SIAM J. ALG. DISC. METH.
Vol. 1, No. 4, December 1980


0196-5212/80/0104-0004 $01.00/0

# ON ADDITIVE BASES AND HARMONIOUS GRAPHS*

R. L. GRAHAM† AND N. J. A. SLOANE†

**Abstract.** This paper first considers several types of *additive bases*. A typical problem is to find $n_\gamma(k)$, the largest $n$ for which there exists a set $\{0 = a_1 < a_2 < \cdots < a_k\}$ of distinct integers modulo $n$ such that each $r$ in the range $0 \leqq r \leqq n-1$ can be written *at least* once as $r \equiv a_i + a_j$ (modulo $n$) with $i<j$. For example, $n_\gamma(8) = 24$, as illustrated by the set $\{0, 1, 2, 4, 8, 13, 18, 22\}$. The other problems arise if *at least* is changed to *at most*, or $i<j$ to $i \leqq j$, or if the words modulo $n$ are omitted. Tables and bounds are given for each of these problems. Then a closely related graph labeling problem is studied. A connected graph with $n$ edges is called *harmonious* if it is possible to label the vertices with distinct numbers (modulo $n$) in such a way that the edge sums are also distinct (modulo $n$). Some infinite families of graphs (odd cycles, ladders, wheels, $\cdots$) are shown to be harmonious while others (even cycles, most complete or complete bipartite graphs, $\cdots$) are not. In fact most graphs are not harmonious. The function $n_\gamma(k)$ is the size of the largest harmonious subgraph of the complete graph on $k$ vertices.

**1. Additive bases.** This paper is mostly concerned with *modular* versions of certain additive bases for the integers $\{1, 2, \cdots, n\}$, and with a closely related graph labeling problem, that of determining which graphs are harmonious.

Although our primary interest is in just two of these function ($n_\gamma$ and $v_\gamma$), it is most convenient to begin by defining eight closely related functions. Our notation is that $[1, n] := \{1, 2, \cdots, n\}$, $\mathbb{Z}_n$ denotes the integers modulo $n$, and $k \geqq 2$ is a natural number. The first four functions are concerned with *covering* $[1, n]$ or $\mathbb{Z}_n$ with sums.

- $n_\alpha(k)$ (resp. $n_\beta(k)$) is the *largest* number $n$ such that there exists a $k$-element set $A = \{0 = a_1 < a_2 < \cdots < a_k\}$ of integers with the property that each $r \in [1, n]$ can be written in *at least* one way as $r = a_i + a_j$, with $i < j$ (resp $i \leqq j$).
- $n_\gamma(k)$ (resp. $n_\delta(k)$) is the *largest* number $n$ such that there exists a subset $A = \{0 = a_1 < a_2 < \cdots < a_k\}$ of $\mathbb{Z}_n$ with the property that each $r \in \mathbb{Z}_n$ can be written in *at least* one way as $r = a_i + a_j$ with $i<j$ (resp. $i \leqq j$).

Since this does not assign a value to $n_\gamma(2)$ we define $n_\gamma(2) = 1$. The other four functions are concerned with *packing* $[0, v]$ or $\mathbb{Z}_v$ with sums.

- $v_\alpha(k)$ (resp. $v_\beta(k)$) is the *smallest* number $v$ such that there exists a $k$-element set $A = \{0 = a_1 < a_2 < \cdots < a_k\}$ of integers with the property that the sums $a_i + a_j$ for $i<j$ (resp. $i \leqq j$) belong to $[0, v]$ and represent each element of $[0, v]$ *at most* once.
- $v_\gamma(k)$ (resp. $v_\delta(k)$) is the *smallest* number $v$ such that there exists a subset $A = \{0 = a_1 < a_2 < \cdots < a_k\}$ of $\mathbb{Z}_v$ with the property that each $r \in \mathbb{Z}_v$ can be written in *at most* one way as $r = a_i + a_j$ with $i<j$ (resp. $i \leqq j$).

Although $n_\gamma$ and $v_\gamma$ do not seem to have been studied before, the other functions have an extensive literature. For example $n_\beta$ is the subject of a series of papers by Rohrbach, Moser, Hämmerer, Hofmeister, and others ([36], [37], [46], [59]–[61], [65], [79]) who refer to the set $A$ as an *interval basis* (*Abschnittsbasis*), or 2-*basis*, and by Lunnon and others ([1], [1a], [43a], [56], [76]) under the name of the *postage stamp problem*. $n_\delta$ was briefly mentioned by Rohrbach in [66]. The functions $v_\beta$ and $v_\delta$ have been studied by Singer, Erdös, Turán, Bose, Chowla and others (see [11], [21], [40, Chapt. II]). The set $A$ associated with $v_\beta$ is often called a $B_2$-sequence. Other types of additive bases have been defined in [14], [19], [40], [45], [51]. (Since this paper impinges on many different parts of combinatorics we have attempted to include a fairly complete bibliography.)

---

* Received by the editors April 1, 1980.

† Bell Laboratories, Murray Hill, New Jersey 07974.

Our interest in $v_\gamma$ stems from its application to error-correcting codes. Let $A(k, 2d, w)$ denote the largest possible number of binary vectors, each containing $w$ 1's and $k-w$ 0's, such that any two vectors differ in at least $2d$ places ([6], [57]). It can be shown ([28], [29]) that

$$A(k, 6, w) \geqq \frac{1}{v_\gamma(k)} \binom{k}{w},$$

and there is a similar bound for $A(k, 2d, w)$ using sets in which all sums of $d-1$ distinct elements are distinct modulo $v$. When combined with Theorem 1, this implies

$$A(k, 6, w) \geqq \frac{k^{w-2}}{w!}(1+o(1)) \quad \text{as } k \to \infty,$$

which is stronger than any previously known bound (see [28]). We should also point out that the function $A(k, 2d, w)$ has been studied under another guise in extremal set theory by Erdös, Hanani, Schönheim, Kalbfleisch, Stanton and others (see [20], [73], [77]) in the following context. Let $D(t, k, s)$ denote the maximum number of $k$-element subsets of an $s$-element set $S$ such that every $t$-element subset of $S$ is contained in at most one of the $k$-element subsets. Then $D(t, k, s) = A(s, 2k-2t+2, k)$.

We shall justify our interest in $n_\gamma$ in § 3.

**2. Tables, bounds and properties.** Tables I–IV give values of these eight functions, and examples of the sets $A$ which attain them. Usually the (lexicographically) first

TABLE I
*$n_\alpha(k)$ and $n_\beta(k)$.*

| $k$ | $n_\alpha(k)$ | An example of the set $A$. |
|---|---|---|
| 2 | 1 | {0, 1} |
| 3 | 3 | {0, 1, 2} |
| 4 | 6 | {0, 1, 2, 4} |
| 5 | 9 | {0, 1, 2, 3, 6} |
| 6 | 13 | {0, 1, 2, 3, 6, 10} |
| 7 | 17 | {0, 1, 2, 3, 4, 8, 13} |
| 8 | 22 | {0, 1, 2, 3, 4, 8, 13, 18} |
| 9 | 27 | {0, 1, 2, 3, 4, 5, 10, 16, 22} |
| 10 | 33 | {0, 1, 2, 3, 4, 5, 10, 16, 22, 28} |
| 11 | 40 | {0, 1, 2, 4, 5, 6, 10, 13, 20, 27, 34} |
| 12 | 47 | {0, 1, 2, 3, 6, 10, 14, 18, 21, 22, 23, 24} |
| 13 | 56 | {0, 1, 2, 4, 6, 7, 12, 14, 17, 21, 30, 39, 48} |
| 14 | 65 | {0, 1, 2, 4, 6, 7, 12, 14, 17, 21, 30, 39, 48, 57} |
| $k$ | $n_\beta(k)$ | An example of the set $A$. |
| 2 | 2 | {0, 1} |
| 3 | 4 | {0, 1, 2} |
| 4 | 8 | {0, 1, 3, 4} |
| 5 | 12 | {0, 1, 3, 5, 6} |
| 6 | 16 | {0, 1, 3, 5, 7, 8} |
| 7 | 20 | {0, 1, 2, 5, 8, 9, 10} |
| 8 | 26 | {0, 1, 2, 5, 8, 11, 12, 13} |
| 9 | 32 | {0, 1, 2, 5, 8, 11, 14, 15, 16} |
| 10 | 40 | {0, 1, 3, 4, 9, 11, 16, 17, 19, 20} |
| 11 | 46 | {0, 1, 2, 3, 7, 11, 15, 19, 21, 22, 24} |
| 12 | 54 | {0, 1, 2, 3, 7, 11, 15, 19, 23, 25, 26, 28} |
| 13 | 64 | {0, 1, 3, 4, 9, 11, 16, 21, 23, 28, 29, 31, 32} |
| 14 | 72 | {0, 1, 3, 4, 9, 11, 16, 20, 25, 27, 32, 33, 35, 36} |

TABLE II
*$n_\gamma(k)$ and $n_\delta(k)$.*

| $k$ | $n_\gamma(k)$ | An example of the set $A$. | $k$ | $n_\delta(k)$ | An example of the set $A$. |
|---|---|---|---|---|---|
| 2 | 1 | — | 2 | 3 | {0, 1} |
| 3 | 3 | {0, 1, 2} | 3 | 5 | {0, 1, 2} |
| 4 | 6 | {0, 1, 2, 4} | 4 | 9 | {0, 1, 3, 4} |
| 5 | 9 | {0, 1, 2, 4, 7} | 5 | 13 | {0, 1, 2, 6, 9} |
| 6 | 13 | {0, 1, 2, 3, 6, 10} | 6 | 19 | {0, 1, 3, 12, 14, 15} |
| 7 | 17 | {0, 1, 2, 3, 4, 8, 13} | 7 | 21 | {0, 1, 2, 3, 4, 10, 15} |
| 8 | 24 | {0, 1, 2, 4, 8, 13, 22} | 8 | 30 | {0, 1, 3, 9, 11, 12, 16, 26} |
| 9 | 30 | {0, 1, 2, 4, 10, 15, 17, 22, 28} | 9 | 35 | {0, 1, 2, 7, 8, 11, 26, 29, 30} |
| 10 | 36 | {0, 1, 2, 3, 6, 12, 19, 20, 27, 33} | | | |

