Lovely New Problems from the Past Year

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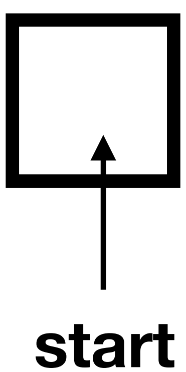
Outline

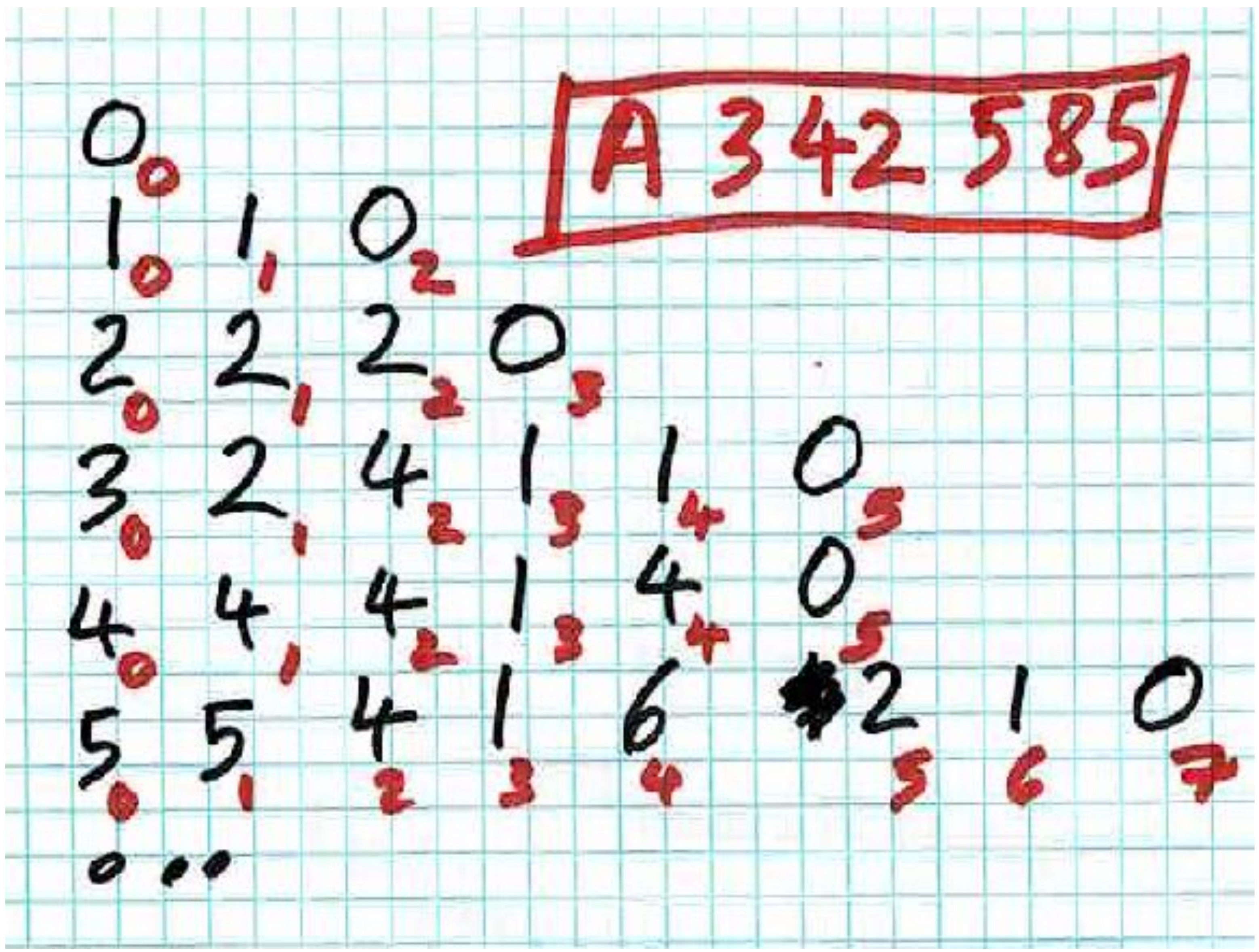
- The Inventory Sequence
- New Stained Glass Window Problem
- Stan Wagon’s Problem of the Week POW 1321
- Stepping Stones Puzzle
- New L.E.S. in EKG, Yellowstone, Enots Wolley Family
- Gerrymandering

The Inventory Sequence A342585

Joseph Rozhenko, March 2021

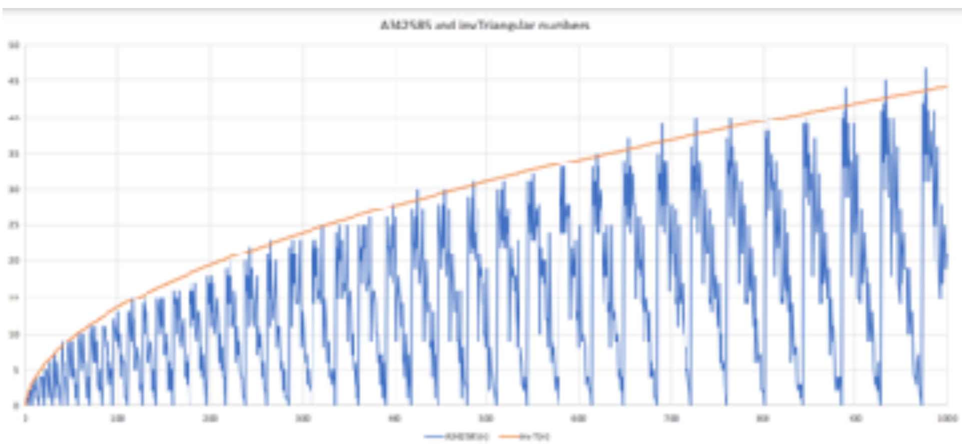
Each row says how many 0s, 1s, 2s, … in sequence so far.
Start new row after seeing a 0.

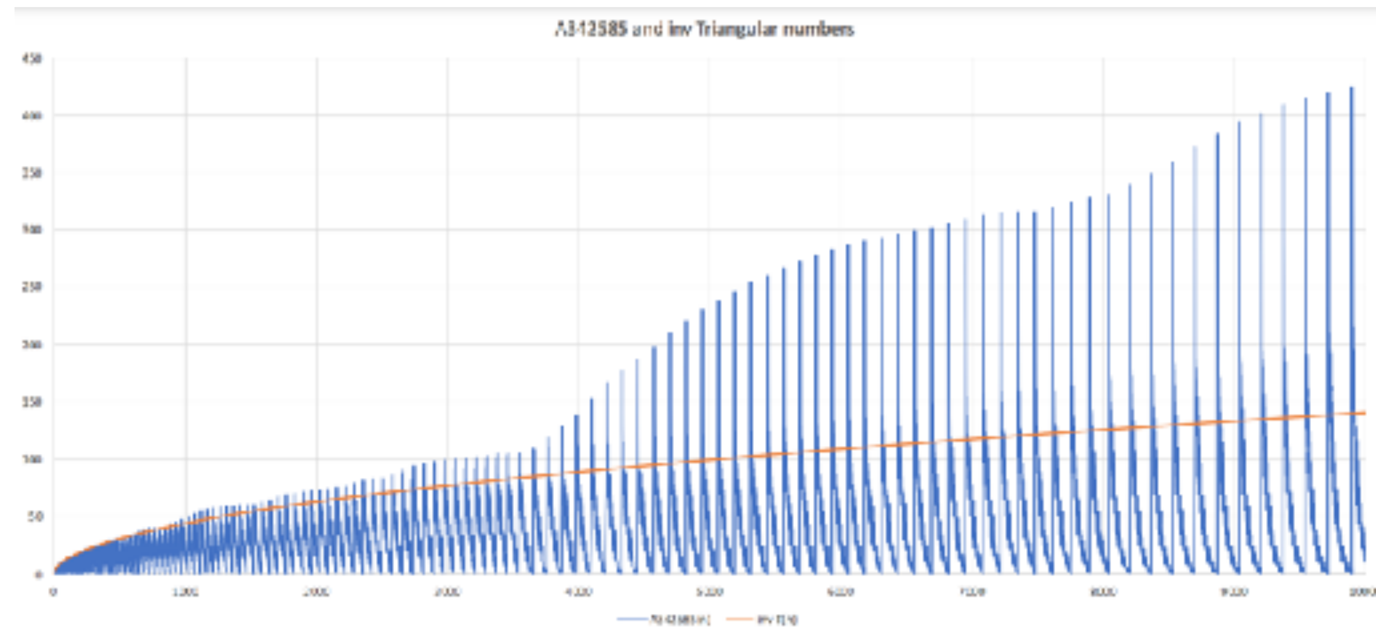




Inventory Sequence A342585 cont.

Have lots of data from Hans Havermann, Peter Munn, Hugo Pfoertner, Rémy Sigrist, Jan van Eck, and others. But essentially nothing is known.





