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SOME PROPERTIES OF MODIFIED LAH NUMBERS

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ABSTRACT

A modification of Lah numbers is suggested in this paper by defining them in relation to the rising factorial coefficients instead of the falling factorial coefficients. Some of their properties are then developed, particularly those in relation to Bernoulli and Stirling numbers and Laguerre polynomials. A partial recurrence relation for the modified Lah numbers is also studied.

Keywords: Bernoulli numbers, Fibonacci polynomials, Laguerre polynomials, Lah numbers, Lucas polynomials, Stirling numbers.

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1 INTRODUCTION

Koutras [9] has developed a unified approach to the study of Eulerian numbers in which the common properties of the various quantities were displayed. In particular, relevant properties of Stirling and Lah numbers and Laguerre polynomials were canvassed there. It is the purpose of this paper to consider modified Lah numbers based on references in the literature which complement the bibliography of Koutras. These modified Lah numbers also include generalized Stirling numbers. The idea for them was provided by a paper of Gould [8].

Lah numbers are defined by

$$(-x)^{\underline{n}} = \sum_{k=0}^{n} \mathrm{L}_{nk} x^{\underline{k}} \tag{1.1}$$

and its inverse

$$x^{\underline{n}} = \sum_{k=0}^{n} \mathrm{L}_{nk} (-x)^{\underline{k}}$$

where

$$x^{\underline{n}} = x(x-1)\dots(x-n+1),$$

is the falling factorial coefficient which is related to the rising factorial coefficient by

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}}$$

It follows that
$$(-1)^n(-x)^{\underline{n}} = \sum_{k=0}^{n}(-1)^k l_{nk} x^{\underline{k}} \tag{1.2}$$

and
$$(-1)^n x^{\underline{n}} = \sum_{k=0}^{n}(-1)^k l_{nk}(-x)^{\underline{k}} \tag{1.3}$$

where the l_{nk} are modified Lah numbers defined by
$$(-x)^{\overline{n}} = \sum_{k=0}^{n} l_{nk} x^{\overline{k}} \tag{1.4}$$

2 PROPERTIES OF MODIFIED NUMBERS

The ordinary Lah numbers and the modified Lah numbers are related by
$$l_{nk} = (-1)^{n+k} \mathrm{L}_{nk} \tag{2.1}$$

with
$$l_{no} = (-1)^n \delta_{no} = \delta_{no}$$

and
$$l_{nk} = (-1)^k \frac{n!}{k!}\binom{n-1}{k-1}, \quad n \geq 1. \tag{2.2}$$

where δ_{nk} is the Kronecker delta.

The proof of the last statement follows after the lemma.

Write
$$l_k(t) = \sum_{n=0}^{\infty} \frac{l_{nk} t^n}{n!}$$

and
$$\mathrm{L}_k(t) = \sum_{n=0}^{\infty} \frac{\mathrm{L}_{nk} t^n}{n!}.$$

Lemma:
$$\sum_{k=0}^{\infty}(-x)^{\underline{n}}\frac{t^n}{n!} = \sum_{k=0}^{\infty}\binom{x}{k}\left(\frac{-t}{1+t}\right)^k.$$

proof:
$$\begin{aligned}\sum_{k=0}^{\infty}\binom{x}{k}\left(\frac{-t}{1+t}\right)^k &= \sum_{k=0}^{\infty}\binom{x}{k}(-t)^k \sum_{r=0}^{\infty}\binom{k+r-1}{r}(-t)^r \\ &= \sum_{n=0}^{\infty}(-t)^n \sum_{m=0}^{n}\binom{x}{n-m}\binom{n-1}{m} \\ &= \sum_{n=0}^{\infty}\binom{x+n-1}{n}(-t)^n \\ &= \sum_{n=0}^{\infty}\frac{(x+n-1)(x+n-2)\ldots(x-1)x}{n!}(-t)^n\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(x+n-1)(x+n-2)\ldots(x-1)x}{n!}(-1)^n t^n$$

$$= \sum_{n=0}^{\infty} \frac{(-x)^{\underline{n}} t^n}{n!} \quad \text{as required.}$$

