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6. Other questions

Many other questions could be asked (what about higher-dimensional analogues, for example?). We conclude with a theorem that identifies which lattices can be embedded in the hexagonal lattice.

Theorem 5. *Let* Γ *have Gram matrix* $\left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$, *where* $a, b, c \in \mathbb{Z}$. *(a) The hexagonal lattice* Λ *contains a scaled copy of* Γ *if and only if*

$$4ac - b^2 = 3m^2, \quad m \in \mathbb{Z} \ . \tag{13}$$

(b) Λ *contains a copy of* Γ *if and only if (13) holds and* a *is of the form (12).*

Proof. (a) The question is equivalent to asking if Λ and Γ are rationally equivalent, and the result is immediate (cf. [7], Chap. 15, Th. 3; [14], Th. 78). (b) The conditions are clearly necessary, and sufficiency follows from the formula

$$4a \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3m^2 \end{pmatrix} \begin{pmatrix} 2a & b \\ 0 & 1 \end{pmatrix} ,$$

which shows that $4a\Gamma$ (and hence Γ) is contained in Λ. ∎

Acknowledgements

We are extremely grateful to an anonymous referee for pointing out an error in our formula for $f_1(N)$ in the original version of this manuscript.

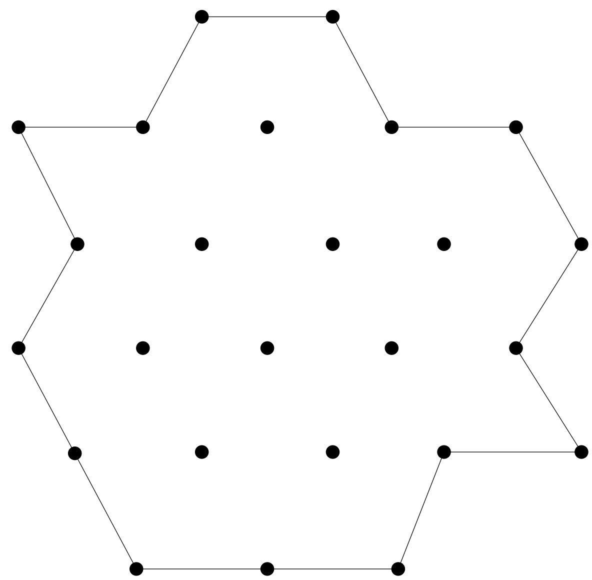

Figure 3: Minimal energy configuration for $N = 25$.

Table 3: Minimal energy configurations $\mathcal{P}$ and corresponding energy M. The lattices are the same as in Table 2, except for $N = 21$ (use $\begin{pmatrix} 21 & 0 \\ 9 & 1 \end{pmatrix}$) and $N = 24$ (use $\begin{pmatrix} 24 & 0 \\ 10 & 1 \end{pmatrix}$).

N	M	$\mathcal{P}$	N	M	$\mathcal{P}$
1	0	1	17	40	$3\overline{4}3\overline{4}3$
2	.5	2	18	46.28	$3\overline{4}43\overline{4}$
3	1	$1\overline{2}$	19	48	$3\overline{45}43$
4	2	$2\overline{2}$	20	56	$3\overline{\overline{5}}4\overline{5}3$
5	3.4	$2\overline{3}$	21	60.71	$3\overline{455}4$
6	4.83	$2\overline{2}2$	22	67.04	$4\overline{55}44$
7	6	$2\overline{3}2$	23	75.09	$5\overline{55}4\overline{5}$
8	8.62	$3\overline{3}2$	24	79	$4\overline{56}54$
9	11	$3\overline{3}3$	25	86	$2\overline{\overline{5}}5\overline{5}53$
10	13.5	$3\overline{4}3$	26	94.27	$4\overline{5}4\overline{5}53$
11	17.82	$3\overline{4}4$	27	99	$4\overline{56}543$
12	19	$3\overline{4}32$	28	108	$3\overline{\overline{5}6}653$
13	23	$3\overline{4}42$	29	121.28	$3\overline{\overline{4}55}4\overline{5}3$
14	26.5	$3\overline{4}43$	30	123.5	$4\overline{56}654$
15	32	$2\overline{\overline{4}}3\overline{4}2$	31	132	$2\overline{\overline{5}6}5\overline{6}52'$
16	35	$1\overline{\overline{4}4}43$	32	142.78	$3\overline{\overline{5}6}55\overline{5}3$