TABLE III
*$v_\alpha(k)$ and $v_\beta(k)$.*

| $k$ | $v_\alpha(k)$ | An example of the set $A$. | $k$ | $v_\beta(k)$ | An example of the set $A$. |
|---|---|---|---|---|---|
| 2 | 1 | {0, 1} | 2 | 2 | {0, 1} |
| 3 | 3 | {0, 1, 2} | 3 | 6 | {0, 1, 3} |
| 4 | 6 | {0, 1, 2, 4} | 4 | 12 | {0, 1, 4, 6} |
| 5 | 11 | {0, 1, 2, 4, 7} | 5 | 22 | {0, 1, 4, 9, 11} |
| 6 | 19 | {0, 1, 2, 4, 7, 12} | 6 | 34 | {0, 1, 4, 10, 12, 17} |
| 7 | 31 | {0, 1, 2, 4, 8, 13, 18} | 7 | 50 | {0, 1, 4, 10, 18, 23, 25} |
| 8 | 43 | {0, 1, 2, 4, 8, 14, 19, 24} | 8 | 68 | {0, 1, 4, 9, 15, 22, 32, 34} |
| 9 | 63 | {0, 1, 2, 4, 8, 15, 24, 29, 34} | 9 | 88 | {0, 1, 5, 12, 25, 27, 35, 41, 44} |
| 10 | 80 | {0, 1, 2, 4, 8, 15, 24, 29, 34, 46} | 10 | 110 | {0, 1, 6, 10, 23, 26, 34, 41, 53, 55} |

TABLE IV
*$v_\gamma(k)$ and $v_\delta(k)$.*

| $k$ | $v_\gamma(k)$ | An example of the set $A$. |
|---|---|---|
| 2 | 2 | {0, 1} |
| 3 | 3 | {0, 1, 2} |
| 4 | 6 | {0, 1, 2, 4} |
| 5 | 11 | {0, 1, 2, 4, 7} |
| 6 | 19 | {0, 1, 2, 4, 7, 12} |
| 7 | 28 | {0, 1, 2, 4, 8, 15, 20} |
| 8 | 40 | {0, 1, 5, 7, 9, 20, 23, 35} |
| 9 | 56 | {0, 1, 2, 4, 7, 13, 24, 32, 42} |
| 10 | 72 | {0, 1, 2, 4, 7, 13, 23, 31, 39, 59} |

| $k$ | $v_\delta(k)$ | An example of the set $A$. |
|---|---|---|
| 2 | 3 | {0, 1} |
| 3 | 7 | {0, 1, 3} |
| 4 | 13 | {0, 1, 3, 9} |
| 5 | 21 | {0, 1, 4, 14, 16} |
| 6 | 31 | {0, 1, 3, 8, 12, 18} |
| 7 | 48 | {0, 1, 3, 15, 20, 38, 42} |
| 8 | 57 | {0, 1, 3, 13, 32, 36, 43, 52} |
| 9 | 73 | {0, 1, 3, 7, 15, 31, 36, 54, 63} |
| 10 | 91 | {0, 1, 3, 9, 27, 49, 56, 61, 77, 81} |
| 11 | ? | |
| 12 | 133 | {0, 1, 3, 12, 20, 34, 38, 81, 88, 94, 104, 109} |

example of $A$ is given. The entries in the $n_\beta$ table are taken from [1], [56], and [76], and the entries in the $v_\delta$ table which come from difference sets (see (8)) are taken from [2a, Table 6.1]. The other entries are believed to be new.

The best bounds presently known for these functions are as follows.

THEOREM 1.

(1) $$\tfrac{5}{18}(k-1)^2 < n_\alpha(k),\ n_\beta(k) < .4802k^2 + O(k),$$

(2) $$\tfrac{5}{18}(k-1)^2 < n_\gamma(k),\ n_\delta(k) < \tfrac{1}{2}k^2 + O(k),$$

(3) $$2k^2 - O(k^{3/2}) < v_\alpha(k),\ v_\beta(k) < 2k^2 + O(k^{36/23}),$$

(4) $$k^2 - O(k) < v_\gamma(k) < k^2 + O(k^{36/23}),$$

(5) $$k^2 - k + 1 \leqq v_\delta(k) < k^2 + O(k^{36/23}).$$

*Discussion of Proof.* Hämmerer and Hofmeister [36] showed that $n_\beta(k) > 5(k-1)^2/18$, and it is not difficult to modify their proof to give $n_\alpha(k) > 5(k-1)^2/18$. The lower bounds in (2) then follow from $n_\alpha(k) \leqq n_\gamma(k)$ and $n_\beta(k) \leqq n_\delta(k) - 1$ (see Lemma 2 below). The upper bound in (1) is due to Klotz [46]. Since there are $\binom{k}{2}$ sums $a_i + a_j (i<j)$ from a $k$-element set $A$, we have immediately that

(6) $$n_\gamma(k) \leqq \binom{k}{2} \leqq v_\gamma(k),$$

and similarly

(7) $$n_\delta(k) \leqq \binom{k+1}{2} \leqq v_\delta(k),$$

which imply the upper bound in (2). Notice that if equality holds on either side of (6) then it holds on both sides, and similarly in (7).

The lower bounds in (3) follow from a straightforward modification of the Erdös-Turán argument ([21], [40, Chapt. II, § 3, Theorem 4]); we omit the details. The lower bound in (4) will be proved at the end of this section. The lower bound in (5) holds because if the sums $a_i + a_j$ $(1 \leqq i \leqq j \leqq k)$ are distinct modulo $v$, then so are the $k(k-1)$ nonzero differences $a_i - a_j$; hence $v - 1 \geqq k^2 - k$. It follows that the equality signs can only hold in (7) when $k = 2$; thus

$$n_\delta(k) < \binom{k+1}{2} < v_\delta(k) \quad \text{for } k > 2.$$

We shall see in Theorem 5 that the equality signs can only hold in (6) when $k = 2$, 3 or 4. The upper bounds in (3)–(5) are all obtained by using Singer difference sets and the fact that (see [43]), whenever $x$ is sufficiently large, there is a prime $p$ with

$$x < p \leqq x + x^{13/23}$$

(compare [40, Chapt. II, § 3, Theorem 6]. In particular, difference sets attain the lower bound in (5), so

(8) $$v_\delta(k) = k^2 - k + 1, \quad \text{whenever } k-1 \text{ is a prime power.}$$

A projective plane of order 6 would have implied $v_\delta(7) = 43$, but since this plane does not exist we may regard the cyclic shifts of $A = \{0, 1, 3, 15, 20, 38, 42\}$ modulo 48 (corresponding to $v_\delta(7) = 48$) as giving, in a sense, the best approximation to such a plane. Other approximations are described in [40a] and [56a].

The following properties of these functions are easily established.

LEMMA 2.

$$n_\alpha(k) \leqq v_\alpha(k), \qquad n_\beta(k) \leqq v_\beta(k),$$
$$n_\gamma(k) \leqq v_\gamma(k), \qquad n_\delta(k) \leqq v_\delta(k),$$
$$n_\alpha(k) \leqq n_\beta(k), \qquad n_\gamma(k) \leqq n_\delta(k),$$
$$v_\alpha(k) \leqq v_\beta(k) - 1, \qquad v_\gamma(k) \leqq v_\delta(k),$$
$$n_\alpha(k) \leqq n_\gamma(k), \qquad n_\beta(k) \leqq n_\delta(k) - 1,$$
$$v_\alpha(k) \geqq v_\gamma(k), \qquad v_\beta(k) \geqq v_\delta(k) - 1.$$

LEMMA 3.

(a) *If* $A = \{0 = a_1 < a_2 < \cdots < a_k\}$ *attains* $n_\alpha(k)$, *then* $a_2 = 1, a_3 = 2, a_4 = 3$ *or* 4, *and* $a_k \leqq n_\alpha(k-1) + 1$. *Furthermore,*

$$n_\alpha(k) + 3 \leqq n_\alpha(k+1), \quad \text{for } k \geqq 3.$$

(b) *If* $A = \{0 = a_1 < a_2 < \cdots < a_k\}$ *attains* $n_\beta(k)$, *then* $a_2 = 1, a_3 = 2$ *or* 3, $a_4 = 3, 4$ *or* 5, *and* $a_k \leqq n_\beta(k-1) + 1$. *Furthermore,*

$$n_\beta(k) + 2 \leqq n_\beta(k+1).$$

(c) *If* $A = \{0 = a_1 < a_2 < \cdots < a_k\}$ *attains* $n_\gamma(k)$ (*or* $n_\delta(k)$, $v_\gamma(k)$ *or* $v_\delta(k)$) *and if* $n_\gamma(k)$ (*or* $n_\delta(k)$, $v_\gamma(k)$, $v_\delta(k)$) *is of the form* $p^r q^s$, *with* $p$, $q$ *prime*, $r, s \geqq 0$, *then we may assume that* $a_2 = 1$.

*Proof.* (a) and (b) are straightforward. (c) Suppose $A = \{0 = a_1 < a_2 < \cdots < a_k\}$ is such that the sums $a_i + a_j$ $(i < j)$ cover $\mathbb{Z}_n$, where $n = n_\gamma(k) = p^r q^s$. If some $a_i$ is relatively prime to $n$ then $A' = a_i^{-1} A$ contains 0 and 1, and also attains $n_\gamma(k) = n$. If not, since not all the $a_i$ can be divisible by $p$, nor by $q$, we can find $a_t$ and $a_u$ such that $p | a_t$, $q \nmid a_t$, $p \nmid a_u$, $q | a_u$. Then $a_t - a_u$ is relatively prime to $n$, and $(a_t - a_u)^{-1}(A - a_u)$ contains 0 and 1 and attains $n_\gamma(k) = n$. Similarly for $n_\delta$, $v_\gamma$ and $v_\delta$. Q.E.D.

Parts (a) and (b) of this Lemma simplify the computation of $n_\alpha$ and $n_\beta$ (and the absence of similar results for the other six functions makes their calculation more difficult). The calculations are further simplified by the next lemma.

LEMMA 4. *If there is no k-element set A such that the sums* $a_i + a_j$ $(i < j)$ *cover* $[1, m]$, *then* $n_\alpha(k) \leqq m - 1$; *and similarly for* $n_\beta(k)$. *If there is no k-element set A such that the sums* $a_i + a_j$ $(i < j)$ *belong to* $[0, m]$ *and are distinct, then* $v_\alpha(k) \geqq m + 1$; *and similarly for* $v_\beta(k)$.

