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A CURIOUS PROBLEM INVOLVING GEOMETRIC SERIES

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1. INTRODUCTION

The aim of this note is to extend some results established in [1], [2], [3] and [4] where closed-form expressions were found for the real positive numbers that preserve their fractional parts when raised to certain integral powers. More precisely, after defining

$$S_k(x) := \sum_{r=0}^{\infty} x^{kr} \tag{1.1}$$

where $0 < x < 1$ is an indeterminate and k is a natural number, we pose ourselves the following problem.

Problem. Characterize the sequence $\{x_n(k)\}$ $(n = 1, 2, 3, \ldots)$ of all the real *positive* numbers x for which x and $S_k(x)$ have the same fractional part, and find (if any) the closed-form expressions for its terms.

The solution of this problem quite easy. Due to the conditions on x, the series (1.1) converges, and its closed-form expression is

$$S_k(x) = 1/(1-x^k) \quad (0 < x < 1). \tag{1.2}$$

Consequently, the numbers $x_n(k)$ must satisfy the equation

$$1/(1-x^k) - x = n \tag{1.3}$$

where n is a natural number. Eqn. (1.3) can be rewritten as

$$x^{k+1} + nx^k - x - n + 1 = 0\,. \tag{1.4}$$

The numbers $x_n(k)$ are given by the positive roots of (1.4), and their closed-form expressions can be found for $k = 1, 2$ and 3 by using the well-known formulas for the solution of second-, third- and fourth-degree equations. This will be done in Section 2, whereas an extension to negative values of k will be presented in Section 3. Some particular solutions for $k > 3$ are also found. The algebraic manipulations involved are not difficult, but some care must be put when one faces the case $k = 3$.

The results presented in this note might be of interest to high school students and mathematics teachers, and, perhaps, to a wider audience. The reader may enjoy using a PC, or even a 10-digit pocket calculator, to check the correctness of the results from the numerical point of view.

2. CLOSED FORM EXPRESSIONS FOR $x_n(k)$

2.1. $k = 1$

For $k = 1$, eqn. (1.4) becomes

$$x^2 + (n-1)x - n + 1 = 0 . \tag{2.1}$$

The positive roots of (2.1) are

$$x_n(1) = \frac{1 - n + \sqrt{n^2 + 2n - 3}}{2} \quad (n \geq 2). \tag{2.2}$$

Remark 1. It can be immediately seen that $x_1(1) = 0$. This solution has to be disregarded as we imposed that x is positive. On the other hand, $x = 0$ is a root of (1.4) for all k, when $n = 1$.

As a special case, we have

$$x_2(1) = \alpha - 1 \quad (\alpha = (1 + \sqrt{5})/2 \text{ the golden section}). \tag{2.3}$$

2.2. $k = 2$

For $k = 2$, eqn. (1.4) becomes

$$x^3 + nx^2 - x - n + 1 = 0. \tag{2.4}$$

The positive roots of (2.4) are

$$x_n(2) = \frac{2(n^2+3)^{1/2}}{3} \cos\left[\frac{1}{3}\cos^{-1}\frac{-2n^3 + 18n - 27}{2(n^2+3)^{3/2}}\right] - \frac{n}{3} \quad (n \geq 1). \tag{2.5}$$

Remark 2. The trigonometric expression (2.3) comes from the fact that the discriminant of (2.4) is negative.

As a special case, we have

$$x_1(2) = \alpha - 1 \quad [= x_2(1)]. \tag{2.6}$$

Proof of (2.6). For $n = 1$, the l.h.s. of eqn. (2.4) factors as $x(x^2 + x - 1)$. The r.h.s. of (2.6) is the positive root of the above second-degree polynomial.

2.3. *k* = 3

For $k = 3$, eqn. (1.4) becomes

$$x^4 + nx^3 - x - n + 1 = 0. \tag{2.7}$$

First, let us consider the case $n = 1$ for which the l.h.s. of eqn. (2.7) factors as $x(x^3 + x^2 - 1)$. The number $x_1(3)$ is the positive root of the above cubic factor. Namely, we get

$$x_1(3) = \sqrt[3]{\frac{25}{54} + \sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{25}{54} - \sqrt{\frac{23}{108}}} - \frac{1}{3}. \tag{2.8}$$

For $n \geq 2$ in (2.7), we get the positive roots

$$x_n(3) = \frac{-n - \sqrt{n^2 + 4u_n} + \sqrt{2n^2 - 4u_n + 2n\sqrt{n^2 + 4u_n} + 8\sqrt{u_n^2 + 4n - 4}}}{4}, \tag{2.9}$$

where

$$u_n = \sqrt[3]{\frac{-n^3 + n^2 + 1}{2} + \sqrt{D_n}} + \sqrt[3]{\frac{-n^3 + n^2 + 1}{2} - \sqrt{D_n}} \tag{2.10}$$

and

$$D_n = \frac{27n^2(n^4 - 2n^3 + n^2 - 14) + 576n - 229}{108}. \tag{2.11}$$

It is worth mentioning that u_n is the real root of the cubic

$$x^3 + (3n - 4)x + n^3 - n^2 - 1 = 0 \tag{2.12}$$

associated to the quartic (2.7). As a special case, we have

$$x_2(3) = (-1 + \sqrt{4\alpha + 1})/2. \tag{2.13}$$

Proof of (2.13). Since the cubic (2.12) is satisfied for $n = 2$ and $x = -1$, it is plain that $u_2 = -1$. Put $n = 2$ and $u_2 = -1$ in (2.9) to obtain (2.13).