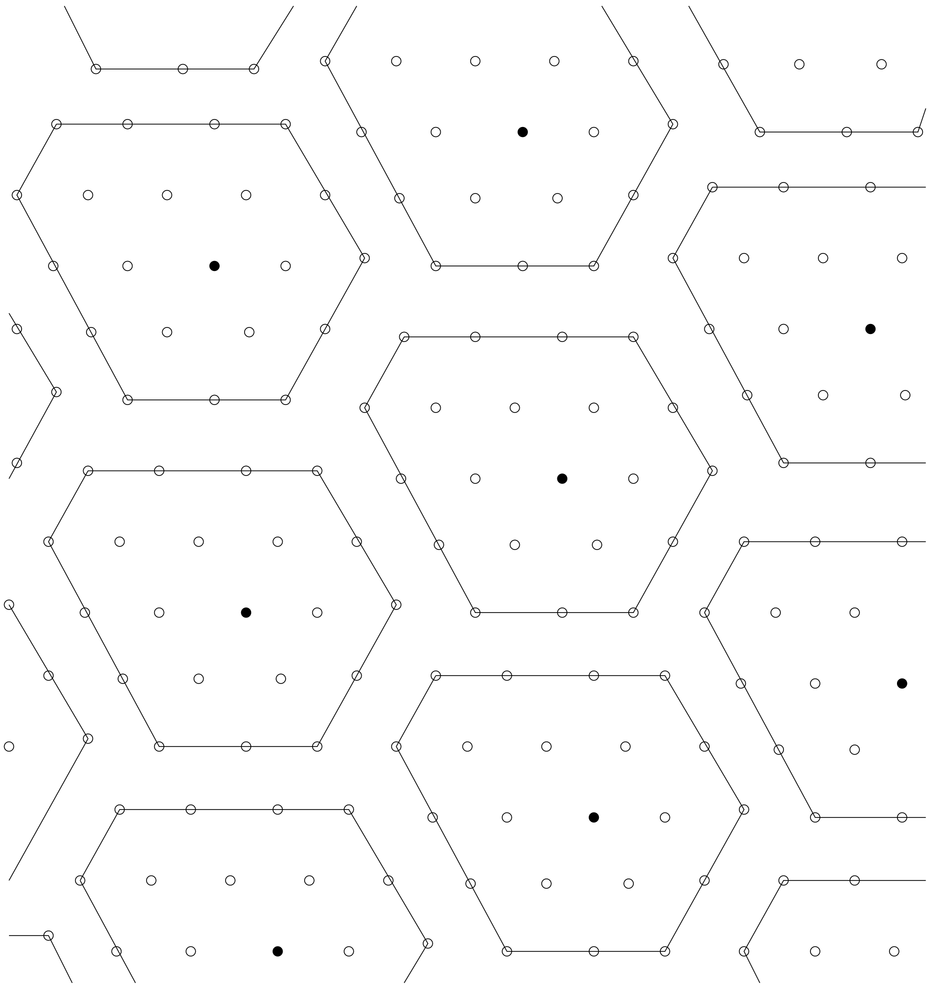

Figure 2: For $N = 21$ the minimal energy configuration (the points on the boundary of and inside any one of these regions) is achieved in a lattice which is not ideal.

indicates successive rows of w, x, $y, \ldots$ points in Λ, where each row begins just to the left or right of the previous row according as the letter does or does not have a bar on it. A double bar indicates an extra left shift. The prime in the symbol for $N = 31$ indicates an extra right shift. For example the configuration for $N = 21$ shown in Fig. 2 is described by $4\bar{5}543$. The notation unfortunately conceals the often curious shapes of these configurations. For example the symbol $2\bar{\bar{5}}5\bar{5}53$ represents the configuration shown in Fig. 3.

For $N = 1, \ldots, 10$, 12, 13, 14, 16, 19, 21, 27, 31, the optimal configurations here coincide with the minimal energy configurations found in [12] (where the points were not constrained to form a fundamental region for a sublattice of Λ). For the other values of $N \leq 32$ the two problems have different solutions, and in these cases the values of M in Table 3 are worse than the corresponding entries in Table 1 of [12].

