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NECESSARY AND SUFFICIENT CONDITIONS FOR SIMPLE $\mathfrak{A}$-BASES

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Let $\mathfrak{A} = \{A_i\}_{i\geq 1}$ where A_i is a set of m_i distinct integers with $m_i \geq 2$ and $0 \in A_i$ for each i. It is shown that $\mathfrak{A}$ possesses a simple $\mathfrak{A}$-base if and only if A_i is a complete residue system modulo m_i for each i and the elements of $\bigcup_{i=h}^{\infty} A_i$ are relatively prime for every positive integer h.

The notions of simple and non-simple $\mathfrak{A}$-bases are due to Long and Woo [3] and generalize those of simple and non-simple A-bases due to de Bruijn [1].

DEFINITION 1. Let $\mathfrak{A} = \{A_i\}_{i\geq 1}$ where each A_i is a set of m_i distinct integers with $m_i \geq 2$ and $0 \in A_i$ for each i. The integer sequence $B = \{b_i\}_{i\geq 1}$ is called an $\mathfrak{A}$-base for the set of integers provided every integer n can be written uniquely in the form

$$n = \sum_{i=1}^{r(n)} a_i b_i \quad , \quad a_i \in A_i \quad \forall_i .$$

If, with possible rearrangement, B can be written in the form $B = \{d_i M_{i-1}\}_{i\geq 1}$ where the d_i are integers and where $M_0 = 1$ and $M_i = \Pi_{j=1}^{i} m_j$ for $i \geq 1$, then it is called a simple $\mathfrak{A}$-base.

DEFINITION 2. Let A be a set of m distinct integers with $m \geq 2$ and $0 \in A$. If $A_i = A$ for all i, the integer sequences of Definition 1 are called A-bases and simple A-bases respectively.

Properties of A- and $\mathcal{A}$-bases were studied by de Bruijn and by Long and Woo but, until recently, necessary and sufficient for the existence of bases were not known. In [5] we gave necessary and sufficient conditions for the existence of simple A-bases. In the present paper we give necessary and sufficient conditions for the existence of simple $\mathcal{A}$-bases. Necessary and sufficient conditions for the existence of non-simple A- and $\mathcal{A}$-bases are not known, but finding meaningful conditions seems unlikely since Swenson [4] has shown that any two sets C and D such that $c - c' \neq d - d'$ for all $c, c' \in C$ and $d, d' \in D$ can be extended to form a non-simple A-base.

In [3], Long and Woo point out that the sequence $\{d_i M_{i-1}\}_{i \geq 1}$ forms an $\mathcal{A}$-base only if A_i is a complete residue system modulo m_i and $(d_i, m_i) = 1$ for each i. They also observe that it is necessary that the elements of the set $\bigcup_{i=1}^{\infty} A_i$ be relatively prime and that $\{d_i M_{i-1}\}_{i \geq 1}$ is an $\mathcal{A}$-base if and only if $\{d_{i+1} M_i / M_s\}_{i \geq s}$ is an $\mathcal{A}'$-base where $\mathcal{A}' = \{A_{i+1}\}_{i \geq s}$ for all $s \geq 0$. Of course, this further implies that it is necessary that the elements of $\bigcup_{i=h}^{\infty} A_i$ be relatively prime for all $h \geq 1$. In the present paper we show that these conditions on $\mathcal{A}$ are both necessary and sufficient.

We first prove two lemmas.