But these properties need not hold for the modular functions. Consider for example the problem of determining $n_\gamma(8)$. The set $A = \{0, 1, 2, 3, 4, 8, 13, 18\}$ covers $\mathbb{Z}_n$ for all $n$ in the range $8 \leqq n \leqq 22$, but no 8-element set covers $\mathbb{Z}_{23}$. Nevertheless $A = \{0, 1, 2, 4, 8, 13, 18, 22\}$ covers $\mathbb{Z}_{24}$, and $n_\gamma(8) = 24$. Similarly when determining $n_\delta(6)$ we find that $A = \{0, 1, 2, 5, 7, 11\}$ covers $\mathbb{Z}_n$ for $6 \leqq n \leqq 15$, $A = \{0, 1, 2, 4, 9, 14\}$ covers $\mathbb{Z}_{16}$, $A = \{0, 1, 2, 3, 8, 12\}$ covers $\mathbb{Z}_{17}$, no 6-element set covers $\mathbb{Z}_{18}$, $A = \{0, 1, 3, 12, 14, 15\}$ covers $\mathbb{Z}_{19}$, and $n_\delta(6) = 19$.

We conclude this section by determining when the equality signs can hold in (6).

THEOREM 5.

$$n_\gamma(k) = \binom{k}{2} = v_\gamma(k) \tag{9}$$

*if and only if* $k = 2, 3$ *or* 4; *otherwise* $n_\gamma(k) < \binom{k}{2} < v_\gamma(k)$.

*Proof.*

(i) If (9) holds and $k \equiv 0$ or 1 (modulo 4) then $k$ must be a perfect square (cf. [80]). For in this case $n = \binom{k}{2}$ is even, and so the parity of an element of $\mathbb{Z}_n$ is well defined. Let $A \subseteq \mathbb{Z}_n$ attain $n_\gamma(k) = v_\gamma(k) = \binom{k}{2}$, and suppose $\alpha$ of the $a_i$ are odd and $\beta$ are even, with $\alpha + \beta = k$. The number of odd sums $a_i + a_j$ $(i<j)$ is $\alpha\beta = \frac{1}{2}\binom{k}{2}$, hence $(\alpha-\beta)^2 = k^2 - k(k-1) = k$ is a perfect square.

(ii) Equation (9) holds for $n = 2, 3, 4$ (see Tables II, IV), part (i) eliminates $k = 5$, 8, 12 and 13, and a computer search elminated $k = 6, 7, 9, 10$ and 11.

(iii) The values of $k \geqq 14$ are eliminated by the following lemma. Q.E.D.

LEMMA 6. *Suppose $A = \{0 = a_1 < a_2 < \cdots < a_k\}$ is a subset of $\mathbb{Z}_n$ such that the sums $a_i + a_j$ $(i<j)$ represent each element of $\mathbb{Z}_n$ at most once. Let $u = [n/3]$ and assume*

$$k \leqq u - 1. \tag{10}$$

*Then*

$$k^2u^2 \leqq n\{u(u-1) + 3ku - k(k+1)\}. \tag{11}$$

*Proof.* The proof is a modification of the Erdös-Turán argument ([21], [40, p. 86]). Consider the $n$ subsets

$$\mathcal{I}_m = \{m, m+1, \cdots, m+u-1\}$$

of $\mathbb{Z}_n$, for $0 \leqq m \leqq n-1$, and let $A_m = |\mathcal{I}_m \cap A|$. Since each $a_i$ belongs to exactly $u$ subsets,

$$\sum_{m=0}^{n-1} A_m = ku. \tag{12}$$

Let $T$ be the number of triples $(a_i, a_j, m)$ with $1 \leqq i < j \leqq k$ and $a_i \in \mathcal{I}_m$, $a_j \in \mathcal{I}_m$. The number of pairs $(a_i, a_j)$ contained in $\mathcal{I}_m$ is $\frac{1}{2}A_m(A_m - 1)$, so

$$T = \frac{1}{2}\sum_{m=0}^{n-1} A_m(A_m - 1). \tag{13}$$

From (12), (13) and Cauchy's inequality,

$$T \geqq \frac{k^2u^2}{2n} - \frac{ku}{2}. \tag{14}$$

For $i \leqq j$ let

$$\rho(a_i, a_j) = \min\{a_j - a_i, n - a_j + a_i\}.$$

If $a_i$ and $a_j$, $i<j$, are contained in $\mathcal{I}_m$, $\rho(a_i, a_j)$ is an integer $d$ with $1 \leqq d \leqq u-1$. Conversely, given $d \in [1, u-1]$, how many pairs $(a_i, a_j)$ satisfy $i<j$, $\rho(a_i, a_j) = d$? It is easly seen that the answer is 0, 1 or 2. If there is one solution we call $d$ *ordinary*, if two, *special.* A special $d$ is associated with a unique triple $a_i$, $a_j$, $a_k$ with

$$2a_j = a_i + a_k, \qquad \rho(a_i, a_j) = \rho(a_j, a_k).$$

Since there is at most one special $d$ associated with $a_j$, there are at most $k$ special $d$'s. An ordinary $d$ contributes $u-d$ to $T$ since the unique pair $(a_i, a_j)$ with $\rho(a_i, a_j) = d$ is contained in exactly $u-d$ of the sets $\mathcal{I}_m$. Similarly, a special $d$ contributes $2(u-d)$ to $T$, and so

$$T = \sum_{d \text{ ordinary}} (u-d) + \sum_{d \text{ special}} 2(u-d)$$
$$\leqq \sum_{d=1}^{u-1} (u-d) + \sum_{\nu=1}^{s} (u-d_\nu),$$

where $d_1, \cdots, d_s$ are the special values of $d$, with $s \leqq k$. Using (10) we can bound this by

(15)
$$T \leqq \tfrac{1}{2}u(u-1) + ku - (1+2+\cdots+k)$$
$$= \tfrac{1}{2}u(u-1) + ku - \tfrac{1}{2}k(k+1),$$

and (11) follows from (14) and (15). Q.E.D.

COROLLARIES.

(i) *If we set* $n = \binom{k}{2}$, *then for* $k \geqq 14$ (10) *is satisfied but* (11) *is not, which eliminates the cases* $k \geqq 14$ *of Theorem* 5.

(ii) *For n large*, (11) *implies*

$$k \leqq \sqrt{n} + O(1),$$

*which is equivalent to the lower bound in* (4).

**3. Harmonious graphs.** We call a connected graph with $v$ nodes and $e \geqq v$ edges *harmonious* if it is possible to label the nodes $x$ with distinct elements $\lambda(x)$ of $\mathbb{Z}_e$ in such a way that, when each edge $xy$ is labeled with $\lambda(x)+\lambda(y)$, the resulting edge labels are distinct. If the graph is a tree (with $v$ nodes and $e = v-1$ edges) we require exactly one node label to be repeated. Such a labeling of the nodes and edges is called a *harmonious labeling* of the graph. In a harmonious labeling the node labels are distinct (or contain exactly one duplicate, if the graph is a tree), and the induced edge labels are 0, $1, \cdots, e-1$. Fig. 1 shows some harmonious graphs with 5 nodes, and Fig. 2 gives harmonious labelings of all trees with 7 nodes.

Harmonious graphs arise naturally out of the problems considered in § 1. For if $n_\gamma(v) = v_\gamma(v) = \binom{v}{2}$ is attained by a set $A = \{a_1, \cdots, a_v\}$, for $v \geqq 3$, then $a_1, \cdots, a_v$ is a harmonious labeling of $K_v$, the complete graph on $v$ nodes. From Theorem 5 we obtain:

THEOREM 7. *The complete graph on v nodes is harmonious if and only if* $v \leqq 4$ (*see Fig.* 3).

For larger values of $v$ it is natural to ask how large a subgraph of $K_v$ can be harmonious. From the definition in § 1 we see that the answer is given by:

$n_\gamma(v)$ is the greatest number of edges in
any harmonious graph on $v$ nodes.

For if $A = \{a_1, \cdots, a_v\}$ attains $n_\gamma(v)$, we label the nodes of $K_v$ with $a_1, \cdots, a_v$ and omit any edge whose label has already appeared. Since by definition the sums $a_i + a_j$ $(i<j)$ cover $\mathbb{Z}_e$, every edge label appears at least once. For example Fig. 1(n) shows the largest harmonious graph on 5 nodes, corresponding to the value $n_\gamma(5) = 9$, which is attained by $A = \{0, 1, 2, 4, 7\}$. One of the two edges labeled 2 has been omitted from $K_5$.

Although many other ways of labeling graphs have been studied in the literature ([8], [9], [25], [49], [54], [67]), this one appears to be new. However, there are many similarities between harmonious graphs and what are called graceful graphs. A connected graph with $v$ nodes and $e \geqq v-1$ edges is *graceful* if it is possible to label the nodes $x$ with distinct integers $\mu(x)$ from $\{0, 1, \cdots, e\}$ in such a way that, when each edge $xy$ is labeled with $|\mu(x)-\mu(y)|$, the resulting edge labels are distinct (and therefore

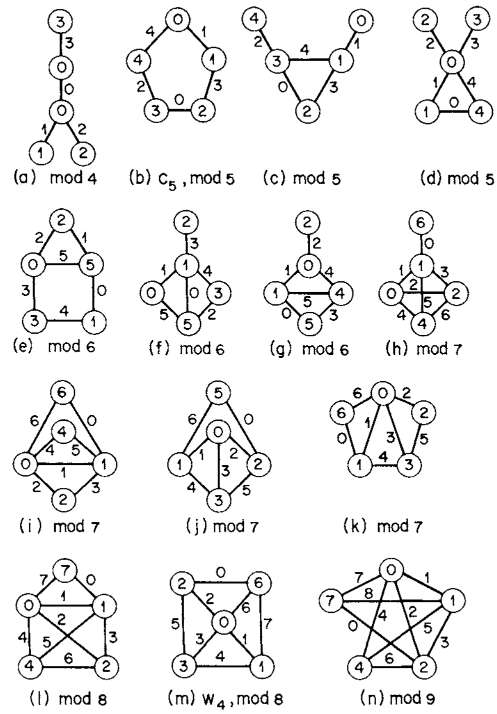


FIG. 1. *Some harmonious graphs with 5 nodes.*

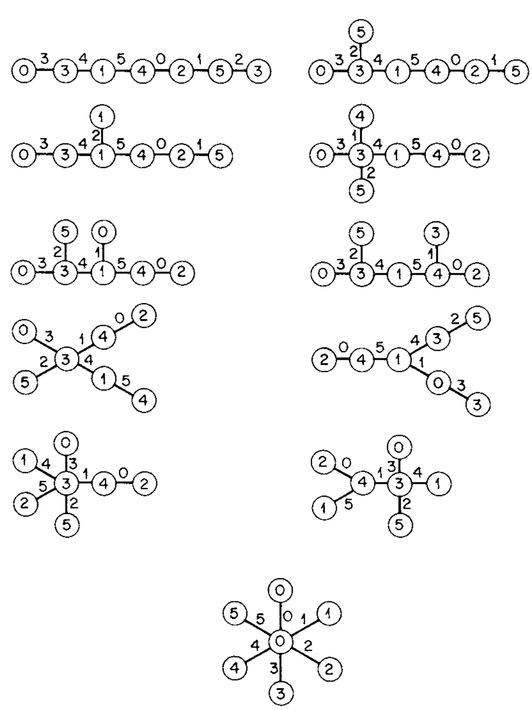

FIG. 2. *Harmonious labelings of the trees with 7 nodes (modulo 6).*

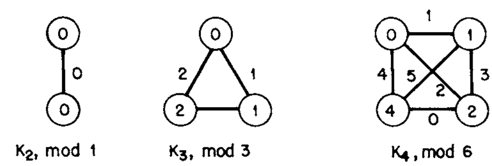


FIG. 3. *The complete graphs* $K_2$, $K_3$ *and* $K_4$.

all values in $\{1, 2, \cdots, e\}$ appear uniquely). A graceful labeling of a graph is also called a *β-valuation* or a *restricted difference basis*. These have an extensive literature ([3]–[5], [7]–[10], [15], [22]–[27], [30], [31], [33]–[35], [42], [44], [47], [48], [50], [53], [57a], [58], [64], [67], [69], [74], [78], [81]).