1000 terms
Brown line = sqrt(2n)

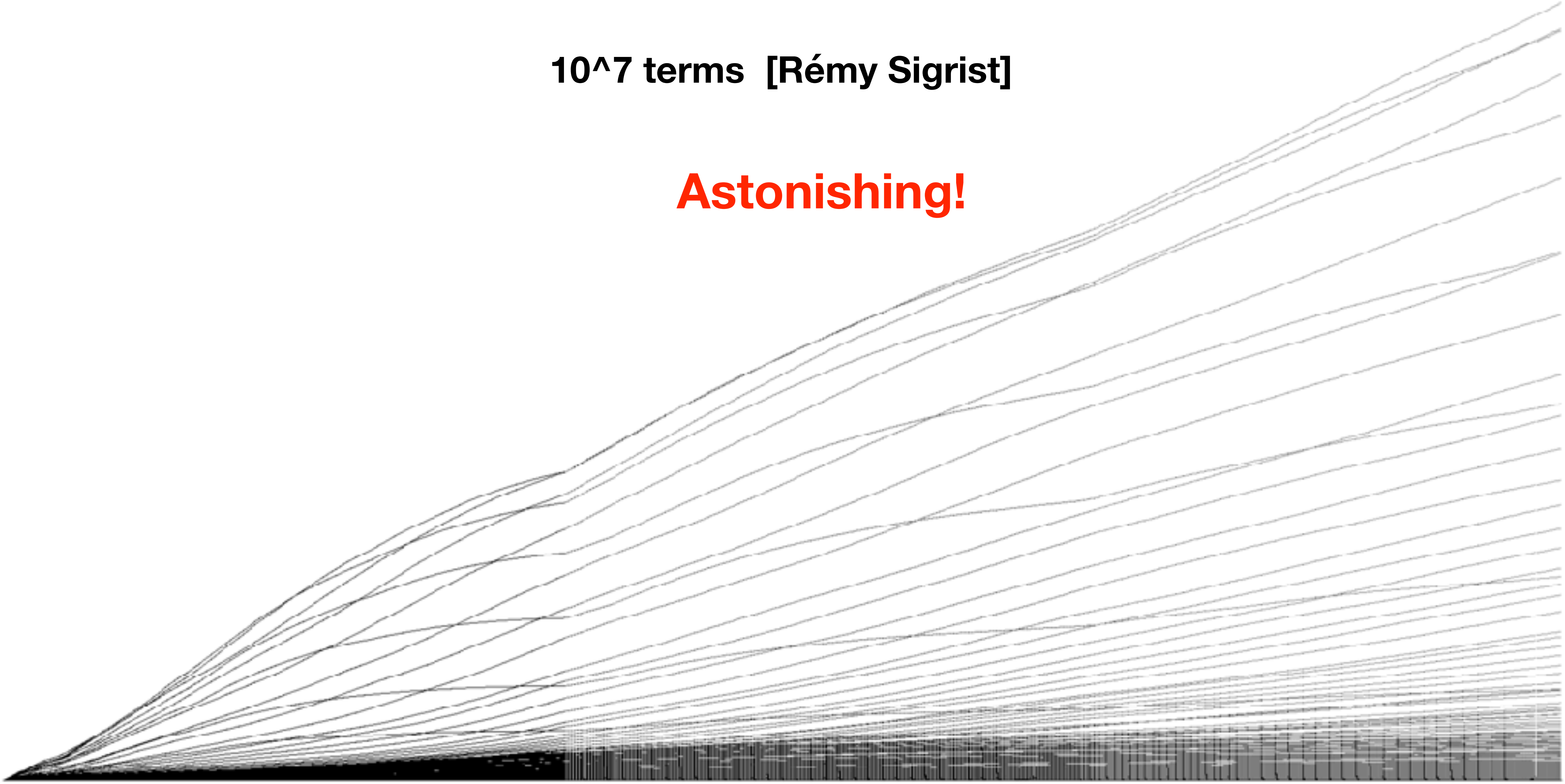
10000 terms

[Jan van Eck]

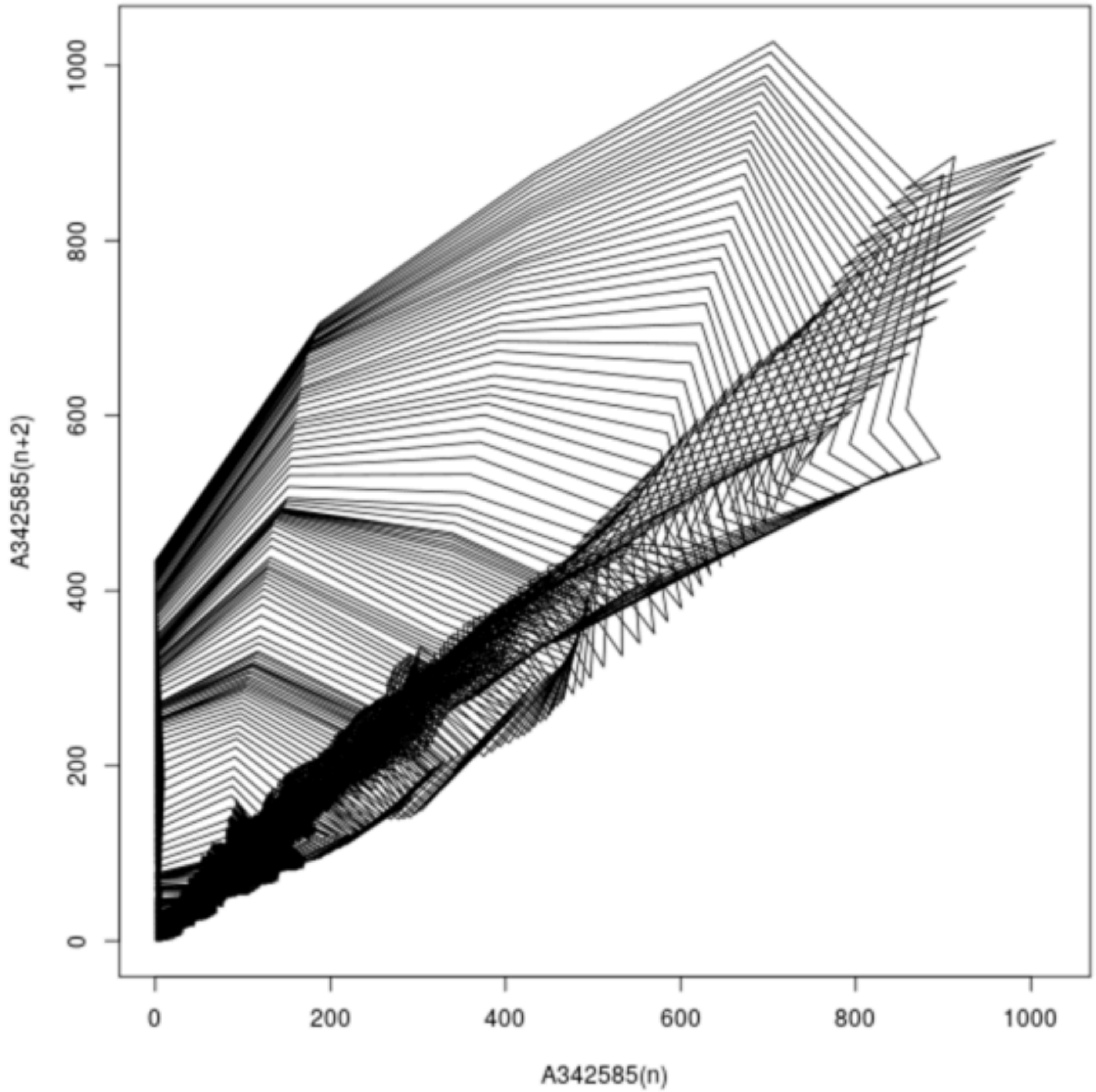
Inventory Sequence A342585 cont.

10^7 terms [Rémy Sigrist]

Astonishing!



Inventory Sequence A342585 cont.

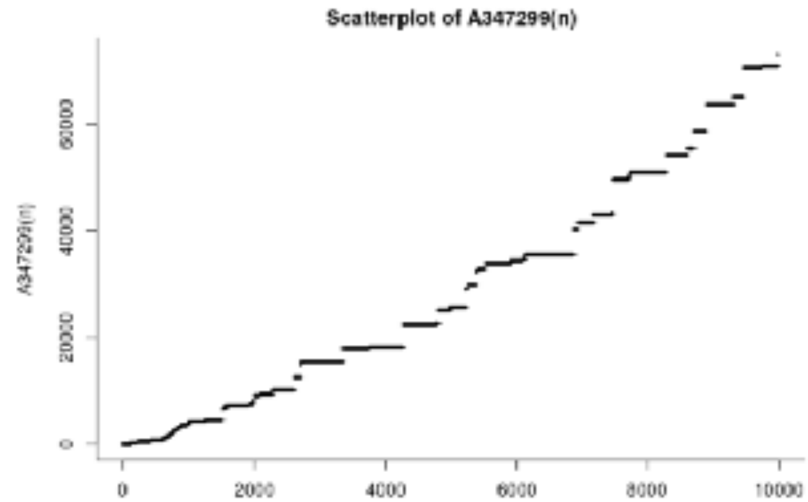


Plot a(n+2) vs a(n) using “Plot 2” [Peter Munn]

Inventory Sequence A342585 cont.

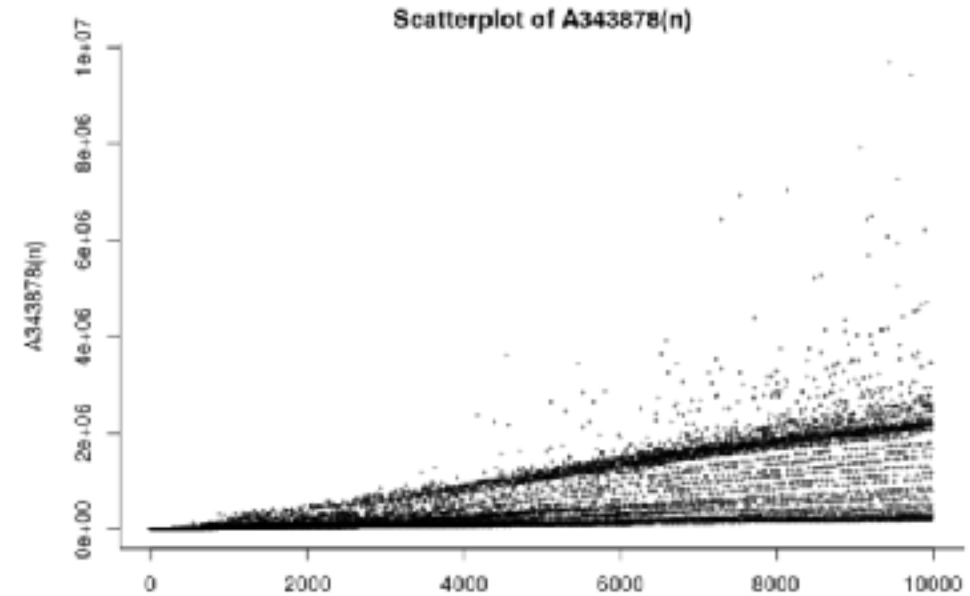
Need a mathematical description of what is happening!