LEMMA 1. Let $\mathcal{A} = \{A_i\}_{i \geq 1}$ where A_i is a complete residue system modulo m_i

and $m_i \geq 2$ for each i. Let $\{M_i\}_{i \geq 0}$ be as above. If the elements of $\bigcup_{i=h}^{\infty} A_i$ are relatively prime for each $h \geq 1$ then, for each $s \geq 1$, there exists an integer $q > s$ and elements $a_i \in A_i$ for $s \leq i \leq q$ such that

$$(1) \qquad \left(a_s \frac{M_{s-1}}{M_{s-1}}, a_{s+1} \frac{M_s}{M_{s-1}}, \cdots, a_q \frac{M_{q-1}}{M_{s-1}} \right) = 1.$$

Proof. Choose an arbitrary element $a_s \in A_s$. If $a_s = \pm 1$, the desired result is clearly true for any q. If $a_s \neq \pm 1$, divide the distinct prime divisors of a_s into two classes:

$$B_1 = \{p : p \mid a_s \text{ and } p \mid m_i \text{ for some } i \geq s\}$$

$$B_2 = \{p : p \mid a_s \text{ and } p \nmid m_i \text{ for any } i \geq s\}$$

If $p \in B_1$, there is a least $i = i(p) \geq s+1$ such that $p \mid m_i$. Since $A_{i(p)}$ is a complete residue system modulo $m_{i(p)}$, there exists a specific element $a_{i(p)} \in A_{i(p)}$ such that $a_{i(p)} \equiv 1 (\text{mod } m_{i(p)})$. Hence, we can choose $a_{i(p)}$ such that $p \nmid a_{i(p)}$ and $p \mid m_{i(p)}$. Of course, we can do this for each of the primes in B_1 and we note that it may be the case that $i(p) = i(p')$ for $p \neq p'$. But then $pp' \nmid a_{i(p)}$ and $pp' \mid m_{i(p)}$. Indeed, if π is the product of all the primes $q \in B_1$ such that $i(q) = i(p)$, then $\pi \nmid a_{i(p)}$ and $\pi \mid m_{i(p)}$. Let $t = \max_{p \in B_1}\{i(p)\}$ and let

$$d = \left(a_s \frac{M_{s-1}}{M_{s-1}}, a_{s+1} \frac{M_s}{M_{s-1}}, \cdots, a_t \frac{M_{t-1}}{M_{s-1}} \right)$$

where the $a_i = a_{i(p)}$ for $p \in B_1$ and a_i is an arbitrary but fixed element of A_i for all other $i, s \leq i \leq t$. Since $p \nmid a_{i(p)} M_{i(p)-1}$ for $p \in B_1$, it follows that $p \mid d$ for any such p. Now consider the primes in B_2. Since, by hypothesis, the elements of $\bigcup_{i=t+1}^{\infty} A_i$ are relatively prime, we may choose specific elements $a_i \in A_i$ for $t+1 \leq i \leq t+u$ such that $(a_{t+1}, a_{t+2}, \cdots, a_{t+u}) = 1$. But then

$$\left(a_s \frac{M_{s-1}}{M_{s-1}}, a_{s+1} \frac{M_s}{M_{s-1}}, \cdots, a_{t+u} \frac{M_{t+u-1}}{M_{s-1}} \right) = 1$$

since $p \in B_1$ implies

$$p \nmid \left(a_s \frac{M_{s-1}}{M_{s-1}}, a_{s+1} \frac{M_s}{M_{s-1}}, \cdots, a_t \frac{M_{t-1}}{M_{s-1}} \right)$$

and $p \in B_2$ implies that

$$p \nmid \left(a_{t+1} \frac{M_t}{M_{s-1}}, a_{t+2} \frac{M_{t+1}}{M_{s-1}}, \cdots, a_{t+u} \frac{M_{t+u-1}}{M_{s-1}} \right).$$

Therefore, setting $t + u = q$, we have

$$\left(a_s \frac{M_{s-1}}{M_{s-1}}, a_{s+1} \frac{M_s}{M_{s-1}}, \cdots, a_q \frac{M_{q-1}}{M_{s-1}} \right) = 1$$

as claimed.

Note that if q satisfies Lemma 1 for a given s, then any larger value of q does also.