We are interested in determining which graphs are harmonious. The principal results we have obtained are summarized in Table V, which shows which graphs are harmonious ($H$) and for comparison which are graceful ($G$). The entries in the table are explained in the remaining sections.

TABLE V
*Comparison of harmonious and graceful graphs.*

| Graph | Harmonious? | Graceful? |
|---|---|---|
| Caterpillars | $H$ (§ 5) | $G$ [15], [67] |
| Trees | Conjectured to be $H$, true for ≦9 nodes | Conjectured to be $G$, true for ≦16 nodes [7], [67] |
| Cycle $C_{4m}$ | Not $H$ (§ 6) | $G$ [10], [64] |
| Cycle $C_{4m+1}$ | $H$ (§ 6) | Not $G$ [10], [64]. |
| Cycle $C_{4m+2}$ | Not $H$ (§ 6) | Not $G$ [10], [64] |
| Cycle $C_{4m+3}$ | $H$ (§ 6) | $G$ [10], [64] |
| Ladder $L_n$ | $H$ iff $n \geqq 3$ (§ 7) | $G$ [10, p. 121] |
| Friendship graph $F_n$ | $H$ iff $n \not\equiv 2$ (mod 4) (§ 8) | $G$ iff $n \equiv 0$ or 1 (mod 4) [4], [5] |
| Fan $f_n$ | $H$ (§ 9) | $G$ (§ 9) |
| Wheel $W_n$ | $H$ (§ 10) | $G$ [22], [44] |
| Complete graph $K_n$ | $H$ iff $n \leqq 4$ (Theorem 7) | $G$ iff $n \leqq 4$ [25], [74] |
| Complete bipartite $K_{m,n}$ | $H$ iff $m$ or $n = 1$ (Theorem 19) | $G$ [25], [67] |
| Small graphs | All with ≦5 nodes are $H$ except for 5 (Fig. 15) | All with ≦5 nodes are $G$ except for 3 [25] |
| Petersen | $H$ (Fig. 16) | $G$ [25] |
| Cube, octahedron | Not $H$ (Theorem 22) | $G$ [25] |
| Icosahedron | $H$ (Fig. 17) | $G$ [23] |
| Dodecahedron | $H$ (Fig. 18) | $G$ [23] |
| Most graphs | Not $H$ (Theorem 23) | Not $G$ (Theorem 24) |

Several of these families of graphs were suggested by the following application. Consider a network of transmitting stations, each of which must be able to communicate with certain others—those to which it is linked in the network. The total bandwidth available is divided into $e$ channels, where $e$ is the number of links in the network, and each station $x$ is assigned a number $\lambda(x)$ from $\mathbb{Z}_e$. When $x$ and $y$ communicate they use channel number $\lambda(x)+\lambda(y)$. If the numbering is harmonious, each channel is assigned to exactly one link.

Harmonious graphs are also interesting because they lead to modular versions of various combinatorial problems. For example, a harmonious labeling of the friendship graph $F_n$ (see § 8) may be regarded as a modular generalization of the Langford-Skolem problem (see [2], [4], [17], [18], [32], [41], [52], [55], [62], [63], [68], [70], [75]), a version of that problem which does not seem to have been discussed before. Harmonious labelings of fans, wheels, complete bipartite graphs, etc. (see below) also have interesting combinatorial interpretations.

To conclude this section we mention that there is a curious geometric interpretation of the condition that a graph $G$ be harmonious. Let $P_e$ denote a fixed regular $e$-gon embedded in the plane. Then $G$ is harmonious if and only if the nodes of $G$ can be embedded into the nodes of $P_e$ so that no two edges of the embedded copy of $G$ are parallel. This follows from the observation that if the nodes of $P_e$ are labeled cyclically with $0, 1, \cdots, e-1$, then the direction of the chord joining $i$ and $j$ depends only on $i+j$ (modulo $e$). (The condition must be modified slightly if $G$ is a tree.) For example, Fig. 4 shows the graph of Fig. 1(f) embedded in a regular hexagon.

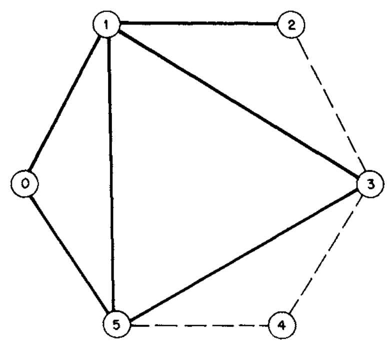


FIG. 4. *The harmonious labeling of Fig.* 1(*f*) *corresponds to an embedding of this graph in a regular hexagon in such a way that no two edges are parallel.*

**4. General properties of harmonious graphs.** The first property concerns equivalent labelings of the same graph.

THEOREM 8. *If $\lambda$ is a harmonious labeling of the nodes of a graph with e edges, then so is $a\lambda+b$, where a is an invertible element of $\mathbb{Z}_e$ and b is any element of $\mathbb{Z}_e$.*

*Proof.* The edge labels $\lambda(x)+\lambda(y)$ are changed to $a(\lambda(x)+\lambda(y))+2b$, but remain distinct. Q.E.D.

COROLLARY.

(i) *Any node in a harmonious graph can be assigned the label* 0.

(ii) *The repeated node label in a harmonious tree can be any element of $\mathbb{Z}_e$.*

On the other hand one harmonious graph may lead to others via the following constructions, which have the effect of moving an edge with a given label from one part of the graph to another.

THEOREM 9. *Let G be a harmoniously labeled graph containing* (i) *an edge wx with label $\lambda(w)+\lambda(x)$, and* (ii) *a pair of nodes y, z not joined by an edge but satisfying $\lambda(w)+\lambda(x)=\lambda(y)+\lambda(z)$. Then deleting the edge wx and inserting yz changes G to another harmonious graph.*

For example we can move the edge labeled 4 in Fig. 1(b) and obtain Fig. 1(c).

THEOREM 10. *Let G be a harmoniously labeled tree containing an edge wx labeled $\lambda(w)+\lambda(x)$, where x is an endpoint* (*of degree* 1), *and $\lambda(x)$ is the repeated node label. If y is any other node in G, we may delete edge wx and node x and replace them with a new node z and edge yz where z is labeled with $\lambda(z)=\lambda(w)+\lambda(x)-\lambda(y)$.*

For example the second and third trees in Fig. 2 are obtained from the first by moving the edge labeled 2.

The last theorem in this section gives a necessary condition for certain graphs to be harmonious.

THEOREM 11. *If a harmonious graph has an even number $e$ of edges and the degree of every node is divisible by $2^\alpha (\alpha \geqq 1)$, then $e$ is divisible by $2^{\alpha+1}$.*

*Proof.* Let node $x$ have label $\lambda(x)$ and degree $\delta(x)$. The sum of the *edge* labels is $\sum_x \delta(x)\lambda(x) \equiv 0+1+\cdots+(e-1) \equiv \frac{1}{2}e(e-1) \equiv e/2$ (modulo $e$); hence $2^\alpha$ divides $e/2$ and so $2^{\alpha+1}$ divides $e$. Q.E.D.

For example the 1-skeleton of the octahedron has 12 edges and 6 nodes, each of degree 4, so is not harmonious.

**5. Are all trees harmonious?** It is easy to see that paths and stars are harmonious (see the first and last examples in Fig. 2). More generally, let a *caterpillar* be a tree with the property that the removal of its endpoints leaves a path.

THEOREM 12. *Any caterpillar is harmonious.*

*Proof.* Draw the caterpillar as a bipartite graph, as shown in Fig. 4a, with say $l$ nodes on the left and $r$ on the right. There are $e = l + r - 1$ edges. If $e$ is odd, or if $e$ is even and $r$ is odd, choose $a \in \mathbb{Z}_e$ so that $2a = r - 1$ (in $\mathbb{Z}_e$). If $e$ and $r$ are both even, then $l$ is odd and we choose $a$ so that $2a = 1 - l$. We label the left-hand nodes $a, a+1, \cdots, a+l-1$ and the right-hand nodes $-a, 1-a, \cdots, r-1-a$, as in Fig. 4a. The full set of node labels is $\{0, 1, \cdots, e-1\}$ with either $a$ repeated (if $2a = r-1$) or $-a$ repeated (if $2a = 1-l$). The edge labels are $\{0, 1, \cdots, e-1\}$, and the graph is harmonious. Q.E.D.

We shall usually just specify the node labels and leave to the reader the straightforward verification that the labeling is harmonious.

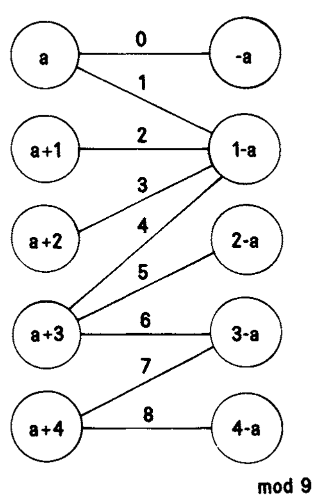


FIG. 4a. *A caterpillar with $e = 9$ edges, drawn as a bipartite graph with $l = 5$ nodes on the left and $r = 5$ nodes on the right. We obtain a harmonious labeling by choosing $a = 2$, so that $2a = r - 1$.*

By repeatedly applying the constructions of Theorems 9 and 10 to caterpillars, it is easy to generate large numbers of harmonious trees. Those with 7 nodes are shown in Fig. 2, and in the same way we have established the following theorem, whose proof is omitted.