Lengths of rows A347299



Profile of rows? Does it converge?

Go to A342585, click "Listen". What is the theme that the rows are variations on?



When k first appears A343878

[Added after the talk: A347317 is a version of the inventory sequence that ends the rows after the final zero, not the first, but this has the drawback that most of terms in the sequence are ultimately zeros]

New Stained Glass Window Problem

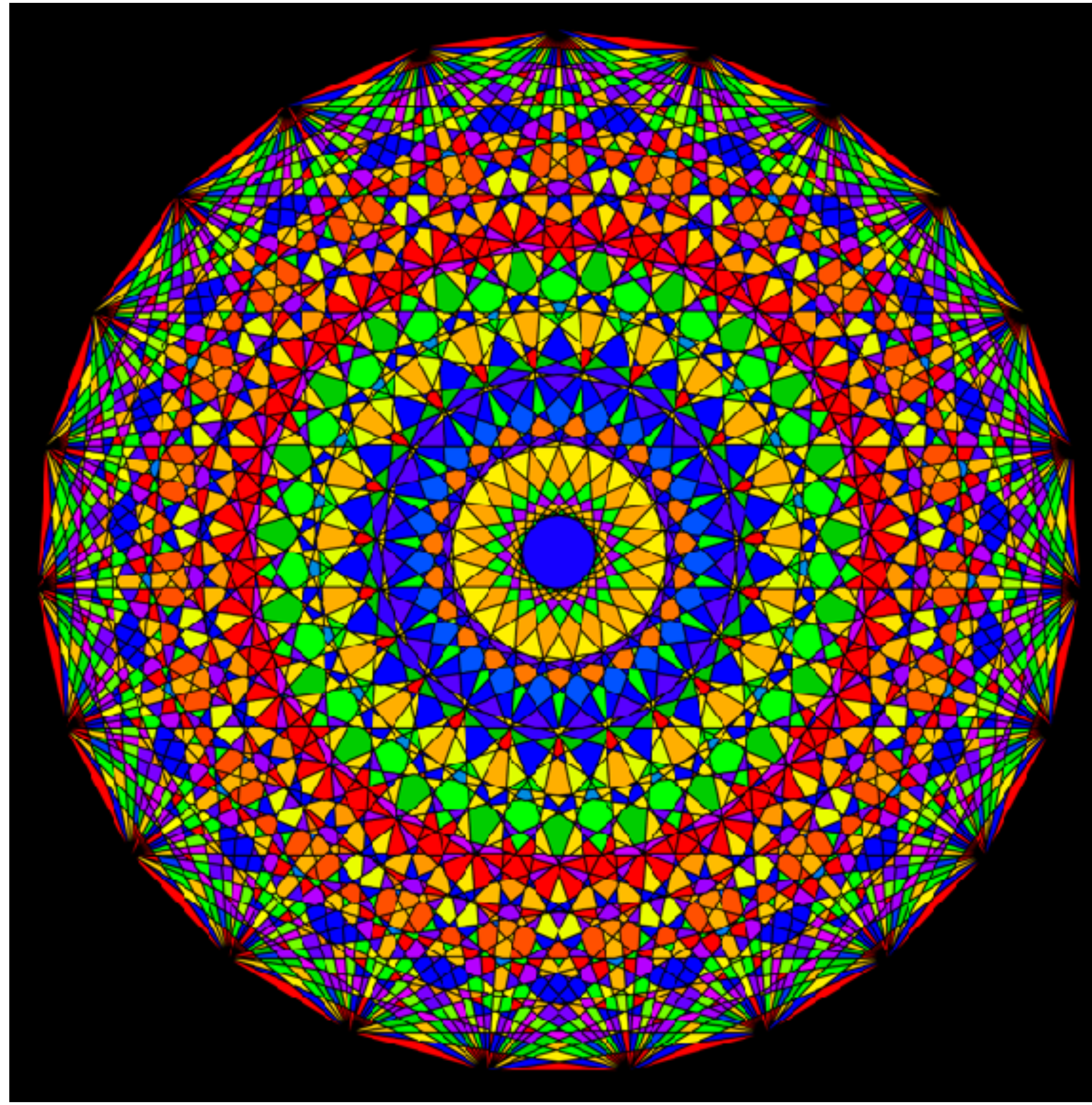
Background

Poonen and Rubinstein enumerated vertices and cells in K_n with all chords drawn.

(Formulas are simpler if n odd)

Blomberg, Shannon, NJAS, Graphical enumeration and stained glass windows I, Integers, 2022.

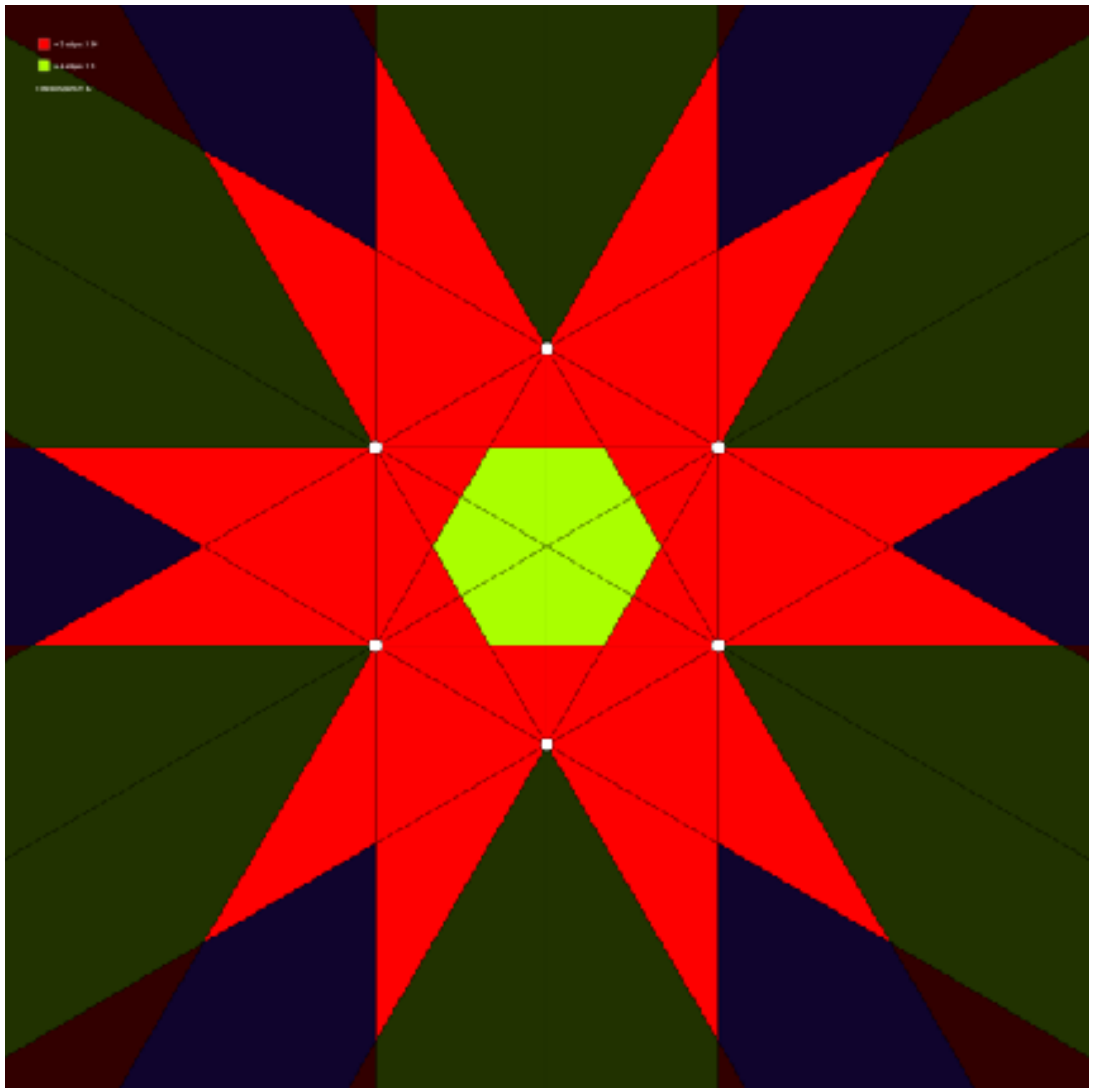
See A007678 for formulas, many more pictures