LEMMA 2. Let $\mathcal{A} = \{A_i\}_{i \geq 1}$ and $\{M_i\}_{i \geq 0}$ be as in Lemma 1 and let $0 \in A_i$ for each i. Then, for any $s \geq 1$, every integer n can be represented in the form

$$n = a_s d_s \frac{M_{s-1}}{M_{s-1}} + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} + \cdots + a_q d_q \frac{M_{q-1}}{M_{s-1}} \tag{2}$$

where $q > s$ and $d_s, d_{s+1}, \cdots, d_q$ are integers with $(d_i, m_i) = 1$ and $a_i \in A_i$ for each i.

Proof. Of course, 0 is trivially representable in the desired form. For $n \neq 0$, we distinguish two cases.

Case 1. $n = 1$.

Since A_s is a complete residue system modulo m_s, there exists $a \in A_s$ such that $a \equiv 1 (\text{mod } m_s)$. By Lemma 1, there exists an integer $q > s$ and elements $a_i \in A_i$ for

$s \leq i \leq q$ such that

$$\left(a_s \frac{M_{s-1}}{M_{s-1}}, a_{s+1} \frac{M_s}{M_{s-1}}, \cdots, a_q \frac{M_{q-1}}{M_{s-1}}\right) = 1.$$

Thus, the diophantine equation

$$1 = a_s \frac{M_{s-1}}{M_{s-1}} x_s + a_{s+1} \frac{M_s}{M_{s-1}} x_{s+1} + \cdots + a_q \frac{M_{q-1}}{M_{s-1}} x_q \tag{3}$$

has a solution $(d'_s, d'_{s+1}, \cdots, d'_q)$. This implies that

$$a_s d'_s \equiv 1 \equiv a (\text{mod } m_s)$$

and hence that $(d'_s, m_s) = 1$. We now set $d_s = d'_s$ and proceed by a limited induction to determine $d_{s+1}, d_{s+2}, \cdots, d_q$ satisfying (2) and such that $(d_i, m_i) = 1$ for $s \leq i \leq q$. Since we have found d_s, we assume that we have determined d_i for $s \leq i < k$ where k is fixed and $k \leq q$. Thus, we have

$$1 = a_s d_s \frac{M_{s-1}}{M_{s-1}} + \cdots + a_{k-1} d_{k-1} \frac{M_{k-2}}{M_{s-1}} + a_k d'_k \frac{M_{k-1}}{M_{s-1}} + \cdots + a_q d'_q \frac{M_{q-1}}{M_{s-1}} \tag{4}$$

with $(d_i, m_i) = 1$, for $s \leq i < k$. If $(d'_k, m_k) = 1$, we set $d_k = d'_k$. If $(d'_k, m_k) \neq 1$, set

$$e_k = \left(d'_k, \quad a_s d_s \frac{M_{s-1}}{M_{s-1}} + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} + \cdots + a_{k-1} d_{k-1} \frac{M_{k-2}}{M_{s-1}}\right) \tag{5}$$

so that

$$\left(\frac{d'_k}{e_k}, \quad \frac{1}{e_k}\left(a_s d_s \frac{M_{s-1}}{M_{s-1}} + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} + \cdots + a_{k-1} d_{k-1} \frac{M_{k-2}}{M_{s-1}}\right)\right) = 1.$$

Then, by Dirichlet's theorem, there exists r_k such that

$$\frac{d'_k}{e_k} - \frac{r_k}{e_k}\left(a_s d_s \frac{M_{s-1}}{M_{s-1}} + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} + \cdots + a_{k-1} d_{k-1} \frac{M_{k-2}}{M_{s-1}}\right) = p_k$$

where p_k is a prime and $p_k \nmid (M_q/M_{s-1})$. Also set

$$\begin{aligned} d_k &= d'_k - r_k \left(a_s d_s \frac{M_{s-1}}{M_{s-1}} + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} + \cdots + a_{k-1} d_{k-1} \frac{M_{k-2}}{M_{s-1}}\right) \\ &= p_k e_k. \end{aligned} \tag{6}$$