THEOREM 13. *All trees with $\leqq 9$ nodes are harmonious.*

We conjecture that all trees are harmonious (cf. [7]).

**6. Cycles.**

THEOREM 14. *The cycle $C_n$ with n nodes, $n \geqq 3$, is harmonious if and only if n is odd.*

*Proof.* If $n$ is odd we label the nodes $0, 1, \cdots, n-1$ (see Fig. 1(b)). If $n=2m$ is even, suppose $a_0, a_1, \cdots, a_{2m-1}$ is a harmonious labeling of $C_{2m}$. The numbers $a_0+a_1, a_1+a_2, \cdots, a_{2m-1}+a_0$ are congruent (modulo $2m$) to some permutation of $0, 1, 2, \cdots, 2m-1$. Adding these numbers we obtain $2S \equiv S$ (modulo $2m$), where $S=0+1+2+\cdots+2m-1 \equiv m$ (modulo $2m$). Hence $m \equiv 0$ (modulo $2m$), a contradiction. Q.E.D.

**7. Ladders are harmonious.** The *ladder* $L_n$ ($n \geqq 2$) is the product graph $P_2 \times P_n$, and contains $2n$ nodes and $3n-2$ edges (Figs. 5, 6).

THEOREM 15. *All ladders except $L_2$ are harmonious.*

*Proof.* $L_2 = C_4$ is not harmonious by the previous theorem. $L_{2a+1}$ ($a \geqq 1$) is harmonious: label one path $0, a+1, 1, a+2, 2, a+3, \cdots$ and the other $3a+1, 2a+1, 3a+2, 2a+2, 3a+3, 2a+3, \cdots$ (Fig. 5). $L_4$ is harmonious: label the paths 0, 5, 1, 9 and 2, 6, 3, 4. Finally Fig. 6 shows a harmonious labeling of $L_{2a}$ for $a \geqq 3$. Q.E.D.

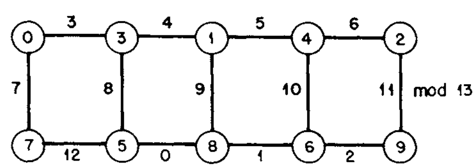


FIG. 5. *The ladder $L_5$.*

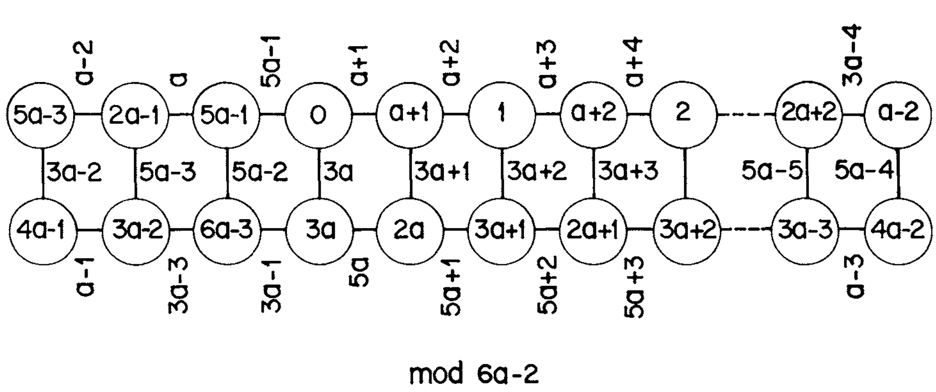


FIG. 6. *The ladder $L_{2a}$, $a \geqq 3$.*

The labeling of $L_{2a+1}$ is exceptionally pleasant since the edges are numbered consecutively. Furthermore by simply joining the ends of the ladder we obtain a harmonious labeling of the *prism* $P_2 \times C_{2a+1}$ (Fig. 7), and the pattern may be continued to produce a harmonious labeling of any $P_m \times C_{2a+1}$ (Fig. 8). The cube $P_2 \times C_4$ is not harmonious (Theorem 22 below), but $P_3 \times C_4$ is (Fig. 9).

**8. Friendship graphs.** The *friendship graph* $F_n$ ($n \geqq 1$) consists of $n$ triangles with a common vertex (see Fig. 10).

THEOREM 16. *$F_n$ is harmonious except when $n \equiv 2$* (modulo 4).

*Proof.* If $n \equiv 2$ (modulo 4), $F_n$ is not harmonious by Theorem 11. If $n \equiv 0$ or 1 (modulo 4) it was shown by Skolem [75] that the numbers $\{1, 2, \cdots, 2n\}$ may be partitioned into $n$ pairs $(a_r, b_r)$ with $b_r - a_r = r$, for $r = 1, \cdots, n$. Then a harmonious

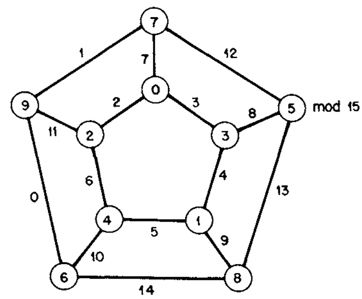


FIG. 7. *The prism* $P_2 \times C_5$.

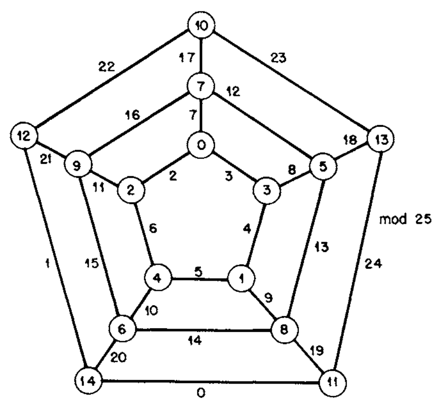


FIG. 8. *The prism* $P_3 \times C_5$.

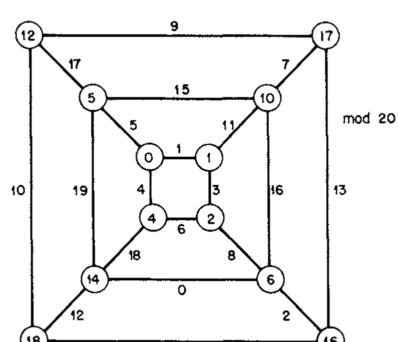


FIG. 9. *The prism* $P_3 \times C_4$.

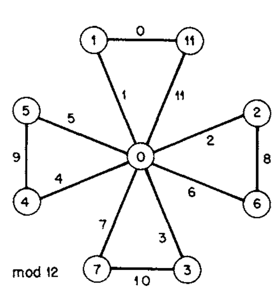


FIG. 10. *The friendship graph* $F_4$.

labeling of $F_n$ is obtained by labeling the vertices of the triangles with $(0, r, n+a_r)$, for $r = 1, \cdots, n$ (see Fig. 10). If $n \equiv 3$ (modulo 4) it is known [4, Th. 1, the case $d = 3$] that $\{1, 2, \cdots, 2n-6\}$ may be partitioned into $n-3$ pairs $(a_r, b_r)$ with $b_r - a_r = r+2$, for $r = 1, \cdots, n-3$. We label the triangles of $F_n$ with $(0, 1, 3n-1)$, $(0, 2, 3n-6)$, $(0, 3n-2, 3n-3)$, and $(0, r+2, n+a_r)$ for $r = 1, \cdots, n-3$. Q.E.D.

**9. Fans are harmonious.** The *fan* $f_n$ $(n \geqq 2)$ is obtained by joining all nodes of $P_n$ to a further node called the *center*, and contains $n+1$ nodes and $2n-1$ edges.

THEOREM 17. *$f_n$ is harmonious.*

*Proof.* Let $m = [n/2]$ and label the center with 0 and the nodes of the path with $m, n, m+1, n+1, m+2, \cdots$ (see Fig. 11).

*Remarks.*

(i) $f_{2m}$ may also be harmoniously labeled in such a way that the endpoints of the path are 1 and $-1$: label the nodes of the path with $1, 2, 5, 6, 9, 10, \cdots, 4m-3, 4m-2$.

(ii) $f_n$ is also graceful, although this fact does not seem to have been mentioned before: label the center with 0 and the nodes of the path with $2n-1$, 1, $2n-3$, 3, $2n-5, \cdots$.

(iii) Let $g_n (n \geqq 2)$ be the graph with $n+2$ nodes and $3n-1$ edges obtained by joining all nodes of $P_n$ to *two* additional nodes. A harmonious labeling of $g_{2m}$ is obtained by labeling the path with $2, 4, 8, 10, 14, 16, \cdots, 6m-4, 6m-2$, and the two additional nodes with 0 and 1 (Fig. 12). But $g_{2m+1}$ does not seem to have such a simple labeling.

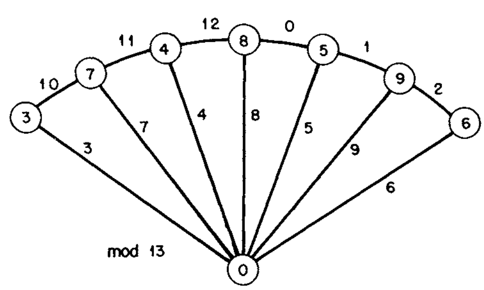


FIG. 11. *The fan $f_7$.*

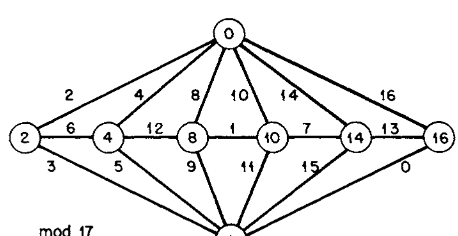


FIG. 12. *The graph $g_6$.*

**10. Wheels are harmonious.** The *wheel* $W_n$ $(n \geqq 3)$ is obtained by joining all nodes of $C_n$ to a further node called the center, and contains $n+1$ nodes and $2n$ edges (see Fig. 13). A harmonious labeling of $W_n$ is equivalent (by Theorem 8) to finding a subset $\{a_1, \cdots, a_n\}$ of $\mathbb{Z}_{2n}$ with the property that

$$a_1, \cdots, a_n, a_1 + a_2, a_2 + a_3, \cdots, a_n + a_1$$

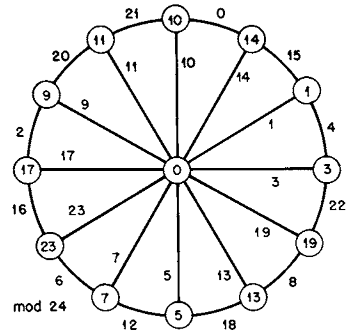


FIG. 13. *The wheel* $W_{12}$.

comprise all the elements of $\mathbb{Z}_{2n}$ (for then we may label the cycle with $a_1, \cdots, a_n$ and the center of the wheel with 0).

