairs and Some Auxiliary Constructions

In the sense of the preceding example, let n have factorization structure (r_1 , …, r_l) and again let $r = r_1 + \cdots + r_l$ be the sum of the corresponding type counts. Then

$$G(x_1,\ldots,x_l) = \prod_{i \le j} \frac{1}{1 - x_i x_j} = \prod_{i \le j} \sum_k x_i^k x_j^k$$

is the generating function for the α_n in the sense that α_n is the coefficient of the monomial component of G whose exponents are, respectively, r_1 , …, r_l . (The product is taken over all pairs (i, j) with $i \le j \le l$, and the sum over all nonnegative k.) (See [T] for a general introduction to this technique.) Note that in the definition of G and subsequent associated polynomials, in the spirit of object-oriented programming (polymorphism), we overload the function name. Thus according to this abuse of notation, $G(x_1 , x_2)$ and $G(x_1 , x_2, x_3)$ are taken as distinct expressions that both use the identifier G, much in the same way that the determinant function is designated det for square matrices of all sizes.

EXAMPLE 1, REVISITED. Again suppose that n has prime factorization structure given by the unordered triple (3, 2, 1). Then $l = 3$, and a direct calculation shows that the coefficient of the $x_1^3 x_2^2 x_3$-term of the generating function

$$G(x_1,x_2,x_3) = \frac{1}{1-x_1^2} \cdot \frac{1}{1-x_1x_2} \cdot \frac{1}{1-x_2^2} \cdot \frac{1}{1-x_1x_3} \cdot \frac{1}{1-x_2x_3} \cdot \frac{1}{1-x_3^2}$$

is (again) 3.

We now direct our attention to the structure of this family of generating functions. The general idea is that we begin with a natural factorization of $G(x_1, \ldots, x_l)$ that gives rise to a family of multivariable polynomials which in turn reveal a key structural feature. It is this feature that allows us to compute the coefficients of $G(x_1, \ldots, x_l)$ with a certain degree of efficiency and elegance.

Let $l \geq 1$. Then again overloading the notation, we define a family of functions F by

$$F(x_1, \ldots, x_l) = \prod_{i \leq l} \frac{1}{1 - x_i x_l}.$$

Then, perhaps after a repentant admission that the indexed product notation fails in the face of this particular instance of polymorphism, we have

$$G(x_1, \ldots, x_l) = F(x_1) \cdot F(x_1, x_2) \cdots F(x_1, x_2, \ldots, x_l), \tag{1}$$

which already suggests a lurking recursive structure. We need introduce only one more pair of supplementary constructions.

For any $l \geq 1$, the formal expansion

$$F(x_1, \ldots, x_{l+1}) = \sum_j A_j(x_1, \ldots, x_l) x_{l+1}^j \tag{2}$$

defines, for each $j \geq 0$, a sequence of polynomials $A_j(x_1, \ldots, x_l)$. Moreover, for $l = 0$, we may identify the corresponding A_j (a sequence of functions of no variables) with an alternating sequence of 1's and 0's:

$$\{A_j\} = \{1,0,1,0,1,0,\ldots\}.$$

Thus with respect to our current notational conventions, we take a polynomial in no variables to be no more than a constant. Note that each of the polynomials $A_j(x_1, \ldots, x_l)$ is symmetric (because each corresponding F is symmetric in all but the last variable) and that for $A_0(x_1, \ldots, x_l) = 1$ for *any* valid l.

To understand the polynomials $A_j(x_1, \ldots, x_l)$, we make use of the associated family of polynomials $R(x_1, \ldots, x_l)$ defined by the relation

$$\frac{\partial F(x_1, \ldots, x_l)}{\partial x_l} = R(x_1, \ldots, x_l) F(x_1, \ldots, x_l), \tag{3}$$

and we may compute at once that

(4) $$R(x_1,\ldots,x_l)=\sum_{i=1}^{l-1}\frac{x_i}{(1-x_ix_l)}+\frac{2x_l}{(1-x_l^2)}\quad .$$

One must take special care with the implicit function overloading here: in all cases R is defined via the partial derivative of its final variable. For reference, we record the Taylor expansion at zero of $R(x_1, \ldots, x_l)$ in its final variable:

(5) $$R(x_1,\ldots,x_l)=\sum_{j=0}^{\infty}\left\{\left(\sum_{i=1}^{l-1}x_i^{2j+1}\right)x_l^{2j}+\left(2+\sum_{i=1}^{l-1}x_i^{2j+2}\right)x_l^{2j+1}\right\}.$$

The Recursive Structure of the Polynomials $A_j\,(x_1,\ldots,x_l)$

In this section we prove the following result, which was suggested by working out a fair number of examples via automated symbolic manipulation. The proof is by algebraic calculation and induction; we do not know if there is any direct interpretation of the formula as a natural recursive counting procedure.

THEOREM. *The polynomials* $A_j\,(x_1, \ldots, x_l)$ *are given by the relations*

$$\begin{aligned}A_j(x_1,\ldots,x_l)&=\sum_{i=0}^{j}A_{j-i}(x_1,\ldots,x_{l-1})x_l^i\\&=A_{j-1}(x_1,\ldots,x_l)x_l+A_{j-1}(x_1,\ldots,x_{l-1})\end{aligned}$$

for all $j \geq 0$, *and* $l \geq 1$.

PROOF. The proof goes by induction on l. For $l = 1$, the assertion amounts to the pair of formulas

$$\begin{aligned}A_{2j}(x_1)&=1+x_1^2+\cdots+x_1^{2j}\\A_{2j+1}(x_1)&=x_1+x_1^3+\cdots+x_1^{2j+1}\end{aligned}$$

for all $j \geq 0$, and these in turn may be read directly off the following factorization of $F(x_1, x_2)$:

$$\begin{aligned}F(x_1,x_2)&=\sum_{i=0}^{\infty}(x_1x_2)^i\cdot\sum_{i=0}^{\infty}x_2^{2i}\\&=\left\{\sum_{i=0}^{\infty}(x_1x_2)^{2i}+\sum_{i=0}^{\infty}(x_1x_2)^{2i+1}\right\}\cdot\sum_{i=0}^{\infty}x_2^{2i}\\&=(1+x_1x_2)\cdot\sum_{i=0}^{\infty}(x_1x_2)^{2i}\cdot\sum_{i=0}^{\infty}x_2^{2i}\quad .\end{aligned}$$

So assume now that l is greater than 1 and observe that the assertion of the theorem is trivial for $j = 0$. Then using equations 2, 3 and 4, we find

$$\begin{aligned}\sum_{j=1}^{\infty} jA_j(x_1,\ldots,x_l)x_{l+1}^{j-1} &= R(x_1,\ldots,x_{l+1})F(x_1,\ldots,x_{l+1}) \\ &= \frac{1}{1-x_1x_{l+1}} \cdot \{x_1F(x_2,\ldots,x_{l+1}) + R(x_2,\ldots,x_{l+1})F(x_2,\ldots,x_{l+1})\} \\ &= \sum_{j=0}^{\infty}(x_1x_{l+1})^j \cdot \left\{x_1\sum_{j=0}^{\infty} A_j(x_1,\ldots,x_l)x_{l+1}^j + \sum_{j=1}^{\infty} jA_j(x_2,\ldots,x_l)x_{l+1}^j\right\}\end{aligned}$$

and this allows us to bring the induction hypothesis to bear on the first summation of the second factor. Thus appealing to the symmetry of the polynomials A_j, we find that

$$\sum_{j=1}^{\infty} jA_j(x_1,\ldots,x_l)x_{l+1}^{j-1} = \sum_{j=0}^{\infty}(x_1x_{l+1})^j \cdot \left\{x_1\sum_{j=0}^{\infty}\sum_{m=0}^{j} x_1^m A_{j-m}(x_2,\ldots,x_l)x_{l+1}^j + \sum_{j=1}^{\infty} jA_j(x_2,\ldots,x_l)x_{l+1}^j\right\}. \tag{6}$$