Q2 but not Q3 is at $N = 20$, when the SNR for $\begin{pmatrix} 20 & 0 \\ 5 & 1 \end{pmatrix}$ is greater than that for $\begin{pmatrix} 20 & 0 \\ 6 & 1 \end{pmatrix}$, even though they both have minimal norm 16. In the range of Table 2 the best lattice for Q3 is also unique, but we do not know if this will be true in general.

Table 2: The lattice with basis matrix $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ is both the densest sublattice of index N in the hexagonal lattice and has the highest SNR of any such sublattice. μ denotes the minimal norm.

N	a	c	d	μ	SNR	N	a	c	d	μ	SNR
1	1	1	1	1	.671	17	17	4	1	13	25.065
2	2	1	1	1	4.931	18	18	4	1	13	25.423
3	3	2	1	3	10.214	19	19	8	1	19	26.247
4	2	2	2	4	12.713	20	20	5	1	16	26.541
5	5	2	1	3	13.747	21	21	5	1	21	27.116
6	6	3	1	4	15.687	22	22	5	1	19	27.428
7	7	3	1	7	17.574	23	23	5	1	19	27.721
8	8	3	1	7	18.579	24	24	5	1	21	28.122
9	3	3	3	9	19.756	25	5	5	5	25	28.630
10	10	3	1	7	20.267	26	26	5	1	21	28.839
11	11	3	1	7	20.802	27	9	6	3	27	29.299
12	6	4	2	12	22.255	28	14	6	2	28	29.615
13	13	4	1	13	22.951	29	29	5	1	21	29.577
14	14	4	1	12	23.473	30	30	9	1	27	30.147
15	15	4	1	13	24.017	31	31	6	1	31	30.500
16	4	4	4	16	24.754	32	32	6	1	28	30.704

5. Q4: Minimal energy sublattices

To answer Q4 we must choose both a sublattice Γ of index N and a fundamental region for Γ such that the set $\mathcal{P}$ of N lattice points in that region has minimal energy M. For any given N this is a finite problem, which we have solved for $N \leq 32$, and the results are given in Table 3. Unfortunately we have not been able to discover any general results. For $N = 21$ the best $\mathcal{P}$ does not arise from the ideal sublattice, so there is no analogue of Theorems 3 and 4. In fact for $N = 21$ the best lattice has basis matrix $\begin{pmatrix} 7 & 0 \\ 5 & 3 \end{pmatrix}$, and the best configuration $\mathcal{P}$ is shown in Fig. 2, which also displays the corresponding tiling of Λ by copies of $\mathcal{P}$. This lattice has minimal norm $\mu = 19$, whereas the ideal sublattice $\begin{pmatrix} 21 & 0 \\ 5 & 1 \end{pmatrix}$ of course has $\mu = 21$, by Theorem 3 (see also Table 2).

The configurations $\mathcal{P}$ are described in Table 3 in a condensed notation. A symbol $wxy\ldots$

Proof. This follows easily from the facts that the hexagonal lattice is the unique densest lattice packing in dimension 2, and that any endomorphism of this lattice corresponds to multiplication by an Eisenstein integer (cf. [7]).

Theorem 4. *The interference $\sigma(\Gamma)$ for a sublattice Γ of index N satisfies $\sigma(\Gamma) \geq \sigma(\Lambda)/N^2$, with equality if and only if Γ is an ideal sublattice.*

Proof. This follows immediately from Rankin's theorem [15] that the hexagonal lattice minimizes the value of the Epstein zeta-function of a lattice for all $s \geq 1.035$. ∎

In fact it follows from the work of Rankin [15], Cassels [5], Ennola [9] and Diananda [8] that the analogue of Theorem 4 holds when the exponent 4 in (2), (3) is replaced by any number greater than 2.

The interference for the hexagonal lattice itself is easily calculated from the formula ([15])

$$\sum_{\substack{a,b\in\mathbb{Z}\\(a,b)\neq(0,0)}} \frac{1}{(a^2-ab+b^2)^s} = 6\,\zeta(s)L(s), \quad s>1 ,$$

where $\zeta(s)$ is the Riemann zeta function and $L(s)$ is the Dirichlet series

$$1-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{5^s}+\frac{1}{7^s}-\frac{1}{8^s}+\cdots .$$

Setting $s=2$ we obtain

$$\sigma = 7.711145\ldots, \quad SNR = 10\log_{10}\frac{9}{\sigma} = .6712\ldots$$

for Λ itself, and then Theorem 4 implies that if Γ is a sublattice of index N,

$$SNR \leq .6712\ldots + 20\log_{10} N \ , \tag{11}$$

with equality if and only if Γ is an ideal sublattice.