K_23

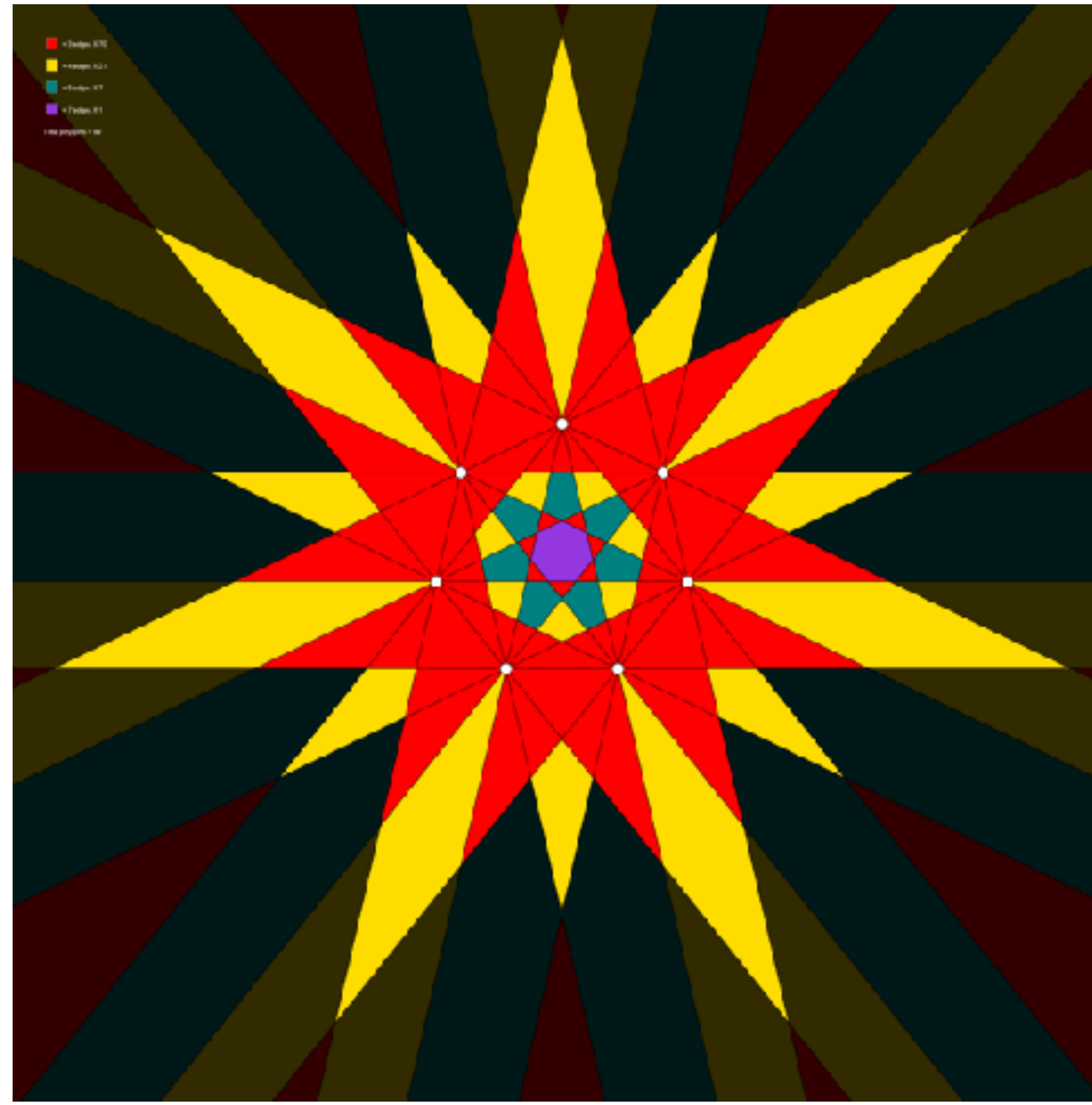
New Stained Glass Window Problem, cont.

Scott Shannon: what if you extend all chords to infinity - how many polygons?



K_6: red = 36 triangles,
green = 6 quadrilaterals
(42 polygons)

White dots = original K_n



K_7: 70 X 3, 21 X 4, 7 X 5, 1 X 7
(99 polygons)

A344857

New Stained Glass Window Problem, cont.

Scott Shannon: A344857 = number of polygons in drawing of K_n with all chords extended

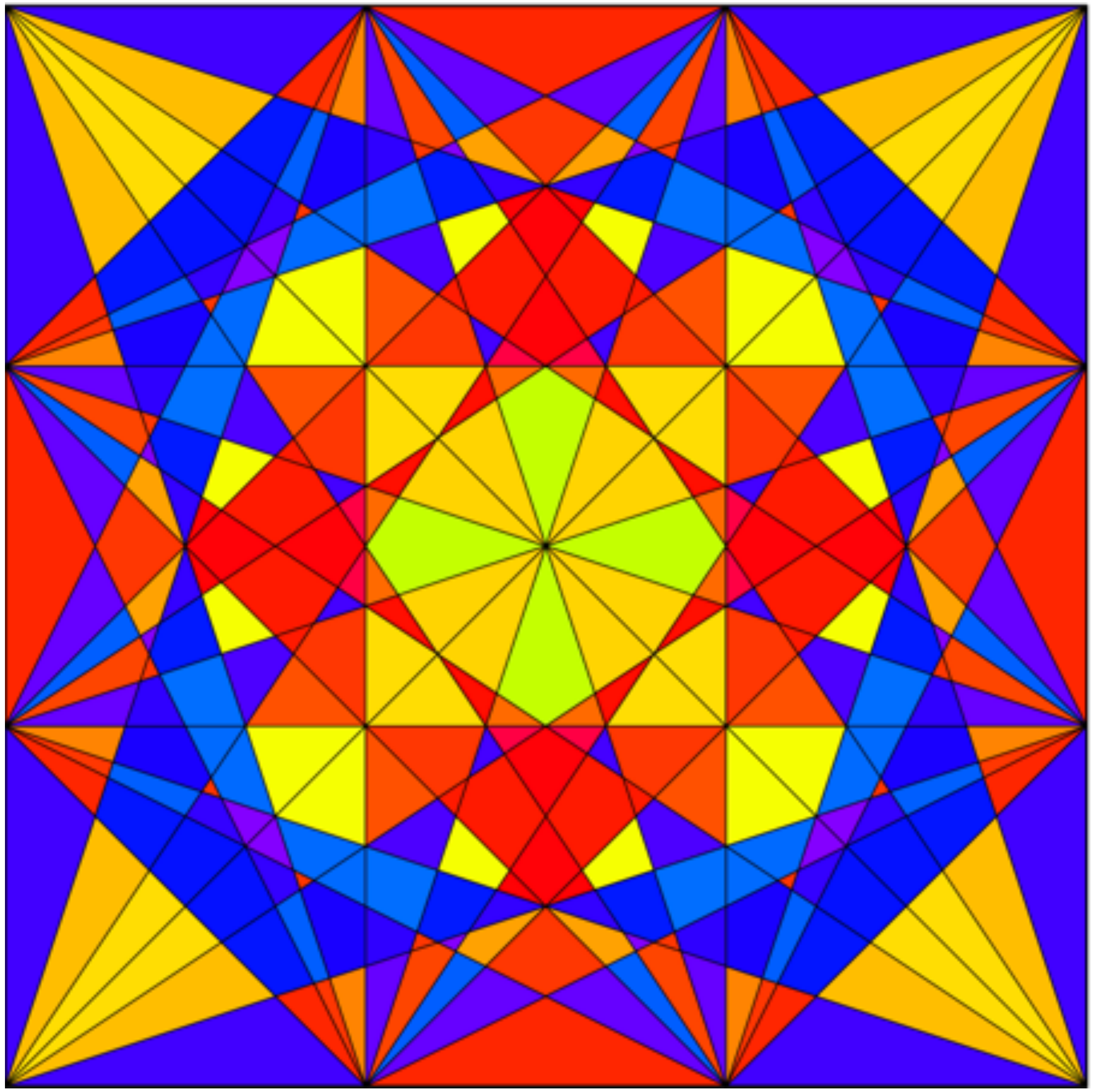
1, 4, 16, 42, 99, 176, 352, 540, 925, 1152, 2016, 2534, 3871, ...

3, 4, 5, 6, 7, ...

Conjecture: for n odd, a(n) ?=? $(n^4 - 7 n^3 + 19 n^2 - 21 n + 8) / 8$

Have 100 terms. Not even a guess for n even!

Two Unsolved Problems from Last Year



BC(3,3)

BC(m,n) = m X n grid of squares with every pair of boundary points joined by a line

BC = “Boundary Chords”

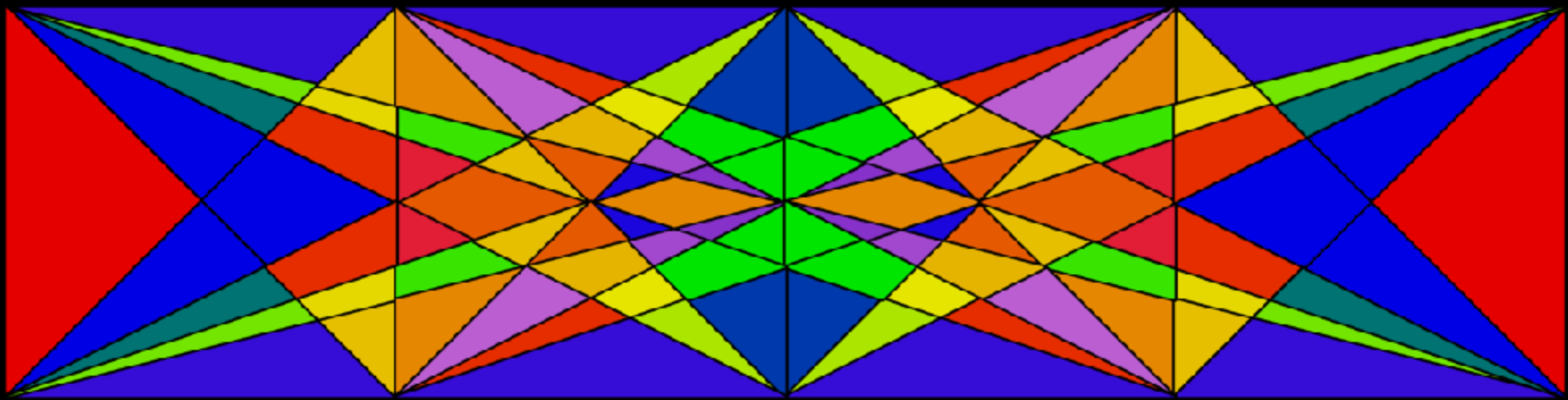
A331452 has many other BC(m,n) stained glass windows.