Now assume that $(d_k, m_k) \neq 1$. Then there exists a prime p such that $p \mid d_k$ and $p \mid m_k$ and hence $p = p_k$ or $p \mid e_k$. But $p \neq p_k$, since $p_k \nmid (M_q/M_{s-1})$ and $p \mid (M_q/M_{s-1})$. Therefore, $p \mid e_k$ and hence, by (5), $p \mid d_k'$ and

$$(7) \qquad p \mid \left(a_s d_s \frac{M_{s-1}}{M_{s-1}} + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} + \cdots + a_{k-1} d_{k-1} \frac{M_{k-2}}{M_{s-1}} \right).$$

But then, by (4) and (7), $p \mid 1$ since $p \mid d_k'$ and $p \mid m_k$. But this is a clear contradiction and it follows that $(d_k, m_k) = 1$. Moreover, using (4) and (6), we have that

$$\begin{aligned}
1 = \; & a_s d_s \frac{M_{s-1}}{M_{s-1}} \left(1 + a_k r_k \frac{M_{k-1}}{M_{s-1}}\right) + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} \left(1 + a_k r_k \frac{M_{k-1}}{M_{s-1}}\right) \\
& + \cdots + a_{k-1} d_{k-1} \frac{M_{k-2}}{M_{s-1}} \left(1 + a_k r_k \frac{M_{k-1}}{M_{s-1}}\right) \\
& + a_k \frac{M_{k-1}}{M_{s-1}} \left[d_k' - r_k \left(a_s d_s \frac{M_{s-1}}{M_{s-1}} + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} + \cdots + a_{k-1} d_{k-1} \frac{M_{k-2}}{M_{s-1}} \right) \right] \\
& + a_{k+1} d_{k+1}' \frac{M_k}{M_{s-1}} + \cdots + a_q d_q' \frac{M_{q-1}}{M_{s-1}} \\
= \; & a_s d_s \frac{M_{s-1}}{M_{s-1}} + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} + \cdots + a_k d_k \frac{M_{k-1}}{M_{s-1}} + a_{k+1} d_{k+1}' \frac{M_k}{M_{s-1}} + \cdots + a_q d_q' \frac{M_{q-1}}{M_{s-1}}.
\end{aligned}$$

This completes the induction and the proof for Case 1.

Case 2. $n \neq 1$.

It suffices to consider n such that all prime factors of n divide infinitely many of the m_i. For suppose $n = n_1 n_2$ where every prime factor of n_1 divides only finitely many of the m_i. Then the q of Lemma 1 and t may be chosen sufficiently large that $(n_1, M_q/M_{t-1}) = 1$. Now suppose that n_2 can be represented in the desired form

$$n_2 = a_t d_t' \frac{M_{t-1}}{M_{s-1}} + a_{t+1} d_{t+1}' \frac{M_t}{M_{s-1}} + \cdots + a_q d_q' \frac{M_{q-1}}{M_{s-1}} \tag{8}$$

with $(d_i', m_i) = 1$ and $a_i \in A_i$ for $t \leq i \leq q$. Then

$$\begin{aligned} n &= n_1 n_2 \\ &= a_t(n_1 d_t') \frac{M_{t-1}}{M_{s-1}} + a_{t+1}(n_1 d_{t+1}') \frac{M_t}{M_{s-1}} + \cdots a_q(n_1 d_q') \frac{M_{q-1}}{M_{s-1}} \\ &= a_s d_s \frac{M_{s-1}}{M_{s-1}} + a_{s+1} d_{s+1} \frac{M_s}{M_{s-1}} + \cdots + a_t d_t \frac{M_{t-1}}{M_{s-1}} + \cdots + a_q d_q \frac{M_{q-1}}{M_{s-1}} \end{aligned}$$

where $a_i = 0$ and $d_i = 1$ for $s \leq i \leq t-1$ and $d_i = n_1 d_i'$ for $t \leq i \leq q$. Then, since $(n_1, M_q/M_{t-1}) = 1$ and $(d_i', m_i) = 1$ for $t \leq i \leq q$, it follows that $(d_i, m_i) = 1$ for $s \leq i \leq q$ and n is represented in the desired form.