Theorems 3 and 4 give the answers to Q2 and Q3 when an ideal sublattice of index N exists, that is when N is of the form

$$N = 3^k \prod_{p_i\equiv 1(3)} p_i^{l_i} \prod_{q_j\equiv -1(3)} q_j^{2m_j} \ , \tag{12}$$

where k, l_i, m_j are integers and p_i, q_j are primes (cf. [3, §4.4]). For other values of N there does not seem to be any general rule to identify which sublattices are best. Table 2 shows the answers to Q2 and Q3 for $N \leq 32$. In this range the best lattice for Q3 is also a best lattice for Q2, and we conjecture that this is always true. The first time there is a lattice which solves

Proof. An arbitrary sublattice Γ of index N in Λ can be written in a unique way as $\Gamma = m\Gamma'$, where Γ' is a primitive sublattice (of index N/m^2 in Λ). ■

The values of $f_1(N)$ and $f(N)$ for $N \le 100$ are given in Table 1. (Altshuler [1] gave a table for $N \le 24$. His value for $f(16)$ is incorrect.)

Table 1: Numbers of primitive sublattices ($f_1(N)$) and all sublattices ($f(N)$) of index N in hexagonal lattice.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$f_1(N)$	1	1	2	2	2	3	3	4	3	4	3	6	4	5	6	6	4	7	5	8
$f(N)$	1	1	2	3	2	3	3	5	4	4	3	8	4	5	6	9	4	8	5	10

N	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
$f_1(N)$	8	7	5	12	6	8	7	10	6	14	7	10	10	10	10	14	8	11	12	16
$f(N)$	8	7	5	15	7	8	9	13	6	14	7	15	10	10	10	20	8	11	12	20

N	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$f_1(N)$	8	18	9	14	14	13	9	20	11	16	14	16	10	19	14	20	16	16	11	28
$f(N)$	8	18	9	17	16	13	9	28	12	17	14	20	10	22	14	25	16	16	11	34

N	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
$f_1(N)$	12	17	18	18	16	26	13	20	18	26	13	28	14	20	22	22	18	30	15	28
$f(N)$	12	17	21	27	16	26	13	24	18	26	13	40	14	20	24	27	18	30	15	38

N	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
$f_1(N)$	19	22	15	36	20	23	22	28	16	38	22	26	24	25	22	36	18	29	26	32
$f(N)$	23	22	15	44	20	23	22	35	16	42	22	31	24	25	22	51	18	30	29	41

4. Q2, Q3: Greatest minimal norm and maximal signal-to-noise ratio

We can identify Λ with the ring of Eisenstein integers $\mathbb{Z}[\omega]$, a principal ideal domain. An ideal of $\mathbb{Z}[\omega]$ has the form $z\Lambda$ for some $z = a + b\omega$, where a, b are rational integers. This ideal corresponds to a sublattice of index $N = z\overline{z} = a^2 - ab + b^2$ which is geometrically similar to Λ. We call these the *ideal* sublattices of Λ. The term is particularly appropriate in view of the next two theorems, which show that when they exist such sublattices answer both Q2 and Q3.

Theorem 3. *The minimal norm μ of any sublattice Γ of index N satisfies $\mu \le N$, with equality if and only if Γ is an ideal sublattice.*

Any sublattice Γ of index N in Λ is obtained by defining a homomorphism ϕ from Λ onto $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ and taking $\Gamma = \ker \phi$. We specify ϕ by its values at the lattice points $1, \omega$ and $\overline{\omega}$; of course we must have $\phi(1) + \phi(\omega) + \phi(\overline{\omega}) = 0$. Clearly multiplication of ϕ by a unit of $\mathbb{Z}_N$ does not affect Γ.

Suppose Γ has rotational symmetry. After multiplication by a unit we may assume $\phi(1) = 1$, $\phi(\omega) = k$, $\phi(\overline{\omega}) = k^2$, where

$$k^2 + k + 1 \equiv 0 \pmod{N} .$$

It is easily seen that the number of solutions to this congruence is ν_1, as given in (6) (see Eq. (4) of [10], where however the formula is marred by an unfortunate misprint). This is the number of lattices with rotational symmetry, and should have been divided by 2 (the number of reflections) rather than 6 — hence the correction term $(1/2 - 1/6)\nu_1 = \nu_1/3$ in (5).