THEOREM 18. *$W_n$ is harmonious.*

*Proof.* The cases $W_{2m+1}$, $W_{4m}$, $W_{8m+2}$ and $W_{8m+6}$ will be handled separately. In each case the center is labeled with 0. For $W_{2m+1}$ the cycle is labeled $1, 3, 5, \cdots, 4m+1$. For $W_{4m}$ we divide the cycle into $2m$ pairs, $m$ of which will be labeled $(4i+1, 4i+3)$, $0 \leqq i \leqq m-1$; $m-1$ of which will be labeled $(4i+7, 4i+1)$, $m \leqq i \leqq 2m-2$; and one which will be labeled $(4m-2, 4m+2)$. The actual labeling of the cycle is

$$
\begin{aligned}
&4m-2, 4m+2;\\
&1, 3; 4m+7, 4m+1; 5, 7; 4m+11, 4m+5;\\
&\quad\cdots \qquad \cdots\\
&4i-7, 4i-5; 4m+4i-1, 4m+4i-7;\\
&4i-3, 4i-1; 4m+4i+3, 4m+4i-3;\\
&4i+1, 4i+3; 4m+4i+7, 4m+4i+1;\\
&\quad\cdots \qquad \cdots\\
&\quad\cdots \qquad \cdots\\
&4m-11, 4m-9; 8m-5, 8m-11;\\
&4m-7, 4m-5; 8m-1, 8m-7;\\
&4m-3, 4m-1.
\end{aligned}
$$

Fig. 13 shows the labeling of $W_{12}$. To verify that this labeling is harmonious we observe that, out of the residues modulo $8m$, all the numbers congruent to 1 or 3 (modulo 4) except $4m+3$ and $8m-3$ appear as spoke labels, and $4m+3$ and $8m-3$, together with all numbers congruent to 0 (modulo 4) appear on the perimeter. Furthermore, the numbers congruent to 2 (modulo 4) appear on the perimeter in the order $\cdots 4m+8i-6, 4m+8i-10, 4m+8i+2, 4m+8i-2, \cdots$.

For $W_{8m+2}$ ($m \geqq 1$) the cycle $C_{8m+2}$ will be labeled modulo 4 as follows:

$$\underbrace{2, 1, 2, 1, \cdots, 2, 1}_{4m+2};\quad \underbrace{1, 1, \cdots, 1}_{2m-1};\quad \underbrace{0, 0, \cdots, 0}_{2m+1}.$$

The actual labels for these three sets of nodes are

$$4m+2, 16m+1, 4m-2, 16m-3, \cdots, 12m+6, 8m+1;$$

$$4m-3, 8m-3, 4m-7, 8m-7, \cdots, 4m+5, 1;$$

and

$$4m, 8m+4, 4m+4, 8m+8, \cdots, 12m, 8m$$

(the last set being 4 times the labels of $f_{2m+1}$ given above). For example $W_{18}$ is labeled (modulo 36) as follows.

nodes: 10 33 6 29 2 25 34 21 30 17;

perimeter: 7 5 35 31 27 23 19 15 11 22

nodes: 5 13 1; 8 20 12 24 16

perimeter: 18 14 9 28 32 0 4 26.

For $W_{8m+6}$ $(m \geqq 0)$ the cycle is labeled

$$12m+2, 12m+5, 12m-2, 12m+1, \cdots, 4m-2, 4m+1;$$

$$16m+5, 4m-3, 16m+1, 4m-7, \cdots, 12m+9, 1;$$

$$16m+8, 16m+4, 16m-8, 16m-12, \cdots, 24, 20, 8, 4$$

(the last set being 4 times the second labeling of $f_{2m+2}$ given above). For example, $W_{14}$ is labeled (modulo 28) as follows.

nodes: 14 17 10 13 6 9 2 5; 21 1; 24 20 8 4

perimeter: 3 27 23 19 15 11 7 26 22 25 16 0 12 18

**11. Complete bipartite graphs.** Let $K_{m,n}$ denote the complete bipartite graph with $m+n$ nodes and $mn$ edges.

THEOREM 19. *$K_{m,n}$ is harmonious if and only if $m$ or $n=1$.*

*Proof.* If $m$ or $n=1$, the graph is a star and is harmonious (see § 5). Suppose $m>1$ and $n>1$. A harmonious labeling of $K_{m,n}$ is equivalent to a direct sum decomposition of $\mathbb{Z}_{mn}=A\oplus B$, where $A$ and $B$ are disjoint subsets of $\mathbb{Z}_{mn}$ with $|A|=m$, $|B|=n$. Since all the sums $a+b$ $(a\in A, b\in B)$ are distinct, so are all the differences $a-b$. But there are $mn$ differences, hence $0=a-b$ must occur exactly once. Therefore $A$ and $B$ are not disjoint, and $K_{m,n}$ is not harmonious. Q.E.D.

The proof has an interesting corollary.

COROLLARY. *If $\mathbb{Z}_n=A\oplus B$ then $|A\cap B|=1$.*

Although many papers have dealt with decompositions of this type ([12], [13], [16], [38], [71], [72]), this result does not seem to have been noticed before.

**12. The one-point union of two complete graphs.** The graph $K_n^{(2)}$ $(n\geqq 3)$ consists of two copies of $K_n$ sharing a common node, and contains $2n-1$ nodes and $n(n-1)$ edges (see Fig. 14). It is known that $K_n^{(2)}$ is never graceful [5].

THEOREM 20. *$K_n^{(2)}$ is harmonious if $n=4$ but is not harmonious if $n$ is odd or $n=6$.*

*Proof.* For $n=4$ see Fig. 14, and for odd $n$ apply Theorem 11. The computer eliminated $n=6$. Q.E.D.

We conjecture that $K_n^{(2)}$ is harmonious only when $n$ is 4.

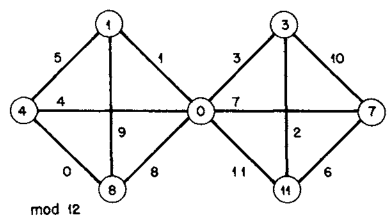


FIG. 14. *The graph $K_4^{(2)}$.*

## 13. Some small graphs.

THEOREM 21. *There are six connected graphs with $\leqq 5$ nodes that are not harmonious—see Fig. 15.*

*Proof.* It has already been shown that $C_4$, $F_2=K_3^{(2)}$, $K_5$ and $K_{2,3}$ are not harmonious, and the other two graphs in Fig. 15 are easily eliminated by hand. Harmonious labelings of most of the other graphs with $\leqq 5$ vertices are given in Fig. 1, and the remainder are easily dealt with. Q.E.D.

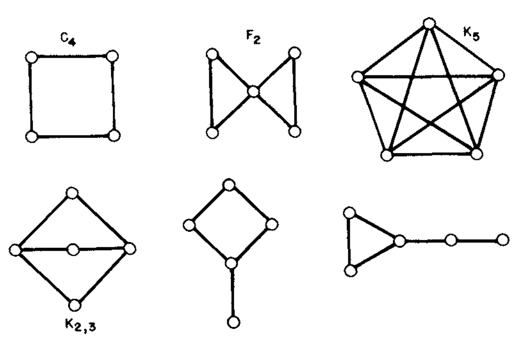


FIG. 15. *The six nonharmonious graphs with $\leqq 5$ nodes.*

For comparison we note that Golomb [25] showed there are three connected graphs with $\leqq 5$ nodes that are not graceful, namely $C_5$, $F_2$ and $K_5$; and Rao Hebbare [64] found that there are six nongraceful connected graphs with 6 nodes.

THEOREM 22. *The Petersen graph and the 1-skeletons of the tetrahedron, icosahedron and dodecahedron are harmonious, while the 1-skeletons of the cube and octahedron are not.*

*Proof.* For the first four see Figs. 16, 1(m), 17 and 18. The octahedron is not harmonious by Theorem 11, and the computer was used to check that the cube is not. Q.E.D.

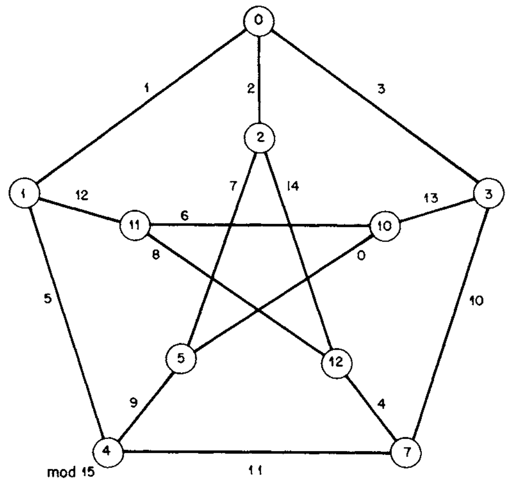


FIG. 16. *The Petersen graph.*

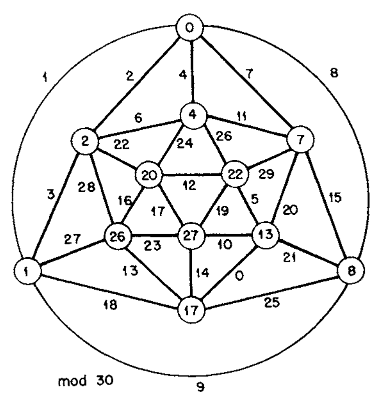


FIG. 17. *The icosahedron.*

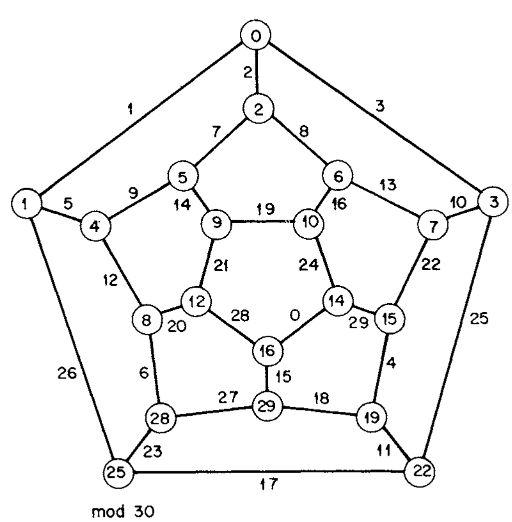


FIG. 18. *The dodecahedron.*

**14. Most graphs.**

THEOREM 23. *Almost all graphs are not harmonious.*

*Proof.* For our model of a random graph with $n$ nodes we assume that each of the $\binom{n}{2}$ possible edges independently exists or does not with probability $\frac{1}{2}$. Fix $\varepsilon \in (0, \frac{1}{2})$, and let $d$ be a fixed integer in the range $[(\frac{1}{2}-\varepsilon)\binom{n}{2}, (\frac{1}{2}+\varepsilon)\binom{n}{2}]$. We shall show that almost no graphs with $n$ nodes and $d$ edges are harmonious (as $n \to \infty$). Since almost all graphs with $n$ nodes have a number of edges in this range, the theorem follows.