A331452

Problem LastYear_1

A306302

BC(1, 4)



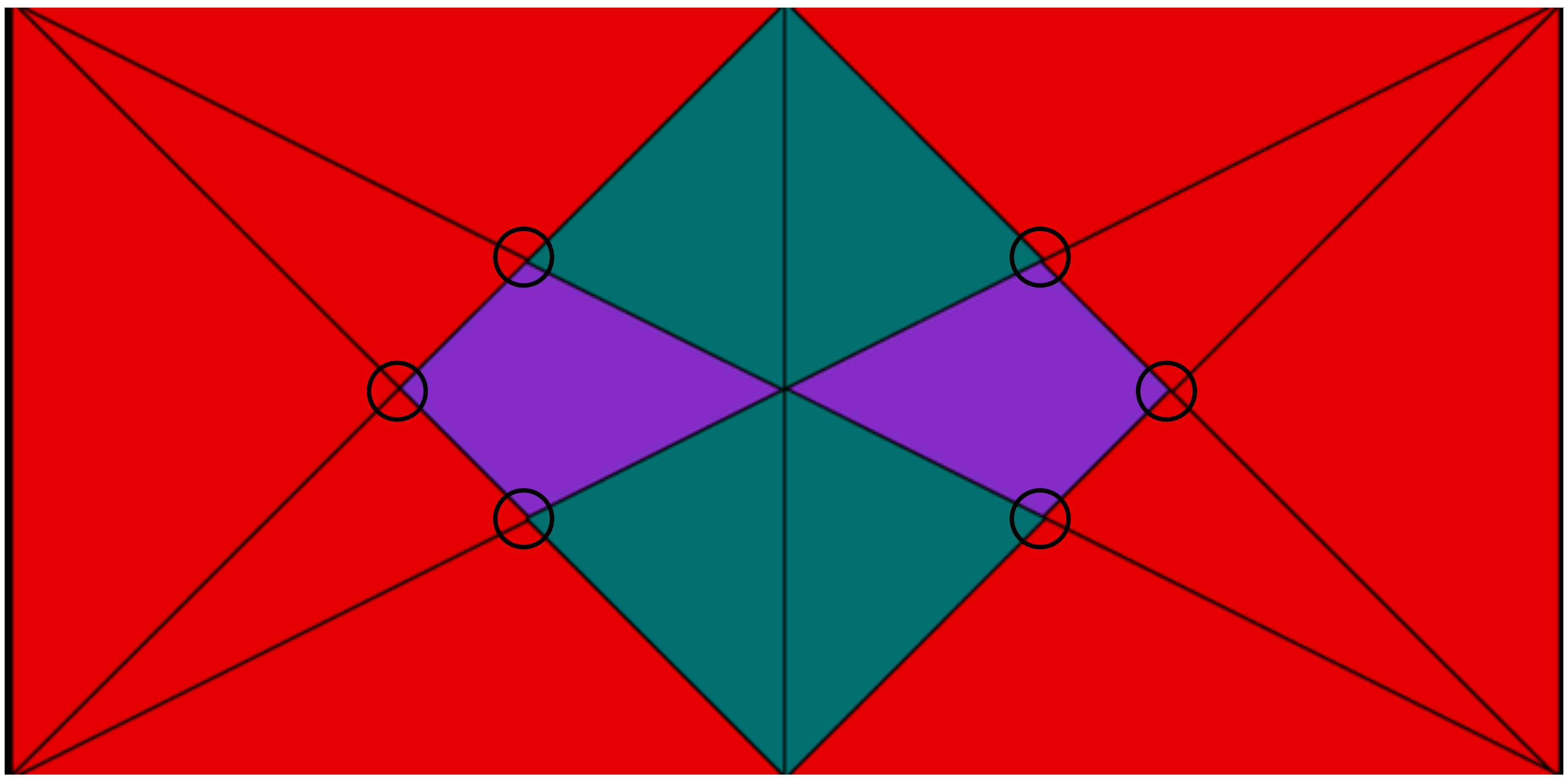
104 cells (70 triangles, 34 quadrilaterals) but no pentagons or hexagons - why?!

Blomberg, Shannon, NJAS, Graphical enumeration and stained glass windows I, Integers, 2022.

Problem LastYear_2

BC(1, 2)

There are six simple interior nodes ○



A334701(2) = 6

Problem LastYear_2

Interior Nodes in BC(1,n)

It appears that most interior nodes in BC(1,n) are "simple", i.e. are where just two chords cross.

For n = 1, 2, 3, … the numbers of simple interior nodes are

1, 6, 24, 54, 124, 214, 382, 598, 950, 1334, …

A334701 has first 500 terms!

Open Problem: Guess a formula.

This is a frequent problem: we have hundreds of terms of a sequence with a simple definition.
Need to improve Superseeker.
Need volunteers!
If interested, contact njasloane@gmail.com

Two Impossible-Sounding Problems

Stan Wagon's Problem of the Week 1321

S = set of n different integers.
$f(S)$ = number of pairs $s < t$ in S such that $s+t$ = a power of 2.
$W(n)$ = max of $f(S)$ over all choices for S.

$W(3) = 3$ from $S = \{-1, 3, 5\}$,
sums $s+t$ are 2, 4, 8

(The powers of 2 do not need to be distinct.)

W(4) onwards not known!

$W(4) >= 4$ from $\{ -3, -1, 3, 5 \}$

$W(5) >= 6$ from $\{ -3, -1, 3, 5, 11 \}$

$W(10) >= 15$ from $\{ -5, -3, -1, 1, 3, 5, 7, 9, 11, 13 \}$

Problem of the Week 1321 (cont.), A347301

Rob Pratt's approach: Only allow numbers in some range, -A to B, and use MILP.

For n <= 100, a convenient range is -n-3 to n+3

Assume elements of S restricted to range -n-3 to n+3, call the answer a(n), now A347301.

This gives conjecturally the true values of W(n) for n <= 100, except for a(5) = 5 from { -4, -3, -1, 3, 5} while W(5) >= 6.

The values of a(n) (A347301) are:

0, 1, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 21, 24, 26, 29, 31, 34, 36, ...

Problem of the Week 1321 (cont.), A347301

General remarks, from Stan Wagon, based on communications from many people:

Upper bound: ~~W(n) = O(n (log n)^2) (Piotr Zielinksi)~~

None known (other than n choose 2)

Lower bound: **Use S = { M-n, M-n+2, M-n+4, …, M+n-2 } for some choice of M.**

If n = 4^p and M = 1 + sqrt(n), then W(n) >= np - n/2 + 3 sqrt(n) / 2 -1

= O(n log n),

and may be optimal.

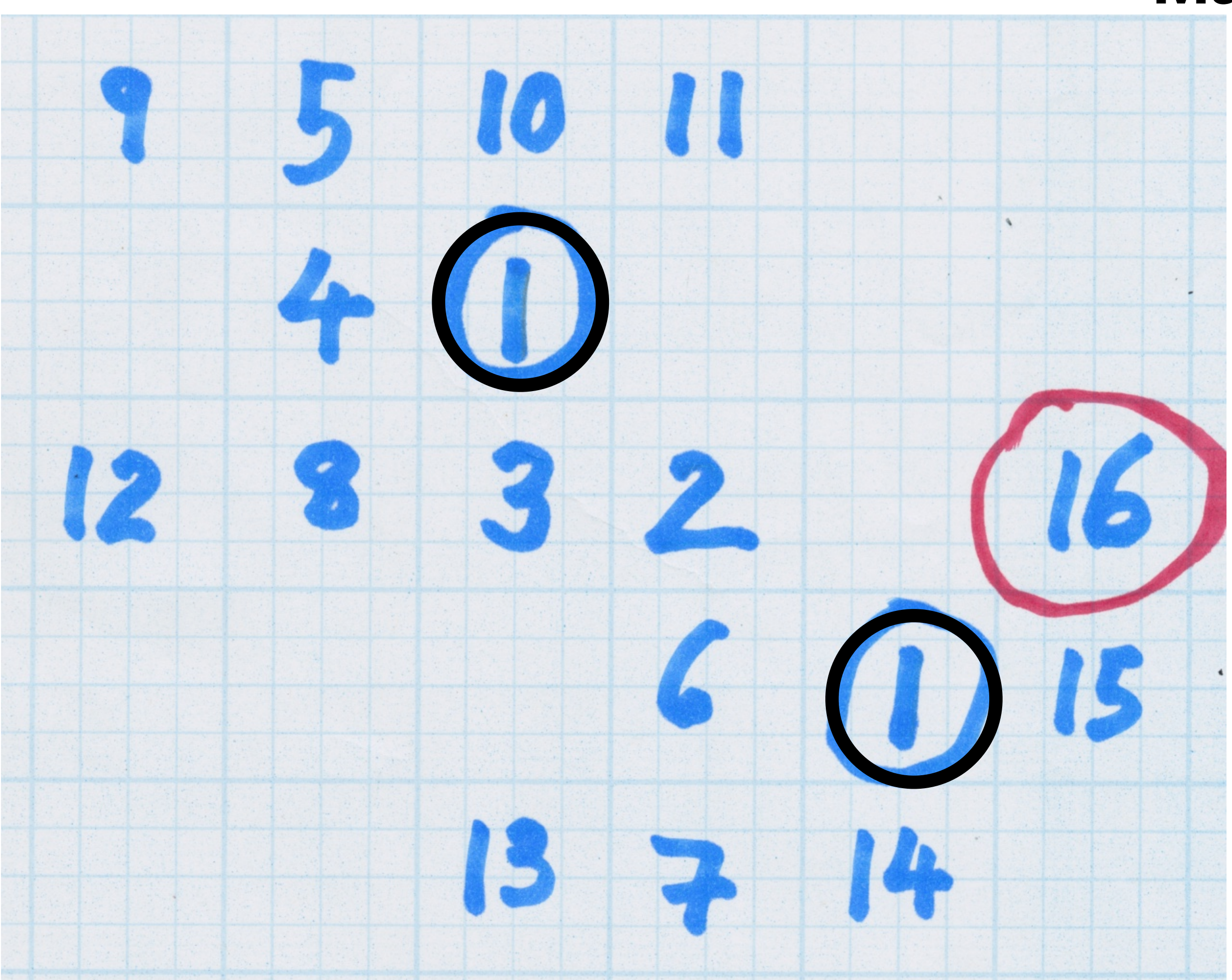
Example: n = 16, p = 2, M = 5,
S is [-11, -9, -7, -5, -3, -1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19]
and 29 sums of pairs are powers of 2

[2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, 16, 16, 16, 16, 16, 16, 32, 32] and W(16) >= a(16) = 29.