Therefore, we must show that all values of n such that all prime factors of n divide infinitely many of the m_i can be represented as in (8). If we asume that n has this property, it follows that the q of Lemma 1 and $t < q$ can be chosen sufficiently large that

$$n \mid \frac{M_{q-1}}{nM_{t-1}} \text{ and } m_i \mid \frac{M_{q-1}}{nM_{t-1}}$$

for $t \leq i \leq q$. Set

$$A' = A_t \frac{M_{t-1}}{M_{s-1}} \oplus A_{t+1} \frac{M_t}{M_{s-1}} \oplus \cdots \oplus A_{q-1} \frac{M_{q-2}}{M_{s-1}}$$

where

$$kA_i = \{b : b = ka, a \in A_i\} \text{ and } A \oplus B = \{c : c = a + b, a \in A, b \in B\}.$$

It is easy to see that A' forms a complete residue system modulo M_{q-1}/M_{t-1}. Thus, there exists $\alpha \in A'$ such that

$$n \equiv \alpha \left(\bmod \frac{M_{q-1}}{M_{t-1}} \right) \tag{9}$$

and there exists as integer r such that

$$n = \alpha + r \frac{M_{q-1}}{M_{t-1}}. \tag{10}$$

Since $\alpha \in A'$, we have that

$$\alpha = a_{\alpha,t} \frac{M_{t-1}}{M_{s-1}} + a_{\alpha,t+1} \frac{M_t}{M_{s-1}} + \cdots + a_{\alpha,q-1} \frac{M_{q-2}}{M_{s-1}} \tag{11}$$

with $a_{\alpha,i} \in A_i$ for $t \leq i \leq q-1$. Since M_{q-1}/nM_{t-1} is an integer, (10) implies that

$$1 = \frac{\alpha}{n} + r \frac{M_{q-1}}{nM_{t-1}} \tag{12}$$

where α/n is an integer, and this implies that $(\alpha/n, r) = 1$. Now, by Case 1, $v > q$ may be chosen so that

$$1 = a_q d'_q \frac{M_{q-1}}{M_{q-1}} + a_{q+1} d'_{q+1} \frac{M_q}{M_{q-1}} + \cdots + a_v d'_v \frac{M_{v-1}}{M_{q-1}} \tag{13}$$

with $a_i \in A_i$ and $(d'_i, m_i) = 1$ for $q \leq i \leq v$. Since $(\alpha/n, r) = 1$, if follows from Dirichlet's theorem that there exists an integer u such that $r + (\alpha/n)u$ is a prime not dividing M_v/M_{q-1}. Thus, from (13), we have that

$$r+\frac{\alpha}{n}u=a_qd_q\frac{M_{q-1}}{M_{q-1}}+a_{q+1}d_{q+1}\frac{M_q}{M_{q-1}}+\cdots+a_vd_v\frac{M_{v-1}}{M_{q-1}} \tag{14}$$

where $d_i=[r+(\alpha/n)u]d_i'$ so that $(d_i,m_i)=1$ for $q\le i\le v$. Moreover, by (12)

$$\frac{\alpha}{n}\left(1-\frac{uM_{q-1}}{nM_{s-1}}\right)+\frac{M_{q-1}}{nM_{s-1}}\left(r+\frac{\alpha u}{n}\right)=\frac{\alpha}{n}+\frac{rM_{q-1}}{nM_{s-1}}=1 \tag{15}$$

and hence,

$$n=\alpha\left(1-\frac{uM_{q-1}}{nM_{s-1}}\right)+\frac{M_{q-1}}{M_{s-1}}\left(r+\frac{\alpha u}{n}\right). \tag{16}$$