2.4. *k* = 4

Since, for $k = 4$ and $n = 1$, the l.h.s. of eqn. (1.4) factors as $x(x^4 + x^3 - 1)$, we can find $x_1(4)$ as the positive root of the above quartic factor. Namely, we get

$$x_1(4) = \frac{-1 - \sqrt{4y+1} + \sqrt{2 - 4y + 2\sqrt{4y+1} + 8\sqrt{y^2+4}}}{4}, \tag{2.14}$$

where

$$y = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{283}{108}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{283}{108}}}. \tag{2.15}$$

3. EXTENSION TO NEGATIVE VALUES OF k

By defining

$$S_{-k}(x) := \sum_{r=0}^{\infty} x^{-kr} \tag{3.1}$$

the theory developed in Section 2 can be extended to cover negative values of k. If $x > 1$, then the series (3.1) converges and its closed-form expression is

$$S_{-k}(x) = x^k / (x^k - 1) \quad (x > 1). \tag{3.2}$$

Consequently, the numbers $x_n(-k)$ must satisfy the equation

$$x - x^k / (x^k - 1) = n \tag{3.3}$$

where n is a nonnegative integer. Eqn. (3.3) can be rewritten as

$$x^{k+1} - (n+1)x^k - x + n = 0. \tag{3.4}$$

The numbers $x_n(-k)$ $(n = 0, 1, 2, \ldots)$ are given by the positive roots of (3.4). From (3.3), it can be readily observed that $S_{-k}[x_0(-k)] = x_0(-k)$.

3.1. $k = 1$

For $k = 1$, eqn. (3.4) becomes

$$x^2 - (n+2)x + n = 0. \tag{3.5}$$

The positive roots of (3.5) are

$$x_n(-1) = \frac{n + 2 + \sqrt{n^2+4}}{2}. \tag{3.6}$$

As special cases, we have

$$x_0(-1) = 2 \quad \text{and} \quad x_1(-1) = \alpha + 1. \tag{3.7}$$

3.2. $k = 2$

For $k = 2$, eqn. (3.4) becomes

$$x^3 - (n+1)x^2 - x + n = 0. \tag{3.8}$$

The positive roots of (3.8) are

$$x_n(-2) = \frac{2(n^2+2n+4)^{1/2}}{3}\cos\left[\frac{1}{3}\cos^{-1}\frac{2n^3+6n^2-12n+11}{2(n^2+2n+4)^{3/2}}\right] + \frac{n+1}{3}. \tag{3.9}$$

As a special case, we have

$$x_0(-2) = \alpha\ . \tag{3.10}$$

3.3. $k = 3$

For $k = 3$, eqn. (3.4) becomes

$$x^4 - (n+1)x^3 - x + n = 0. \tag{3.11}$$

First, let us consider the case $n = 0$ for which the l.h.s. of eqn. (3.11) factors as $x(x^3 - x^2 - 1)$. The number $x_0(-3)$ is the positive root of the above cubic factor. Namely, we get

$$x_0(-3) = \sqrt[3]{\frac{29}{54} + \sqrt{\frac{31}{108}}} + \sqrt[3]{\frac{29}{54} - \sqrt{\frac{31}{108}}} + \frac{1}{3}. \tag{3.12}$$

For $n \geq 1$ in (3.11), we get the positive roots

$$x_n(-3) = \frac{n+1+Q_n+\sqrt{2(n+1)^2-4z_n+2(n+1)Q_n-8\sqrt{z_n^2-4n}}}{4} \tag{3.13}$$

where

$$Q_n = \sqrt{(n+1)^2+4z_n}\ , \tag{3.14}$$

$$z_n = \sqrt[3]{\frac{n^3+2n^2+n+1}{2} + \sqrt{\Delta_n}} + \sqrt[3]{\frac{n^3+2n^2+n+1}{2} - \sqrt{\Delta_n}} \tag{3.15}$$

and

$$\Delta_n = \frac{9n(3n^5+12n^4+18n^3+6n^2+27n+2)+31}{108}. \tag{3.16}$$

3.4. $k = 4$ and 5

Since, for $k = 4$ and $n = 0$, the l.h.s. of eqn. (3.4) factors as $x(x^4 - x^3 - 1)$, we can find $x_0(-4)$ as the positive root of the above quartic factor. Namely, we get

$$x_0(-4) = x_1(4) + (1 + \sqrt{4y+1})/2 \tag{3.17}$$

where y is defined by (2.15). Moreover, for $k = 5$ and $n = 0$, the l.h.s. of eqn. (3.4) factors as $x(x^2 - x + 1)(x^3 - x - 1)$ so that we can find $x_0(-5)$ as the positive root of the above cubic factor. Namely, we get

$$x_0(-5) = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{23}{108}}}\ . \tag{3.18}$$

We do not exclude the possibility that further interesting factorizations can be found.

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