Stepping Stones Puzzle

Thomas Ladouceur and Jeremy Rebenstock, October 2020

Start by placing n 1's on infinite square grid.
Then write 2, 3, 4, …, m subject to condition
that when you write k, the sum of its neighbors must equal k.
Maximize m.



a(2) = 16

A337663

Stepping Stones Puzzle (2)   A337663

Only six values known:

1	2	3	4	5	6
1	16	28	38	49	60

Lower bound (Andrew Howroyd) $a(n) >= 5n-4$

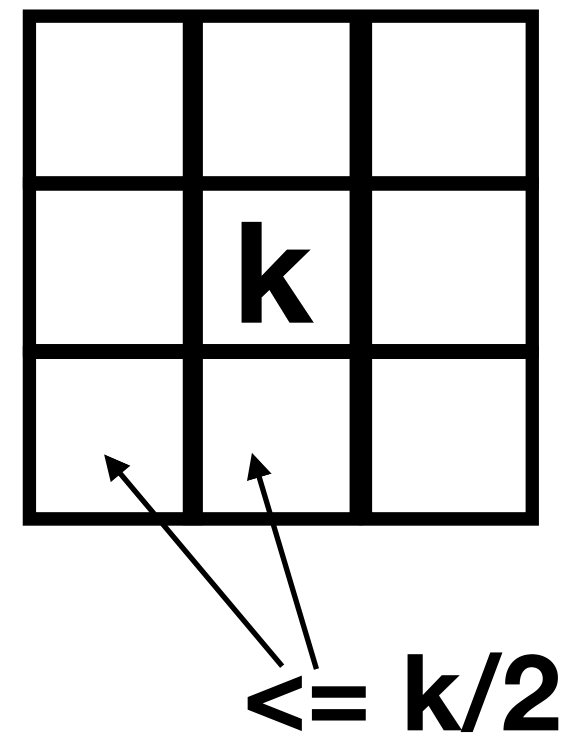
Proof: Continue this pattern!

```
+----+----+----+----+----+----+----+----+----+----+
|    |    | 4  |    |    |    |    |    | 14 |    |
+----+----+----+----+----+----+----+----+----+----+
|    | 3  | 1  | 5  |    |    |    | 13 | 1  | 15 |
+----+----+----+----+----+----+----+----+----+----+
|    | 2  |    | 6  |    |    |    | 12 |    | 16 |
+----+----+----+----+----+----+----+----+----+----+
| 1  |    |    |    |    | 7  | 11 |    |    |    |
+----+----+----+----+----+----+----+----+----+----+
|    |    |    |    |    | 8  | 1  | 10 |    |    |
+----+----+----+----+----+----+----+----+----+----+
|    |    |    |    |    |    | 9  |    |    |    |
+----+----+----+----+----+----+----+----+----+----+
```

Stepping Stones Puzzle (3) A337663

Upper bound (Robert Gerbicz)

$$\boxed{a(n) < 714\,n}$$



Idea of proof. Suppose a(n) = k > 1, and let S denote the square containing k.

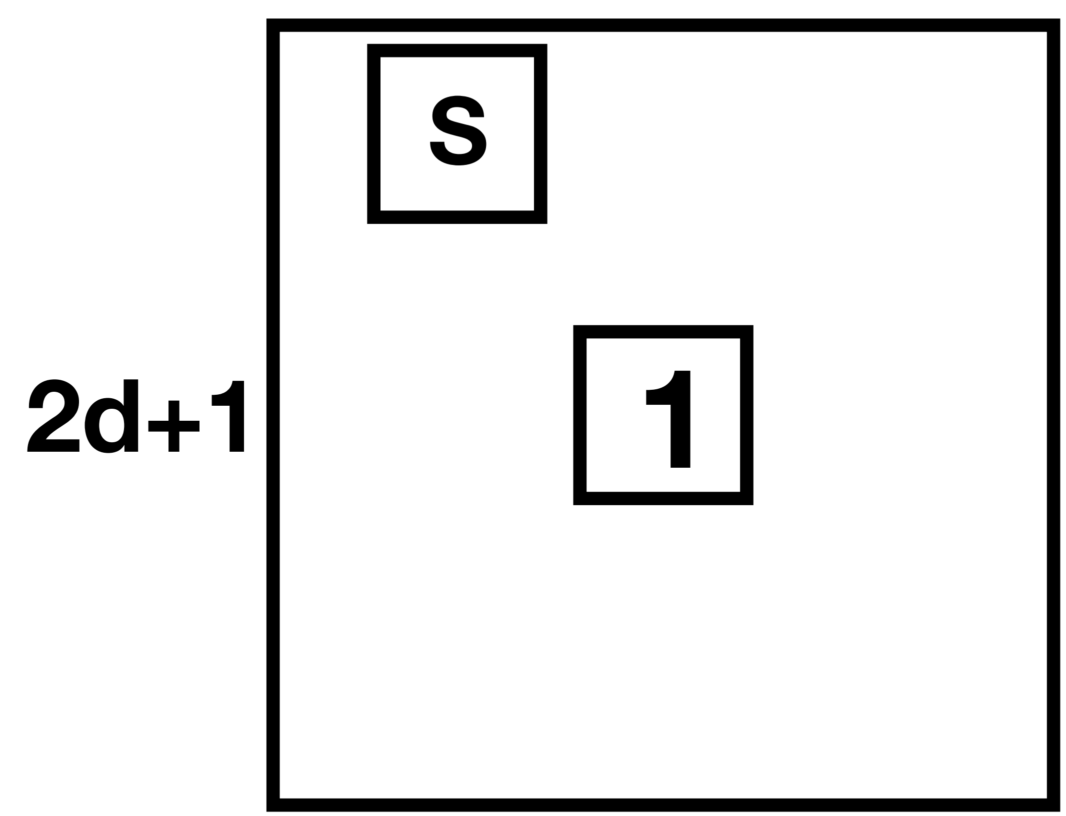
Since S is the sum of its neighbors, there is a nonempty square adjacent to S containing at most k/2.

So if k < 2^(d+1), there is a square within distance d of S containing 1.

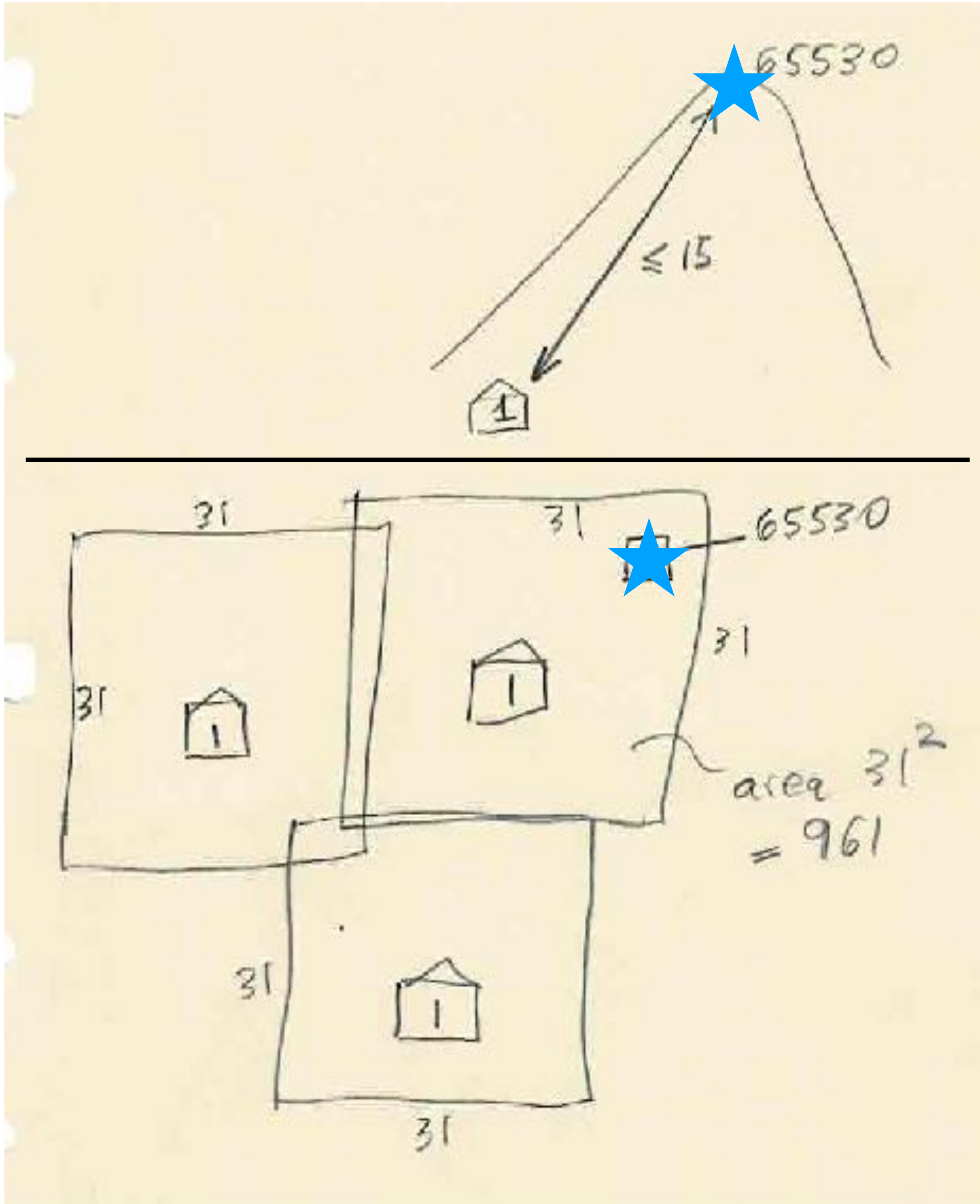
We have n 1's, so S is within one of the n squares of side 2d+1 around the 1's.