There are $\binom{n(n-1)/2}{d}$ labeled graphs with $n$ nodes and $d$ edges, and so at least

$$\frac{1}{n!}\binom{n(n-1)/2}{d}$$

unlabeled graphs with $n$ nodes and $d$ edges.

Let $\lambda$ be a labeling of the $n$ nodes with distinct numbers from $\{0, 1, \cdots, d-1\}$. There are $d(d-1)\cdots(d-n+1) \leqq d^n$ such labelings. Let us consider how many graphs there are for which $\lambda$ is a harmonious labeling. Let $p_i$ be the number of pairs of nodes $\{v, v'\}$ with $\lambda(v)+\lambda(v') \equiv i$ (modulo $d$). Then

$$\sum_{i=0}^{d-1} p_i = \binom{n}{2}.$$

A graph is harmonious with this labeling if it consists of one edge taken from each of the classes counted by $p_i$. Thus there are

$$\prod_{i=0}^{d-1} p_i$$

labeled graphs for which $\lambda$ is a harmonious labeling. This product is maximized by taking the $p_i$'s as equal as possible; in particular

$$\prod_{i=0}^{d-1} p_i \leqq \left(\frac{n(n-1)}{2d}\right)^d.$$

Therefore there are at most

$$d^n\left(\frac{n(n-1)}{2d}\right)^d$$

harmonious labeled graphs. This is also an upper bound on the number of harmonious *unlabeled* graphs. To complete the proof we show that the ratio

$$\rho = \frac{d^n\left(\frac{n(n-1)}{2d}\right)^d}{\frac{1}{n!}\binom{n(n-1)/2}{d}}$$

approaches 0 when $n \to \infty$ and $d$ is in the required range. Write $d = (\frac{1}{2}-\mu)\binom{n}{2}$, with $\mu \in (-\frac{1}{2}, \frac{1}{2})$. Then

$$\rho < \frac{d^n n! \sqrt{8\binom{n}{2}(\frac{1}{2}-\mu)(\frac{1}{2}+\mu)}}{(\frac{1}{2}-\mu)^d 2^{\binom{n}{2} H_2[(1/2)-\mu]}},$$

where $H_2(x) = -x \log_2 x - (1-x)\log_2(1-x)$ (cf. [57, p. 309]). The denominator is equal to

$$2^{-\binom{n}{2}[(1/2+\mu]\log_2[(1/2)+\mu]}$$

and so $\rho \to 0$ as $n \to \infty$. Q.E.D.

The same argument establishes an unpublished result of Erdös (cf. [25]):

THEOREM 24. *Almost all graphs are not graceful.*

**15. Comparison of harmonious and graceful graphs.** A study of Table V suggests that the properties of being harmonious and graceful are roughly similar, although the entries for cycles show that the two properties are in general independent. Comparing the harmonious labelings of the previous sections with graceful labelings of the same graphs (see for example [10], [22], [44]) suggests that harmonious labelings are considerably more complicated. We know that $n_\gamma(v)$, the number of edges in the largest harmonious graph on $v$ nodes, is bounded by (2). On the other hand, if $g(v)$ denotes the number of edges in the largest graceful graph on $v$ nodes, it is known that $\lim_{v\to\infty} g(v)/v^2$ exists and satisfies

$$\tag{16} \tfrac{1}{3} \leqq \lim_{v\to\infty} \frac{g(v)}{v^2} \leqq 0.411,$$

(see [26], [42], [53], [58], [81]). Table VI compares the first few values of $n_\gamma(v)$ and $g(v)$: they are extremely close. We conclude therefore with an open problem: show that $\lim_{v\to\infty} n_\gamma(v)/v^2$ exists, and find improvements to (2) comparable with (16).

TABLE VI

*The size of the largest harmonious graph on $v$ nodes ($n_\gamma(v)$) compared with the size of the largest graceful graph ($g(v)$). The values of $g(v)$ are taken from* [53] *and* [58].

| $v$ | $n_\gamma(v)$ | $g(v)$ |
|---|---|---|
| 2 | 1 | 1 |
| 3 | 3 | 3 |
| 4 | 6 | 6 |
| 5 | 9 | 9 |
| 6 | 13 | 13 |
| 7 | 17 | 17 |
| 8 | 24 | 23 |
| 9 | 30 | 29 |
| 10 | 36 | 36 |
| 11 | | 43 |
| 12 | | 50 |
| 13 | | 58 |
| 14 | | 68 |

**Acknowledgments.** We should like to thank P. Erdös and A. M. Odlyzko for helpful discussions, and F. R. K. Chung for proving Theorem 17 and parts of Theorem 18.

*Note added in proof.* Thom Grace (written communication, June 14, 1980) has shown that $g_{2m+1}$ is harmonious (see § 9): label the path $m, 0, m+1, 1, \cdots, m-1, 2m$, and the two additional nodes $3m$ and $5m+1$ (modulo $6m+2$). E. Levine (written communication, June 24, 1980) has shown that if $K_n^{(2)}$ is harmonious (see § 12) then $n$ is a sum of two squares.

## REFERENCES


[1] R. ALTER AND J. A. BARNETT, *Remarks on the postage stamp problem with applications to computers*, in Proc. 8th Southeastern Conference on Combinatorics, Graph Theory, Computing, Congress Num. XIX, Utilitas Math. Pub., Winnipeg, 1977, pp. 43–59.

[1a] ———, *A postage stamp problem*, Amer. Math. Monthly, 87 (1980), pp. 206–209.

[2] G. BARON, *Über Verallgemeinerungen des Langford'schen Problems*, in Combinatorial Theory and its Applications, P. Erdös et al., eds., Colloq. Math. Soc. János Bolyai, 4, North-Holland, Amsterdam, 3 vols., 1970, pp. 81–92.

[2a] L. D. BAUMERT, *Cyclic Difference Sets*, Lecture Notes in Math., 182, Springer-Verlag, Berlin, 1971.

[3] J.-C. BERMOND, *Problem*, in Combinatorics, A. Hajnal and V. T. Sós, eds., Colloq. Math. Soc. János Bolyai, 18, North-Holland, Amsterdam, 2 vols., 1978, p. 1190.

[4] J.-C. BERMOND, A. E. BROUWER AND A. GERMA, *Systèmes de triplets et différences associées*, Problèmes Combinatoires et Théorie des Graphs, Colloq. Intern. du Centre National de la Rech. Scient., 260, Editions du Centre Nationale de la Recherche Scientifique, Paris, 1978, pp. 35–38.

[5] J.-C. BERMOND, A. KOTZIG AND J. TURGEON, *On a combinatorial problem of antennas in radioastronomy*, in Combinatorics, A. Hajnal and V. T. Sós, eds., Colloq. Math. Soc. János Bolyai, 18, North-Holland, Amsterdam, 2 vols., 1978, pp. 135–149.

[6] M. R. BEST, A. E. BROUWER, F. J. MACWILLIAMS, A. M. ODLYZKO AND N. J. A. SLOANE, *Bounds for binary codes of length less than* 25, IEEE Trans. Information Theory, IT-24 (1978), pp. 81–93.

[7] G. S. BLOOM, *A chronology of the Ringel-Kotzig conjecture and the continuing quest to call all trees graceful*, Ann. N.Y. Acad. Sci., 326 (1979), pp. 32–51.

[8] G. S. BLOOM AND S. W. GOLOMB, *Numbered complete graphs, unusual rulers, and assorted applications*, in Theory and Applications of Graphs, Lecture Notes in Math., 642, Springer-Verlag, New York, 1978, pp. 53–65.

[9] ———, *Applications of numbered undirected graphs*, Proc. IEEE, 65 (1977), pp. 562–570.

[10] R. BODENDIEK, H. SCHUMACHER AND H. WEGNER, *Über graziöse Graphen*, Math.-Phys. Semesterberichte, 24 (1977), pp. 103–126.

[11] R. C. BOSE AND S. CHOWLA, *Theorems in the additive theory of numbers*, Comment. Math. Helvet., 37 (1962–63), pp. 141–147.

[12] N. G. DE BRUIJN, *On the factorization of finite Abelian groups*, Proc. Kon. Ned. Akad. Wetensch. Amsterdam, 56A (1953), pp. 258–264.

[13] ———, *On the factorization of cyclic groups*, ibid., pp. 370–377.

[14] ———, *On number systems*, Nieuw Archief voor Wiskunde, (3), IV (1956), pp. 15–17.

[15] J. CAHIT AND R. CAHIT, *On the graceful numbering of spanning trees*, Information Processing Letters, 3 (1975), pp. 115–118.

[16] L. CARLITZ AND L. MOSER, *On some special factorizations of* $(1-x^n)/(1-x)$, Canad. Math. Bull. 9 (1966), pp. 421–426.

[17] R. O. DAVIES, *On Langford's problem (II)*, Math. Gazette, 43 (1959), pp. 253–255.

[18] J. F. DILLON, *The generalized Langford-Skolem problem*, Proc. 4th Southeastern Conference on Combinatorics, Graph Theory, Computing, Utilitas Math. Pub., Winnipeg, 1973, pp. 237–247.

[19] P. ERDÖS AND R. L. GRAHAM, *Old and new problems and results in combinatorial number theory*, Monographies de l'Enseignement Math., to appear.

[20] P. ERDÖS AND H. HANANI, *On a limit theorem in combinatorical analysis*, Publ. Math. Debrecen, 10 (1963), pp. 10–13.

[21] P. ERDÖS AND P. TURÁN, *On a problem of Sidon in additive number theory, and some related problems*, J. London Math. Soc., 16 (1941), pp. 212–215 and 19 (1944), p. 208.

[22] R. FRUCHT, *Graceful numbering of wheels and related graphs*, Ann. N.Y. Acad. of Sci. 319 (1979), pp. 219–229.