Now since $m_i\mid M_{q-1}/nM_{s-1}$ for $s\le i\le q$, it follows that

$$1=(1-\frac{uM_{q-1}}{nM_{s-1}},m_i) \tag{17}$$

for each i. Thus, from (11), (14), and (16) we have that

$$\begin{aligned}n=\ & a_{\alpha,t}\left(1-\frac{uM_{q-1}}{nM_{s-1}}\right)\frac{M_{t-1}}{M_{s-1}}+a_{\alpha,t+1}\left(1-\frac{uM_{q-1}}{nM_{s-1}}\right)\frac{M_t}{M_{s-1}}+\cdots\\ &+a_{\alpha,q-1}\left(1-\frac{uM_{q-1}}{nM_{s-1}}\right)\frac{M_{q-2}}{M_{s-1}}+\frac{M_{q-1}}{M_{s-1}}\left[a_qd_q\frac{M_{q-1}}{M_{q-1}}+a_{q+1}d_{q+1}\frac{M_q}{M_{q-1}}\right.\\ &\left.+\cdots+a_vd_v\frac{M_{v-1}}{M_{q-1}}\right]\end{aligned}$$

$$= a_{\alpha,t}\left(1-\frac{uM_{q-1}}{nM_{s-1}}\right)\frac{M_{t-1}}{M_{s-1}} + a_{\alpha,t+1}\left(1-\frac{uM_{q-1}}{nM_{s-1}}\right)\frac{M_t}{M_{s-1}} + \cdots$$

$$+a_{\alpha,q-1}\left(1-\frac{uM_{q-1}}{nM_{s-1}}\right)\frac{M_{q-2}}{M_{s-1}} + a_q d_q\frac{M_{q-1}}{M_{s-1}} + a_{q+1}d_{q+1}\frac{M_q}{M_{s-1}}$$

$$+\cdots + a_v d_v\frac{M_{v-1}}{M_{s-1}}$$

$$= a_t d_t\frac{M_{t-1}}{M_{s-1}} + a_{t+1}d_{t+1}\frac{M_t}{M_{s-1}} + \cdots + a_v d_v\frac{M_{v-1}}{M_{s-1}}$$

where $a_i = a_{\alpha,i}$ and $d_i = (1 - uM_{q-1}/nM_{s-1})$ for $s \le i \le q-1$. Of course, for $s \le i \le q-1$,

$$(d_i, m_i) = \left(1 - \frac{uM_{q-1}}{nM_{s-1}}, m_i\right) = 1$$

by (17), and so $(d_i, m_i) = 1$ for $s \le i \le v$ as required. This completes the proof.

We now prove the main result.

THEOREM. Let $\mathcal{A} = \{A_i\}_{i\ge1}$ where A_i is a set of m_i distinct integers with $m_i \ge 2$ and $0 \in A_i$ for all $i \ge 1$, and let $\{M_i\}_{i\ge0}$ be as in definition 1. Then $\mathcal{A}$ has a simple $\mathcal{A}$-base if and only if A_i is a complete residue system modulo m_i for each i and the elements of $\bigcup_{i=h}^{\infty} A_i$ are relatively prime for every positive integer h.

Proof. The necessity follows from [3] as indicated in the introduction.

Suppose that $\mathcal{A}$ satisfies the conditions of the theorem. We must show that there exists an integer sequence $\{d_i\}_{i\geq 1}$ with $(d_i, m_i) = 1$ for all i such that every integer n is uniquely representable in the form

$$n = \sum_{i=1}^{r(n)} a_{n,i} d_i M_{i-1} \quad , \quad a_{n,i} \in A_i \ \ \forall_i. \tag{18}$$

Of course, 0 is trivially representable in the desired form. Also, by Lemma 2, 1 can be represented in the desired form and will, in fact, appear in the sum

$$S_1 = d_1 M_0 A_1 \oplus d_2 M_1 A_2 \oplus \cdots \oplus d_{s_1} M_{s_1 - 1} A_{s_1}$$