These also contain 2,3,...,k-1.

See A337663 for details of proof.



Stepping Stones Puzzle (4)   A337663



a(n) cannot be too big

Think of a mountain peak with n climbers' huts where the 1's are

If a(3) >= 65530, then need 3 X 961 >= 65529 false

[To answer a question raised during the talk, the same argument that proves that there must be a 1 within distance 15 of the 65520 peak also shows that there must be a 1 within distance 15 of any of the numbers 2 through 65529. So all the numbers from 2 though 65530 must be inside the three squares of side 31, which is impossible]

A337663

Illustration for a(4) = 38 from Arnauld Chevallier:

```
 .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
 . 35 18 36  . 23  . 21  . 32  .  .  .  .  .
 .  . 17  1  . 14  9  . 12 20  .  .  .  .  .
 .  . 34 16 15  .  5  4  8  .  . 26 27  .  .
 .  .  .  . 31  . 10  1  3 19 25  .  1 28  .
 .  .  .  .  .  . 11  .  2  6  . 33  . 29  .
 .  .  .  .  .  . 24 13 22  1  7  .  .  .  .
 .  .  .  .  .  . 37  .  . 30 38  .  .  .  .
 .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
```

From Bert Dobbelaere, Nov 01 2020 (Start):
Illustration for a(6) = 60:

```
 .  .  .  .  .  .  .  .  .  .  .  .  . 47 24 48  .
 .  .  .  .  .  .  .  .  .  .  .  .  .  . 23  1 49
 .  .  .  .  .  .  .  .  .  .  .  . 41  . 22  . 50
 .  .  .  .  .  .  .  . 51  . 36  . 20 21 43  .  .
 .  .  .  .  .  .  .  . 34 17  . 19  1  .  .  .  .
 .  .  .  .  .  .  .  . 16  1 18 38 58 59  .  .  .
 .  .  .  .  . 37 30 15 40  . 57  .  .  .  .  .  .
 .  .  .  .  .  .  7  8  .  .  .  .  .  .  .  .  .
 .  .  . 35 46  6  1 25 33  .  .  .  .  .  .  .  .
 . 60 32  .  3  2  9  .  .  .  .  .  .  .  .  .  .
 .  . 28  4  1 31 11 45  . 52  .  .  .  .  .  .  .
 . 42 14 10  5  .  . 12 13 39  .  .  .  .  .  .  .
 . 56  . 29 44  .  .  1 26  .  .  .  .  .  .  .  .
 .  .  .  .  .  . 55 54 27 53  .  .  .  .  .  .  .
```

Code Golf; C; Perl; Python

A New L.E.S.

L.E.S. = Lexicographically Earliest (Infinite) Sequence

New L.E.S. in EKG, Yellowstone, Enots Wolley Family (A347113)

Lexicographic order: 1,2,3,... before 1,2,4,... etc

Usually:

L.E.S. = Lexicographically Earliest Infinite Sequence of distinct positive numbers such that ...

Many famous examples:

EKG sequence: gcd(a(n-1), a(n)) > 1 (A064413). Perm. of pos. numbers? YES

Yellowstone Permutation: gcd(a(n-2), a(n)) > 1; gcd(a(n-1), a(n))) = 1 (A098550). Perm. of pos. numbers? YES

Enots Wolley sequence: gcd(a(n-2), a(n)) = 1; gcd(a(n-1), a(n))) > 1 (A3369570). Perm. of {1, 2, numbers with at least 2 different prime factors }? WE STILL DO NOT KNOW

GRANT OLSON'S SEQUENCE

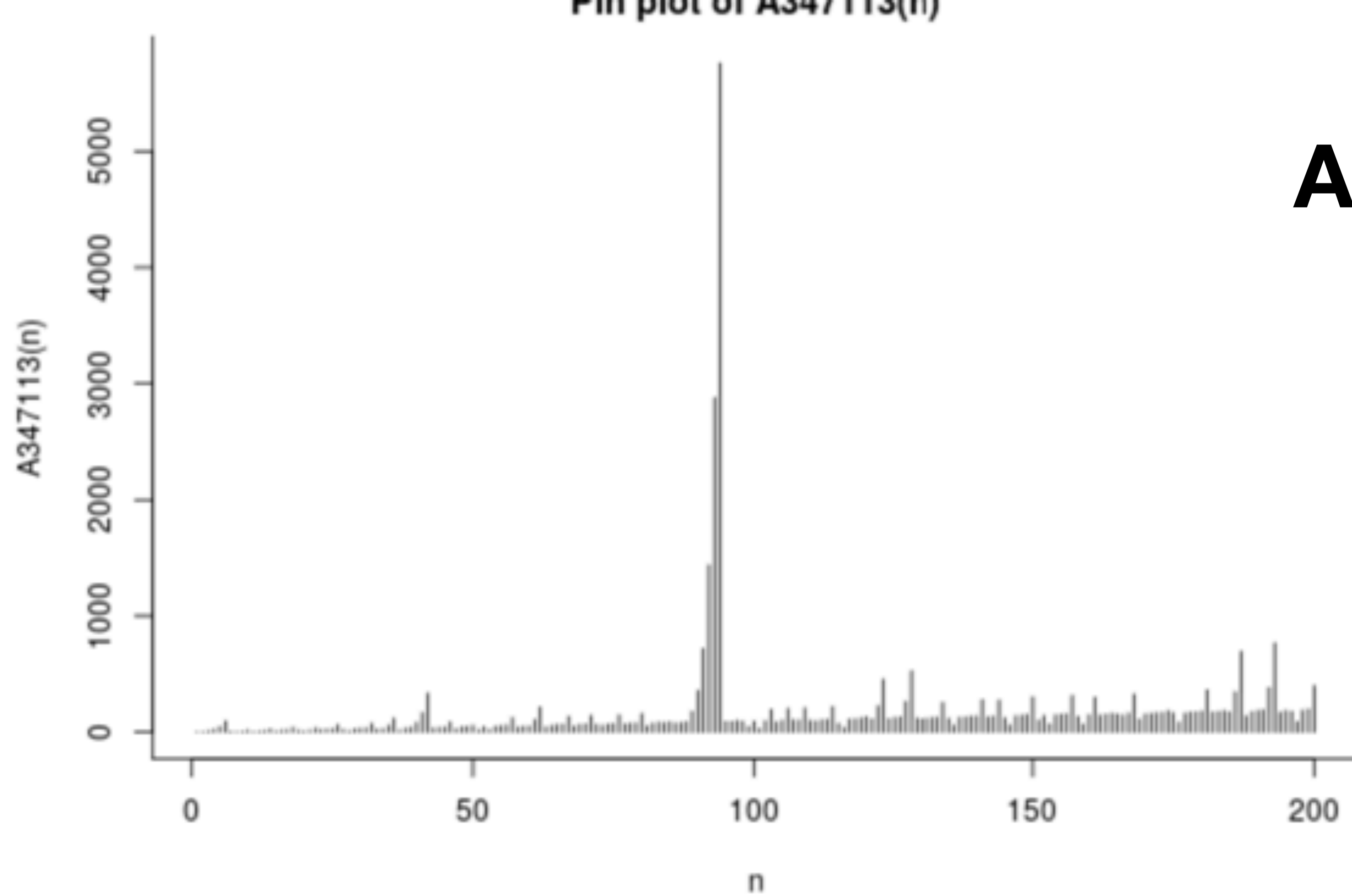
The new member of the family, A347113, from Grant Olson, August 18 2021

L.E.S. sequence of distinct positive numbers such that gcd(a(n-1)+1, a(n)) > 1 but a(n) != a(n-1) + 1.