What then is the jth coefficient on the right hand side as a polynomial in the indeterminate x_{l+1}? We arrange the contributing terms of this formidable-looking product into a triangular pattern that begins with the jth-degree term of the first factor times the constant term of the second; each succeeding row then represents the contribution that results from decrementing the degree of the term drawn from the first factor and incrementing the degree of the reciprocating term drawn from the second factor.

$$\begin{array}{l}
x_1^{j+1}A_0 + x_1^j A_1(x_2,\ldots,x_l) \\
x_1^{j+1}A_0 + x_1^j A_1(x_2,\ldots,x_l) + 2x_1^{j-1}A_2(x_2,\ldots,x_l) \\
x_1^{j+1}A_0 + x_1^j A_1(x_2,\ldots,x_l) + \ x_1^{j-1}A_2(x_2,\ldots,x_l) + 3x_1^{j-2}A_3(x_2,\ldots,x_l) \\
x_1^{j+1}A_0 + x_1^j A_1(x_2,\ldots,x_l) + \ x_1^{j-1}A_2(x_2,\ldots,x_l) + \ x_1^{j-2}A_3(x_2,\ldots,x_l) + 4x_1^{j-3}A_4(x_2,\ldots,x_l) \\
\ \vdots \\
x_1^{j+1}A_0 + x_1^j A_1(x_2,\ldots,x_l) + \ x_1^{j-1}A_2(x_2,\ldots,x_l) + \ x_1^{j-2}A_3(x_2,\ldots,x_l) + \ x_1^{j-3}A_4(x_2,\ldots,x_l) + \cdots + (j+1)A_{j+1}(x_2,\ldots,x_l)
\end{array}$$

All is now clear: if we sum the preceding expressions vertically, we have, by comparison of coefficients in equation 6, that

$$(j+1)A_{j+1}(x_1,\ldots,x_l) = (j+1)\cdot\sum_{i=0}^{j+1} A_{j+1-i}(x_2,\ldots,x_l)x_1^j$$

and, with a final appeal to symmetry, the first equality of theorem is proved. The second then follows immediately.

Computing the Coefficients

Suppose that the integer n has factorization given by $(r_1, \ldots, r_l)$, so that we must compute the coefficient of

$$x_1^{r_1} \cdots x_l^{r_l}$$

in equation 1. We examine a naïve provisional procedure for computing α_n, assuming that the components of the factorization structure of n lie in ascending order. The main point of this analysis is to motivate a subsequent algorithm. (While the following calculations are trivial relative to the underlying symbolic operations, the procedure is impractical insofar as the polynomials grow rapidly in length.)

STEP ONE. Within a pair of nested loops, first over m, then over j, successively compute and store the polynomials $A_j(x_1, \ldots, x_m)$ for $m = 1, \ldots, l-1$ and $j = 0, \ldots, r_l$. Note that it follows from the objective of the calculation and the theorem above that these computations are highly modular in two distinct senses:

(a) The polynomial $A_j(x_1, \ldots, x_m)$ may be computed modulo the monomials

$$x_1^{r_1+1}, \ldots, x_m^{r_m+1}$$

(or more precisely, modulo the ideal generated by these monomials in $\mathbf{Z}[[x_1, \ldots, x_m]]$, the power series ring over the integers generated by $x_1, \ldots, x_m$).

(b) The sequences in j for fixed m eventually repeat with periodicity 2 .

The example below illustrates both principles. Observe that $A_{r_l}(x_1, \ldots, x_{l-1})$ is the only term in the expansion of $F(x_1, \ldots, x_l)$ that will be needed in the calculation of the coefficient of $x_l^{r_l}$ when $G(x_1, \ldots, x_l)$ is identified with an element in $\mathbf{Z}[[x_1, \ldots, x_{l-1}]][[x_l]]$; that is, when $G(x_1, \ldots, x_l)$ is identified with a power series in x_l over the ring of power series generated by its predecessors.

STEP TWO. Decrementing m from l to 1, calculate the coefficient of the monomial

$$x_m^{r_m} \cdots x_l^{r_l} \text{ in the partial product } F(x_1, x_2, \ldots, x_m) \cdots F(x_1, x_2, \ldots, x_l)$$

as a polynomial over $\mathbf{Z}[x_1, \ldots, x_{m-1}]$; again we work modulo the appropriate powers of the indeterminates, and at each stage we need only reference the $F(x_1, \ldots, x_m)$ and the result of the previous iteration. At $m = 1$, we have the value of α_n.