[23] M. GARDNER, *Mathematical games: the graceful graphs of Solomon Golomb, or how to number a graph parsimoniously*, Scientific American, 226, 3 (1972), pp. 108–112; 226, 4 (1972), p. 104; 226, 6 (1972), p. 118.

[24] M. J. E. GOLAY, *Note on the representation of* $1, 2, \cdots, n$ *by differences*, J. London Math. Soc., 4 (1972), pp. 729–734.

[25] S. W. GOLOMB, *How to number a graph*, in Graph Theory and Computing, R. C. Read, ed., Academic Press, New York, 1972, pp. 23–37.

[26] ———, *The largest graceful subgraph of the complete graph*, Amer. Math. Monthly, 81 (1974), 499–501.

[27] S. W. GOLOMB AND G. S. BLOOM, *Multifarious applications of numbered graphs*, in Proc. 2nd Caribbean Conference on Combinatorics and Computing, R. C. Read and C. C. Cadogan, eds., Univ. of West Indies, Cave Hill, Barbados, 1977, pp. 82–95.

[28] R. L. GRAHAM AND N. J. A. SLOANE, *Lower bounds for constant weight codes*, IEEE Trans. Information Theory, IT-26 (1980), pp. 37–43.

[29] ———, *On constant weight codes and harmonious graphs*, Utilitas Math., to appear.

[30] R. K. GUY, *Monthly research problems* 1969–73, Amer. Math. Monthly, 80 (1973), pp. 1120–1128.

[31] ———, *Monthly research problems* 1969–75, ibid., 82 (1975), pp. 995–1004.

[32] ———, *Packing* $[1, n]$ *with solutions of* $ax + by = cz$: *the unity of combinatorics*, in Teorie Combinatorie, Atti dei Convegni Lincei, 17, Vol. II, 1976, pp. 173–179.

[33] ———, *Monthly research problems* 1969–77, Amer. Math. Monthly, 84 (1977), pp. 807–815, and 85 (1978), p. 263.

[34] R. K. GUY AND V. L. KLEE, JR., *Monthly research problems* 1969–71, Amer. Math. Monthly, 78 (1971), pp. 1113–1122.

[35] A. GYÁRFÁS AND J. LEHEL, *A method to generate graceful trees*, Problèmes Combinatoires et Théorie des Graphs, Colloq. Intern. du Centre National de la Rech. Scient. 260, Éditions du Centre National de la Recherche Scientifique, Paris, 1978, pp. 207–209.

[36] N. HÄMMERER AND G. HOFMEISTER, *Zu einer Vermutung von Rohrbach*, J. Reine Angew. Math. 286/287 (1976), pp. 239–247.

[37] E. HÄRTTER, *Basen für Gitterpunktmengen*, J. Reine Angew. Math. 202 (1959), pp. 153–170.

[38] G. HAJÓS, *Sur la factorisation des groupes abéliens*, Časopis Pêst. Mat. Fys., 74 (1950), pp. 157–162.

[39] ———, *Sur le problème de factorisation des groupes cycliques*, Acta Math. Acad. Sci. Hungar., 1 (1950), pp. 189–195.

[40] H. HALBERSTAM AND K. F. ROTH, *Sequences*, Vol. 1, Oxford University Press, Oxford, 1966.

[40a] J. I. HALL, A. J. E. M. JANSEN, A. W. J. KOLEN AND J. H. VAN LINT, *Equidistant codes with distance* 12, Discrete Math., 17 (1977), pp. 71–83.

[41] H. HANANI, *A note on Steiner triple systems*, Math. Scand., 8 (1960), pp. 154–156.

[42] C. B. HASELGROVE AND J. LEECH, *Note on restricted difference bases*, J. London Math. Soc. 32 (1957), pp. 228–231.

[43] R. D. HEATH-BROWN AND H. IWANIEC, *On the difference between consecutive primes*, Bull Amer. Math. Soc., 1 (1979), pp. 758–760.

[43a] R. L. HEIMER AND H. LANGENBACH, *The stamp problem*, J. Recreational Math., 7 (1974), pp. 235–250.

[44] C. HOEDE AND H. KUIPER, *All wheels are graceful*, Utilitas Math., 14 (1978), p. 311.

[45] J. R. ISBELL, *Perfect addition sets*, Discrete Math., 24 (1978), pp. 13–18.

[46] W. KLOTZ, *Eine obere Schranke für die Reichweite einer Extremalbasis zweiter Ordnung*, J. Reine Angew. Math., 238 (1969), pp. 161–168.

[47] K. M. KOH, T. TAN AND D. G. ROGERS, *Two theorems on graceful trees*, Discrete Math., 25 (1979), pp. 141–148.

[48] A. KOTZIG, *On certain vertex-valuations of finite graphs*, Utilitas Math., 4 (1973), pp. 261–290.

[49] A. KOTZIG AND A. ROSA, *Magic valuations of finite graphs*, Canad. Math. Bull., 13 (1970), pp. 451–461.

[50] A. KOTZIG AND J. TURGEON, *β-Valuations of regular graphs with complete components*, in Combinatorics, A. Hajnal and V. T. Sós, eds., Colloq. Math. Soc. János Bolyai, 18, North-Holland, Amsterdam, 2 vols., 1978, pp. 697–703.

[51] C. W. H. LAM, *Nth power residue addition sets*, J. Combinatorial Theory, 20A (1976), pp. 20–33.

[52] C. D. LANGFORD, *Problem*, Math. Gazette, 42 (1958), p. 228.

[53] J. LEECH, *On the representation of* $1, 2, \cdots, n$ *by differences*, J. London Math. Soc., 31 (1956), pp. 160–169.

[54] ———, *Another tree labelling problem*, Amer. Math. Monthly, 82 (1975), pp. 923–925.

[55] E. LEVINE, *On the generalized Langford problem*, Fibonacci Quarterly, 6 (1968), pp. 135–138.

[56] W. F. LUNNON, *A postage stamp problem*, Computer J., 12 (1969), pp. 377–380.

[56a] D. MCCARTHY, R. C. MULLIN, P. J. SCHELLENBERG, R. G. STANTON, AND S. A. VANSTONE, *On approximations to a projective plane of order* 6, Ars Combinatoria 2 (1976), pp. 111–168.

[57] F. J. MACWILLIAMS AND N. J. A. SLOANE, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, 1977.

[57a] M. MAHEO, *Strongly graceful graphs*, Discrete Math., 29 (1980), pp. 39–46.

[58] J. C. P. MILLER, *Difference bases: three problems in additive number theory*, in Computers in Number Theory, A. O. L. Atkin and B. J. Birch, eds., Academic Press, N.Y., 1971, pp. 299–322.

[59] L. MOSER, *On the representation of* $1, 2, \cdots, n$ *by sums*, Acta Arith., 6 (1960), pp. 11–13.

[60] L. MOSER, J. R. POUNDER AND J. RIDDELL, *On the cardinality of h-bases for n*, J. London Math. Soc., 44 (1969), pp. 397–407.

[61] M. B. NATHANSON, *Additive h-bases for lattice points*, Second International Conference on Combinatorial Mathematics, Ann. N.Y. Acad. Sci., 319 (1979), pp. 413–414.

[62] R. NOWAKOWSKI, *Generalizations of the Langford-Skolem problem*, M.Sc. Thesis, University of Calgary, 1975.

[63] C. J. PRIDAY, *On Langford's problem* (I), Math. Gazette, 43 (1959), pp. 250–253.

[64] S. P. RAO HEBBARE, *Graceful cycles*, Utilitas Math. 10 (1976), pp. 307–317.

[65] H. ROHRBACH, *Ein Beitrag zur additiven Zahlentheorie*, Math. Zeit., 42 (1937), pp. 1–30.

[66] ———, *Anwendung eines Satzes der additiven Zahlentheorie auf eine grappentheoretische Frage*, ibid., pp. 538–542.

[67] A. ROSA, *On certain valuations of the vertices of a graph*, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod, Paris, 1967, pp. 349–355.

[68] D. P. ROSELLE, *Distributions of integers into s-tuples with given differences*, Proc. Manitoba Conf. Number. Math., Dept. of Computer Science, University of Manitoba, Winnipeg, 1971, pp. 31–42.

[69] ———, *Comments and complements*, Amer. Math. Monthly, 81 (1974), pp. 1097–1099.

[70] D. P. ROSELLE AND T. C. THOMASSON, JR, *On generalized Langford sequences*, J. Combinatorial Theory, 11A (1971), pp. 196–199.

[71] A. D. SANDS, *On the factorisation of finite abelian groups*, Acta Math. Acad. Sci. Hungar., 8 (1957), pp. 65–86 and 13 (1962), pp. 153–169.

[72] ———, *The factorization of Abelian groups*, Quart. J. Math. Oxford (2), 10 (1959), pp. 81–91.

[73] J. SCHÖNHEIM, *On maximal systems of k-tuples*, Stud. Sci. Math. Hungar., 1 (1966), pp. 363–368.

[74] G. J. SIMMONS, *Synch-sets: a variant of difference sets*, Proc. 5th Southeastern Conference on Combinatorics, Graph Theory and Computing, Utilitas Math. Pub. Co., Winnipeg, 1974, pp. 625–645.

[75] T. SKOLEM, *On certain distributions of integers in pairs with given differences*, Math. Scand., 5 (1957), pp. 57–68.

[76] R. G. STANTON, J. A. BATE AND R. C. MULLIN, *Some tables for the postage stamp problem*, Proc. Fourth Manitoba Conference on Numerical Math., Utilitas Math. Pub., Winnipeg, 1974, pp. 351–356.

[77] R. G. STANTON, J. G. KALBFLEISCH AND R. C. MULLIN, *Covering and packing designs*, Proc 2nd Chapel Hill Conference on Combinatorical Mathematics and its Applications, R. C. Bose et al., eds., Chapel Hill, 1970, pp. 428–450.

[78] R. G. STANTON AND C. R. ZARNKE, *Labelling of balanced trees*, Proc. 4th Southeastern Conference on Combinatorics, Graph Theory, Computing, Utilitas Math. Pub., Winnipeg, 1973, pp. 479–495.

[79] A. STÖHR, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe*, J. Reine Angew. Math., 194 (1955), pp. 40–65 and 111–140.

[80] H. TAYLOR, *Odd path sums in an edge-labeled tree*, Math. Mag., 50 (1977), pp. 258–259.

[81] B. WICHMANN, *A note on restricted difference bases*, J. London Math. Soc., 38 (1962), pp. 465–466.