Suppose Γ has only reflectional symmetry. After multiplication by a unit we have one of the following (necessarily distinct) cases:

$$\begin{aligned} &\phi(1) = 1, \quad \phi(\omega) = k, \quad \phi(\overline{\omega}) = -k-1, \\ &\phi(1) = -k-1, \quad \phi(\omega) = 1, \quad \phi(\overline{\omega}) = k, \\ &\phi(1) = k, \quad \phi(\omega) = -k-1, \quad \phi(\overline{\omega}) = 1, \end{aligned} \tag{9}$$

where $k^2 \equiv 1 \pmod{N}$. Again an easy counting argument show that the number of solutions to this congruence is $2^{n-1+\nu_2}$, where ν_2 is given by (7) (compare Eq. (5) of [10], which gives a formula, again with an omission — the product should only involve odd primes — for the number of solutions to $k^2 + 1 \equiv 0 \pmod{N}$). Therefore the number of distinct sublattices of Λ with reflectional symmetry is $3.2^{n-1+\nu_2}$. This number should have been divided by 3 (the number of rotations) rather by 6, which accounts for the correction term

$$\left(\frac{1}{3} - \frac{1}{6}\right) 3.2^{n-1+\nu_2} = 2^{n-2+\nu_2}$$

in (5). ■

Theorem 2. *The number of inequivalent sublattices of Λ of index N is*

$$f(N) = \sum_{m^2 | N} f_1\left(\frac{N}{m^2}\right) . \tag{10}$$

Question Q4 asks for the minimal-energy arrangements of pennies that are fundamental regions of sublattices of Λ of index N. In other words, find those arrangements of N pennies which have minimal energy subject to admitting a lattice tiling. Rather surprisingly, the lattices that yield the best answers to Q4 are not always the best from the point of view of Q2 or Q3.

3. Q1: The number of sublattices

Let Γ be a sublattice of Λ of index N, with basis vectors $v_1 = a + b\omega$, $v_2 = c + d\omega$, where a, b, c, d are integers with $ad - bc = N$. We call $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ a *basis matrix* for Γ.

Since Λ has rank 2, the quotient group $Q = \Lambda/\Gamma$ is either a cyclic group C_N, in which case we say Γ is *primitive*, or a direct product $C_{N/m} \times C_m$ of a pair of cyclic groups with m dividing N/m, so $m^2|N$.

Theorem 1. *Let $N = \prod_{i=1}^{n} p_i^{k_i}$. The number of inequivalent primitive sublattices of Λ of index N, $f_1(N)$, is*

$$\frac{1}{6} N \prod_{i=1}^{n} \left(1 + \frac{1}{p_i}\right) + \frac{\nu_1}{3} + 2^{n-2+\nu_2} \ , \tag{5}$$

where

$$\nu_1 = \begin{cases} 0 \quad \textit{if} \quad 2|N \quad \textit{or} \quad 9|N \ , \\ \prod\limits_{\substack{i=1 \\ p_i>3}}^{n} \left(1 + \left(\frac{p_i}{3}\right)\right) \quad \textit{otherwise} \ , \end{cases} \tag{6}$$

and

$$\nu_2 = \begin{cases} 2 \quad \textit{if} \quad N \equiv 0 \pmod 8 \ , \\ 1 \quad \textit{if} \quad N \equiv 1, 3, 4, 5 \quad \textit{or} \quad 7 \pmod 8 \ , \\ 0 \quad \textit{if} \quad N \equiv 2 \quad \textit{or} \quad 6 \pmod 8 \ . \end{cases} \tag{7}$$

Proof. The number of primitive sublattices of index N in a generic two-dimensional lattice is

$$N \prod_{i=1}^{n} \left(1 + \frac{1}{p_i}\right) \tag{8}$$

— see [16, p. 134, Theorem 8]. Because we are only counting sublattices of Λ up to rotation and reflection, we should divide (8) by 6, to take the symmetries of Λ into account. The additional terms involving ν_1 and ν_2 in (5) are needed to compensate for the sublattices of Λ which already have rotational or reflectional symmetry: in these cases, we should have divided only by 2 or by 3 respectively. To determine the numbers of such sublattices we argue as follows.

a given index. Surprisingly, the solution does not seem to be given in the literature, although with the help of [18] we were able to locate a paper by Altshuler [1] that treats an equivalent problem (formulated in terms of counting certain Hamiltonian maps on a torus). Altshuler does not however find the answer, which we give in Theorem 2 (see also Table 1).