EXAMPLE 2. We sketch the computation of α_n for $n = 158{,}838{,}853{,}498{,}109{,}885{,}494{,}697$, which has factorization type (1, 1, 2, 3,3). The parameters l and r_l (respectively, 5 and 3), are small enough that the results of the first step of the procedure above may be captured in the following table. (We use the bar notation in the column headers to indicate that the calculations always proceed modulo the appropriate powers of the indeterminates.)

	A_j	$\overline{A}_j(x_1)$	$\overline{A}_j(x_1,x_2)$	$\overline{A}_j(x_1,x_2,x_3)$	$\overline{A}_j(x_1,x_2,x_3,x_4)$
$j=0$	1	1	1	1	1
$j=1$	0	x_1	x_2+x_1	$x_3+x_2+x_1$	$x_4+x_3+x_2+x_1$
$j=2$	1	1	x_1x_2+1	$x_3^2+(x_2+x_1)x_3+x_1x_2+1$	$x_4^2+(x_3+x_2+x_1)x_4+$ $x_3^2+(x_2+x_1)x_3+x_1x_2+1$
$j=3$	0	x_1	x_2+x_1	$(x_2+x_1)x_3^2+(x_1x_2+1)x_3+x_2+x_1$	$x_4^3+(x_3+x_2+x_1)x_4^2+$ $(x_3^2+(x_2+x_1)x_3+x_1x_2+1)x_4+$ $(x_2+x_1)x_3^2+(x_1x_2+1)x_3+x_2+x_1$

Note that the third column of the table goes periodic for $j=0$, the fourth for $j=1$, the fifth for $j=3$, and the final column a little later.

The last entry in the table is the coefficient of x_5^3 in $F(x_1, \ldots, x_5)$, and the fifth column bears the coefficients of $F(x_1, \ldots, x_4)$ (as a power series in x_4) required to compute the coefficient of the $x_4^3x_5^3$-term of the product $F(x_1, \ldots, x_4) \cdot F(x_1, \ldots, x_5)$, which in turn will be a polynomial in x_1, x_2 and x_3. (Remember that all calculations proceed modulo the appropriate powers of the participating indeterminates!) The point is that working to the left in a similar manner we eventually reach the coefficient of $x_1x_2x_3^2x_4^3x_5^3$, as required.

We carry this calculation no further, since a brief examination of the steps required at once reveals a better way to organize it, but before stating a less naïve algorithm, let us introduce the following simplifying notation:

$$a_{mj} = \overline{A}_j(x_1, \ldots, x_m)$$

for all $j \geq 0$, $m \geq 0$. (When m is zero, the a_{mj} revert to integer constants, as above.) Again we assume that the components of the factorization type have been sorted as above.

ALGORITHM. For $m = l-1$ down to 0, use the theorem above to compute p_m, the coefficient of

$$x_{m+1}^{r_{m+1}} \cdots x_l^{r_l} \text{ in the partial product } F(x_1, x_2, \ldots, x_{m+1}) \cdots F(x_1, x_2, \ldots, x_l)$$

expressed formally as a polynomial in x_m with coefficients in $\mathbf{Z}[a_{m-1,j}]_{j \geq 0}$. The value of α_n is then nothing more or less than p_0.

Note that the periodicities of a_{mj}, which are easily anticipated without explicit calculation, should be used wherever possible, as illustrated below. *We emphasize yet again, moreover, that all calculations are modular.*

One might describe this a *just-in-time unpacking* of the $\overline{A}_j(x_1, \ldots, x_m)$ insofar as the underlying variables do not appear explicitly in the symbolic calculations until needed.