Now
$$\sum_{n=0}^{\infty}(-x)^{\underline{n}}\frac{t^n}{n!} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathrm{L}_{nk}x^{\underline{k}}t^n}{n!}$$

$$= \sum_{k=0}^{\infty}\mathrm{L}_k(t)x^{\underline{k}}$$

$$= \sum_{k=0}^{\infty}\mathrm{L}_k(t)\frac{x!}{(x-k)!}.$$

Whence, from the lemma

$$\mathrm{L}_k(t) = \frac{1}{k!}\left(\frac{-t}{1+t}\right)^k,$$

and
$$l_k(t) = \frac{1}{k!}\left(\frac{t}{1+t}\right)^k.$$

It follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathrm{L}_{nk}\left(\frac{t^n}{n!}\right)\frac{x!}{(x-k)!} = \sum_{k=0}^{\infty}\binom{x}{k}\sum_{r=0}^{\infty}(-1)^{k+r}\binom{k+r-1}{r}t^{k+r}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x!}{(x-k)!}\frac{n!}{k!}(-1)^n\binom{n-1}{k-1}\frac{t^n}{n!}$$

and so
$$l_{nk} = (-1)^{n+k}\mathrm{L}_{nk}$$

$$= (-1)^k\frac{n!}{k!}\binom{n-1}{k-1}, \quad n \geq 1, \quad \text{which is (2.2).}$$

The first few values of the l_{nk} are given in Table 1.

	k=1	2	3	4	5
n=1	−1				
2	−2	1			
3	−6	6	−1		
4	−24	36	−12	1	
5	−120	240	−120	20	−1

Table 1. First Five Modified Lah Numbers.

Theorem: $$l_{n+1,k} = (n+k)l_{n,k} - l_{n,k-1}$$

Proof: $$l_{n+1,k} + l_{n,k-1} = (-1)^k \frac{(n+1)!}{k!}\binom{n}{k-1} + (-1)^{k-1}\frac{n!}{(k-1)!}\binom{n-1}{k-2}$$

$$= (-1)^k \frac{n!}{k!}\left(\frac{n(n+1)}{n-k+1} - \frac{k(k-1)}{n-k+1}\right)$$

$$= (-1)^k \frac{n!}{k!}\binom{n-1}{k-1}\frac{(n+k)(n-k+1)}{n-k+1}$$

$$= (n+k)l_{nk}, \text{ as required.}$$

Other modified Lah number results include

$$\sum_{k=0}^{\infty} l_k(t)x^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} l_{nk}x^k \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} l_{n,n-k}x^{n-k}\frac{t^n}{n!}$$

and $$(1-t)\sum_{n=0}^{\infty} \mathrm{L}_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\mathrm{L}_n(x) - n\mathrm{L}_{n-1}(x)).$$

Whence,

$$\mathrm{L}_n(x) - n\mathrm{L}_{n-1}(x) = \sum_{k=0}^{n} l_{n,n-k}x^{n-k}. \tag{2.3}$$

This can be illustrated as follows:

$$\mathrm{L}_1(x) - \mathrm{L}_0(x) = 1 - x - 1 = -x,$$

and $$\sum_{k=0}^{1} l_{1,1-k}x^{1-k} = l_{11}x + l_{10} = -x.$$

3 OTHER RELATIONSHIPS

$$\sum_{k=0}^{\infty} l_k(t)x^k = \exp\left(\frac{xt}{t-1}\right). \tag{3.1}$$

proof: $$\sum_{k=0}^{\infty} l_k(t)x^k = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{xt}{t-1}\right)^k$$

$$= \exp\left(\frac{xt}{t-1}\right).$$

This indicates that the modified Lah numbers are related to the Laguerre polynomials which are defined by

$$\exp\left(\frac{xt}{t-1}\right) = (1-t)\sum_{n=0}^{\infty} \mathrm{L}_n(x)\frac{t^n}{n!};$$

here the $\mathrm{L}_n(x)$ are Laguerre polynomials. Carlitz has dealt with them in [2,7] for example.

$$l_{nk} = \sum_{j=0}^{k}(-1)^{n+j+k} s_{nj} S_{jk} \tag{3.2}$$

where s_{nj} and S_{jk} are Stirling numbers of the first and second kind respectively, defined by

$$t^{\underline{n}} = \sum_{k=0}^{n} s_{nk} t^k,$$

and

$$t^n = \sum_{k=0}^{n} S_{nk} t^{\underline{k}},$$

and used by Carlitz in a number of papers [1,3,4,5,6].