for suitably chosen integers $d_1, d_2, \cdots, d_{s_1}$ with $s_1 > 1$ and $(d_i, m_i) = 1$ for $1 \leq i \leq s_1$. S_1 is easily seen to be a complete residue system modulo M_{s_1} since A_i is a complete residue system modulo m_i and $(d_i, m_i) = 1$ for $1 \leq i \leq s_1$. Of course, all elements of S_1 are represented in the desired form. Let r_1 be the integer of least absolute value such that $r_1 \notin S_1$. If there are two such values, r and $-r$, we set $r_1 = r$. Since S_1 is a complete residue system modulo M_{s_1}, there exists $\sigma \in S_1$ such that $r_1 \equiv \sigma (\text{mod } M_{s_1})$. Thus, $r_1 = \sigma + w M_{s_1}$ for some integer w and, by Lemma 2, there exists an integer $s_2 > 1$ and integers d_{s_1+i} with $(d_{s_1+i}, m_{s_1+i}) = 1$ for $1 \leq i \leq s_2$ such that

$$w = a_{w,s_1+1} d_{s_1+1} \frac{M_{s_1}}{M_{s_1}} + a_{w,s_1+2} d_{s_1+2} \frac{M_{s_1+1}}{M_{s_1}} + \cdots + a_{w,s_1+s_2} d_{s_1+s_2} \frac{M_{s_1+s_2-1}}{M_{s_1}} \tag{19}$$

with $a_{w,s_1+i} \in A_{s_1+i}$ for each i. Also, since $\sigma \in S_1$,

$$\sigma = a_{\sigma,1} d_1 M_0 + a_{\sigma,2} d_2 M_1 + \cdots + a_{\sigma,s_1} d_{s_1} M_{s_1-1} \tag{20}$$

with $a_{\sigma,i} \in A_i$ and $(d_i, m_i) = 1$ for $1 \leq i \leq s_1$. But then, combining (19) and (20),

$$r_1 = \sigma + wM_{s_1}$$

$$= a_{\sigma,1} d_1 M_0 + a_{\sigma,2} d_2 M_1 + \cdots + a_{\sigma,s_1} d_{s_1} M_{s_1-1}$$

$$+M_{s_1} \left(a_{w,s_1+1} d_{s_1+1} \frac{M_{s_1}}{M_{s_1}} + a_{w,s_1+2} d_{s_1+2} \frac{M_{s_1+1}}{M_s} + \cdots + a_{w,s_1+s_2} d_{s_1+s_2} \frac{M_{s_1+s_2-1}}{M_{s_1}} \right)$$

$$= a_{\sigma,1} d_1 M_0 + \cdots + a_{\sigma,s_1} d_{s_1} M_{s_1-1} + a_{w,s_1+1} d_{s_1+1} M_{s_1} + \cdots + a_{w,s_1+s_2} d_{s_1+s_2} M_{s_1+s_2-1}$$

which is a representation of r_1 in the desired form. Now form the set

$$S_2 = d_1 M_0 A_1 \oplus d_2 M_1 A_2 \oplus \cdots \oplus d_{s_1+s_2} A_{s_1+s_2} M_{s_1+s_2-1}.$$

Note that $S_1 \subset S_2$ since $0 \in A_i$ for all i and also note that all elements of S_2 are represented in the desired form. We now iterate with r_2 the integer of least absolute value not in S_2, and so on. In this way we build our $\mathfrak{A}$-base step by step and it is clear that any particular n will be properly represented after at most $2 \mid n \mid$ steps. Since it is clear that such representations are unique, the proof is complete.

REFERENCES

1. N. G. de Bruijn, On bases for the set of integers, *Publ. Math.* (Debrecen), 1 (1950), 232-242.

2. N. G. de Bruijn, Some direct decompositions of the set of integers, *Math. Comp.*, *18* (1964), 537-546.

3. C. T. Long and N. Woo, On bases for the set of integers, *Duke Math. J.*, *38* (1971), 583-590.

4. C. Swenson, Direct sum decompositions of Z, *Pacific J. Math.*, *53* (1974), 629-633.

5. C. Swenson and C. T. Long, Necessary and sufficient conditions for simple A-bases, *Pacific J. Math.*, *126* (1987), 379-384.