(Q2) As in [7] we define the *norm* of a vector v to be its squared length $v \cdot v = |v|^2$, and the *minimal norm* of a lattice Γ to be

$$\min_{v \in \Gamma, v \neq 0} v \cdot v \ . \tag{1}$$

Question Q2 is equivalent to asking for the sublattices of Λ of index N and maximal density (cf. [7], Chap. 1).

(Q3) If the nonzero points of the sublattice Γ are regarded as transmitters which interfere with a transmitter at the origin, a standard measure of the total *interference* is given by

$$\sigma = \sum_{v \in \Gamma, v \neq 0} \frac{1}{|v|^4} \ , \tag{2}$$

and the *signal-to-noise ratio* for this sublattice is

$$SNR = 10 \log_{10} \frac{R^{-4}}{\sigma} db \ , \tag{3}$$

where R ($= 1/\sqrt{3}$ on our standard scale) is the covering radius of Λ. Question Q3 asks which sublattices Γ of index N minimize σ, or equivalently maximize SNR. In the range $N \leq 32$ the best lattices for Q3 are also the best lattices for Q2, but not conversely (since the answer to Q2 is in general not unique).

(Q4) If $\mathcal{P} = \{P_1, \ldots, P_k\}$ is a set of points in $\mathbb{R}^2$ with $|P_i - P_j| \geq 1$ $(i \neq j)$, its *energy* or *second moment* is defined to be

$$M = \sum_{i=1}^{k} |P_i - \overline{P}|^2 \ , \tag{4}$$

where $\overline{P} = \frac{1}{k} \sum_{i=1}^{k} P_i$ is its centroid. The interpretation of M as the energy in $\mathcal{P}$ is a standard one in communications [4], [11]. Conjecturally minimal values of M for $k \leq 100$ (and for some larger k) were given in [12], and subsequent research [6], [17] has failed to improve these values. These sets of points may equivalently be described as minimal-energy arrangements of nonoverlapping circular disks (pennies, for example). For $k \neq 4$ the putatively optimal solutions are subsets of the lattice Λ. (The case $k = 4$ is exceptional because there are infinitely many optimal solutions — see [12], Fig. 2.)

On Sublattices of the Hexagonal Lattice

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1. Questions

Let Λ denote the familiar hexagonal lattice shown in Fig. 1. We shall investigate four questions arising from digital communications, especially cellular radio:

(Q1) How many sublattices does Λ have of index N?

(Q2) Which sublattice Γ of index N has the greatest minimal norm μ (defined in (1))?

(Q3) Which sublattice has the highest signal-to-noise ratio S (defined in (2))?

(Q4) Which sublattice has a fundamental region of minimal energy M (defined in (4))?

Applications of these results will be discussed in a separate paper [2].

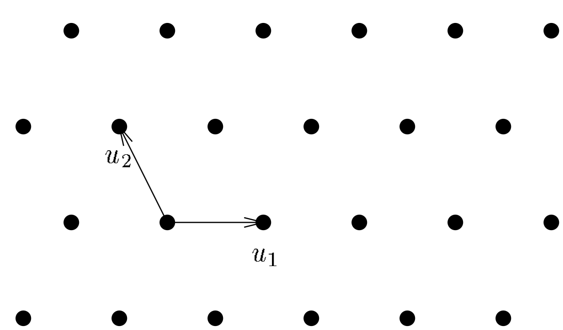

Figure 1: Hexagonal lattice Λ, showing the standard basis $u_1 = (1, 0) = 1$, $u_2 = (-1/2, \sqrt{3}/2) = \omega = e^{2\pi i/3}$.

2. Comments

We assume throughout that Λ is the hexagonal lattice defined in Fig. 1, having Gram matrix $\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$ and determinant $3/4$. The *index* of a sublattice M in a lattice L is the order of the quotient group L/M.

(Q1) Two sublattices of Λ are called *equivalent* if they differ only by a rotation and possibly a reflection that sends Λ to itself. We wish to find the number of inequivalent sublattices of

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Abstract

How many sublattices of index N are there in the planar hexagonal lattice? Which of them are the best from the point of view of packing density, signal-to-noise ratio, or energy? We answer the first question completely and give partial answers to the other questions.

* *Discrete Math.*, accepted.