Proof of (3.2):

$$\sum_{k=0}^{n} e_{nk}(-t)^{\overline{k}} = t^{\overline{n}}(-1)^n(-t)^{\underline{n}}$$

$$= \sum_{j=0}^{n}(-1)^{n+j} s_{nj} t^j$$

$$= \sum_{j=0}^{n}(-1)^{n+j} s_{nj} \sum_{k=0}^{j} S_{jk} t^{\underline{k}}$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{n+j+k} s_{nj} S_{jk}(-t)^{\overline{k}}$$

and so

$$l_{nk} = \sum_{j=0}^{k}(-1)^{n+j+k} s_{nj} S_{jk}.$$

From the generating function for Bernoulli numbers in the even suffix notation, namely,

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \exp(tB) = \frac{t}{e^t - 1} \tag{3.3}$$

in which umbral coefficients are used for the exponential expansion, we can define

$$B_{\overline{n}} = \sum_{k=0}^{n} (-1)^{n+k} s_{nk} B_k, \tag{3.4}$$

which relates $B_{\overline{n}}$, which may be called a 'Bernoulli rising n-factorial', to the Bernoulli numbers and the Stirling numbers of the first kind. The reason for (3.4) can be seen in terms of umbral coefficients:

$$B^{\underline{n}} = (-1)^n(-B)^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n+k} s_{nk} B_k.$$

The $B_{\overline{n}}$ can also be related to the modified Lah numbers as follows. Riordan [10] has shown that

$$B_n = \sum_{k=0}^{n} (-1)^k k! S_{nk}/(k+1).$$

Then

$$\begin{aligned} B_{\overline{n}} &= \sum_{k=0}^{n} (-1)^{n+k} s_{nk} \sum_{j=0}^{k} (-1)^j j! S_{kj}/(j+1) \\ &= \sum_{j=0}^{n} \sum_{k=0}^{j} (-1)^k s_{nk} S_{kj} j!/(j+1) \\ &= \sum_{j=0}^{n} (-1)^{n+j} l_{nj} j!/(j+1) \end{aligned}$$

In conclusion, it is of interest to note that Tauber [11] has developed Lah numbers for Fibonacci and Lucas polynomials.

References

[1] L. Carlitz. "Congruences for generalized Bell and Stirling numbers." *Duke Mathematical Journal* **22** (1955): 193–205.

[2] L. Carlitz. "A note on the Laguerre polynomials." *Michigan Mathematical Journal* **7** (1960): 219–223.

[3] L. Carlitz. "A problem in partitions related to the Stirling numbers." *Bulletin of the American Mathematical Society* **70** (1964): 275–278.

[4] L. Carlitz. "Extended Stirling and exponential numbers." *Duke Mathematical Journal* **32** (1965): 205–224.

[5] L. Carlitz. "Some partition problems related to the Stirling numbers of the second kind." *Acta Arithmetica* **10** (1965): 409–422.

[6] L. Carlitz. "The coefficients in an asymptotic expansion and certain related numbers." *Duke Mathematical Journal* **35** (1965): 83–90.

[7] L. Carlitz. "Some generating functions for Laguerre polynomials." *Duke Mathematical Journal* **35** (1968): 825–828.

[8] H. W. Gould. "The q–Stirling numbers of first and second kinds." *Duke Mathematical Journal* **28** (1961): 281–289.

[9] M. V. Koutras. "Eulerian numbers associated with sequences of polynomials." *Fibonacci Quarterly* **32** (1994): 44–57.

[10] J. Riordan. *Combinatorial Identities.* New York: Wiley, 1968.

[11] S. Tauber. "Lah numbers for Fibonacci and Lucas polynomials." *Fibonacci Quarterly* **6** (1968): 93–99.