EXAMPLE 2, REVISITED AND DISPATCHED. By construction, p_4 is the polynomial in x_4 over $\mathbf{Z}[a_{3,j}]_{j \geq 0}$ that serves as the coefficient of x_5^3 in $F(x_1, \ldots, x_5)$. This is just $A_3(x_1, \ldots, x_4)$, which at this point we should express as follows:

$$p_4 = a_{30}x_4^3 + a_{31}x_4^2 + a_{32}x_4 + a_{33} \ .$$

Since r_4 also has value 3, the part of $F(x_1, \ldots, x_4)$ that matters to our calculation is given by the expression

$$a_{33}x_4^3 + a_{32}x_4^2 + a_{31}x_4 + a_{30} ,$$

whence the coefficient of $x_4^3 x_5^3$ in the product $F(x_1, \ldots, x_4) \cdot F(x_1, \ldots, x_5)$ is

$$a_{30}^2 + a_{31}^2 + a_{32}^2 + a_{33}^2 \ .$$

Expanding the a_{3j} s in turn, we now have p_3:

$$p_3 = (a_{20}^2 + 2a_{20}a_{22} + a_{21}^2 + 2a_{21}a_{23} + a_{22}^2)x_3^2 + 2(a_{20}a_{21} + a_{21}a_{22} + a_{22}a_{23})x_3 + (a_{20}^2 + a_{21}^2 + a_{22}^2 + a_{23}^2)$$

Continuing, since $r_3 = 2$, the part of $F(x_1, x_2, x_3)$ of concern here is

$$a_{22}x_3^2 + a_{21}x_3 + a_{20} .$$

One may accordingly compute the coefficient of $x_3^2 x_4^3 x_5^3$ in the product $F(x_1, x_2, x_3) \cdot F(x_1, \ldots, x_4) \cdot F(x_1, \ldots, x_5)$, which when followed by the expansion of the a_{2j} s yields

$$p_2 = 6(5a_{10}^2 + a_{11}^2)x_2 + a_{10}(6a_{10}^2 + 11a_{11}^2) \ .$$

Repeating the process, we find next that

$$p_1 = 36x_1 + 1,$$

whence $\alpha_n = p_0 = 36$.

Note finally that the numbers are small enough to check this result directly. Factoring $\alpha_n = \alpha(n)$ through the factorization type of n, one makes the following "semi-recursive" calculation

$$\begin{aligned}\alpha(1,1,2,3,3) &= \alpha(2,3,3) + \alpha(1,1,3,3) + 2\alpha(1,2,2,3) \\ &= 6 + 8 + 2 \cdot 11\end{aligned}$$

to find that all is consistent. (This is admittedly tedious and *ad hoc*, but the two leading values of 1 are real lifesavers!)

The Case $k > 2$: Some Observations and Questions

We conclude with two observations and a handful of questions. First note that the Euler product that defined the function $\zeta^{(2)}(s)$ above may be directly generalized to

$$\prod_{p_1 \geq p_2 \geq \ldots \geq p_k} \frac{1}{1 - (p_1 p_2 \cdots p_k)^{-s}}$$

where the product is taken over decreasing sequences of rational primes of length k. The previous convergence argument goes through for $\sigma > k$, to define $\zeta^{(k)}(s)$ in a half plane. (One is tempted, of course, by the possibility of an analytic continuation of $\zeta^{(k)}(s)$ or a functional equation.) The associated coefficients α_n again depend only on the structure of the prime factorization of n and now more generally count unordered k-tuples of r objects with repetition of types. The corresponding generating functions $G(x_1, \ldots, x_l)$ will then take the form

$$G(x_1,\ldots,x_l) = \prod_{i_1 \leq \ldots \leq i_k} \frac{1}{1 - x_{i_1} \cdots x_{i_k}} = \prod_{i_1 \leq \ldots \leq i_l} \sum_k x_{i_1}^k \cdots x_{i_k}^k \ .$$

We have a further factorization as in equation 1, with the $F(x_1, \ldots, x_l)$ given this time by

$$F(x_1,\ldots,x_l) = \prod_{i_1 \leq i_2 \leq \ldots i_{k-1} \leq l} \frac{1}{1 - x_{i_1} x_{i_2} \cdots x_{i_{k-1}} x_l} \ .$$

The interested reader might ask at this point how much of the program above goes through in this more general case. In particular, one wonders if there is an extension of our theorem that is also at least theoretically effective in the computation of the α_n.

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