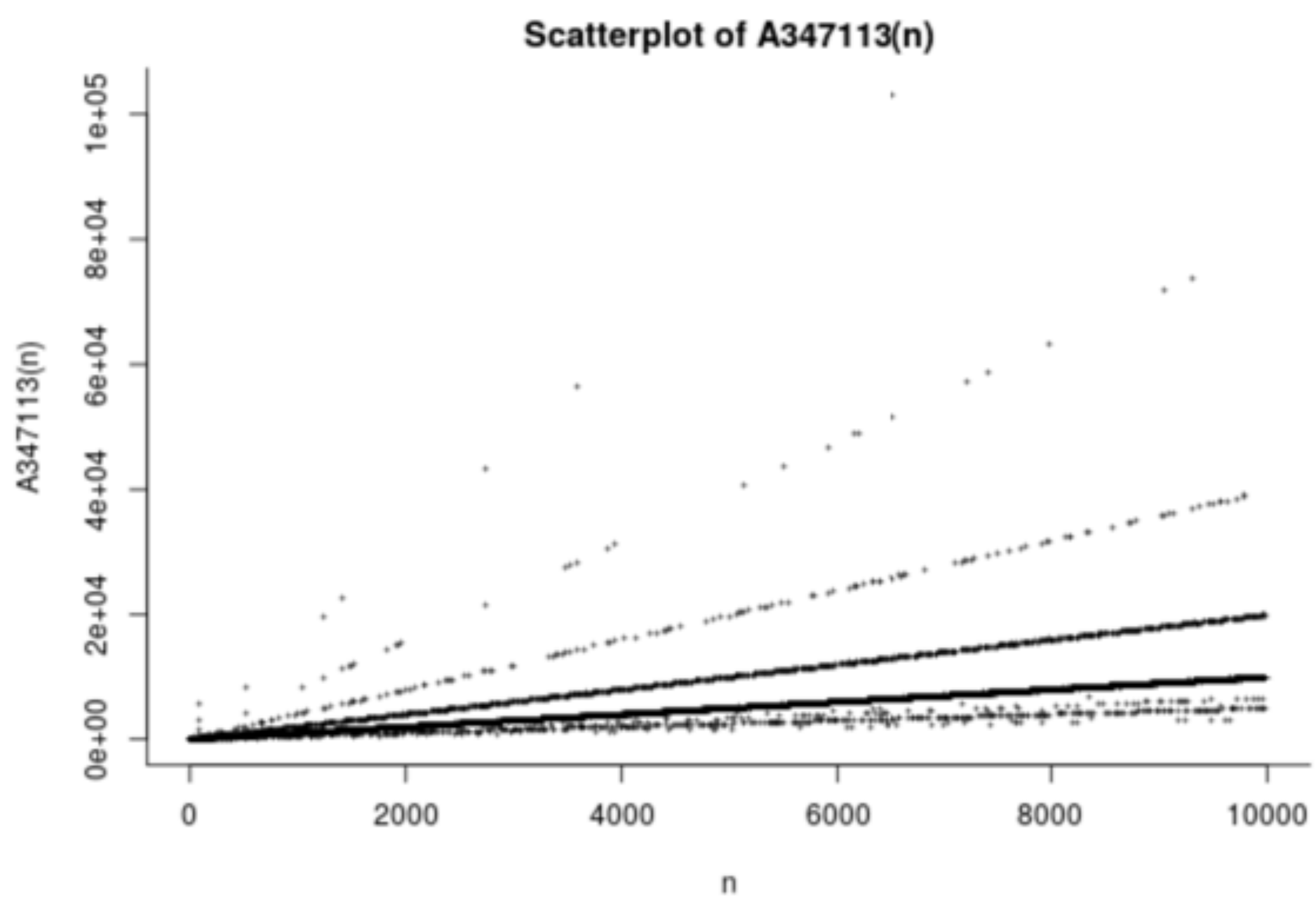
1, 4, 10, 22, 46, 94, 5, 2, 6, 14, 3, 8, 12, 26, 9, 15, 18, 38, 13,

Open problem: Show that this is a permutation of the positive integers.

The graph has some unusual features….


Pin plot of A347113(n)
A347113(n)
0
1000
2000
3000
4000
5000
0
50
100
150
200
n

A347113


Scatterplot of A347113(n)
A347113(n)
0e+00
2e+04
4e+04
6e+04
8e+04
1e+05
0
2000
4000
6000
8000
10000
n

Explanation of spike at n = 94

n	84	85	86	87	88	89	90
a(n)	80	87	77	81	88	178	358
a(n)+1	3^4	8.11	2.3.13	2.41	89	179	359

n	91	92	93	94	95	96	97
a(n)	718	1438	2878	5758	91	90	98
a(n)+1	719	1439	2879	13.443	4.23	7.13	9.11

Cunningham chain of primes

Set-Theory Versions of LES Sequences

5 and 10 are disjoint
101
1010

6 and 10 intersect
110
1010

Set-theory analog of EKG sequence A064413

a(n) = smallest unused number that intersects a(n-1)

n	a(n)	a(n)	set
1	1	1	1
2	11	3	1,2
3	10	2	2
4	110	6	2,3
5	100	4	3
6	101	5	1,3
7	111	7	1,2,3
8	1001	9	1,4
9	1000	8	4
10	1010	10	2,4
11	1011	11	1,2,4
12	1100	12	3,4

...

A115510

Easily seen to be a permutation of pos. integers

Set-theory analog of Yellowstone A098550

a(n) = smallest unused number that intersects a(n-2) but not a(n-1)

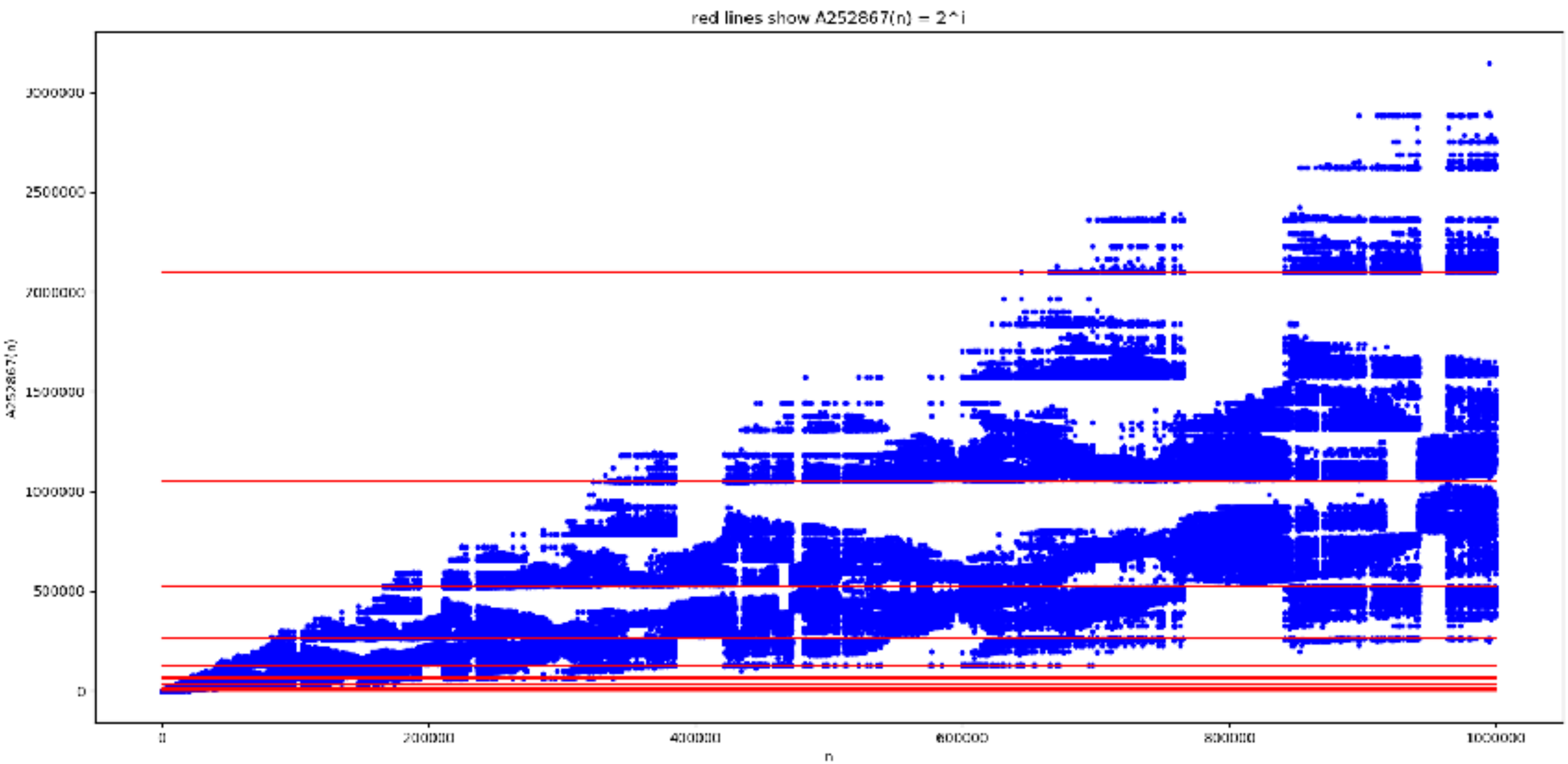
n	a(n)	a(n)	set
1	1	1	1
2	10	2	2
3	101	5	1,3
4	1010	10	2,4
5	100	4	3
6	11	3	1,2
7	1100	12	3,4
8	10001	17	1,5
9	110	6	2,3
10	1001	9	1,4
11	10010	18	2,5
12	1000	8	4

A252867

Open Problem: Show this is a permutation of pos. integers

Set-theory analog of Yellowstone Permutation A252867

Graph of first million terms, from Chai Wah Wu



It is conjectured that this is the graph of a permutation!

The red lines are at powers of 2

Gerrymandering

1. **A341578 (Sean Chorney), A341721 (Don Reble), Feb 2021: Minimum number of votes needed to win with n voters if all districts must have same size**

Rules

Two candidates, a and B, and n voters.
The voters are divided into d equal districts of size n/d.
The districts are winner-takes-all.
Tied districts go to neither candidate.

If there are an even number of districts, it is enough to win half the districts and tie in one further district.

Example: n = 36 voters: optimal strategy is three districts of 12 voters each, and then you can win with only 14 total votes (7+7+0).

EMPTY SQUARES ARE 22 “a” VOTES

B	a	a	a	B	B
a	a	B	a	a	a
B	a	B	a	B	a
a	B	a	B	a	B
a	a	a	a	B	a
B	a	a	B	B	a

36 VOTERS

ONLY 14 B VOTES

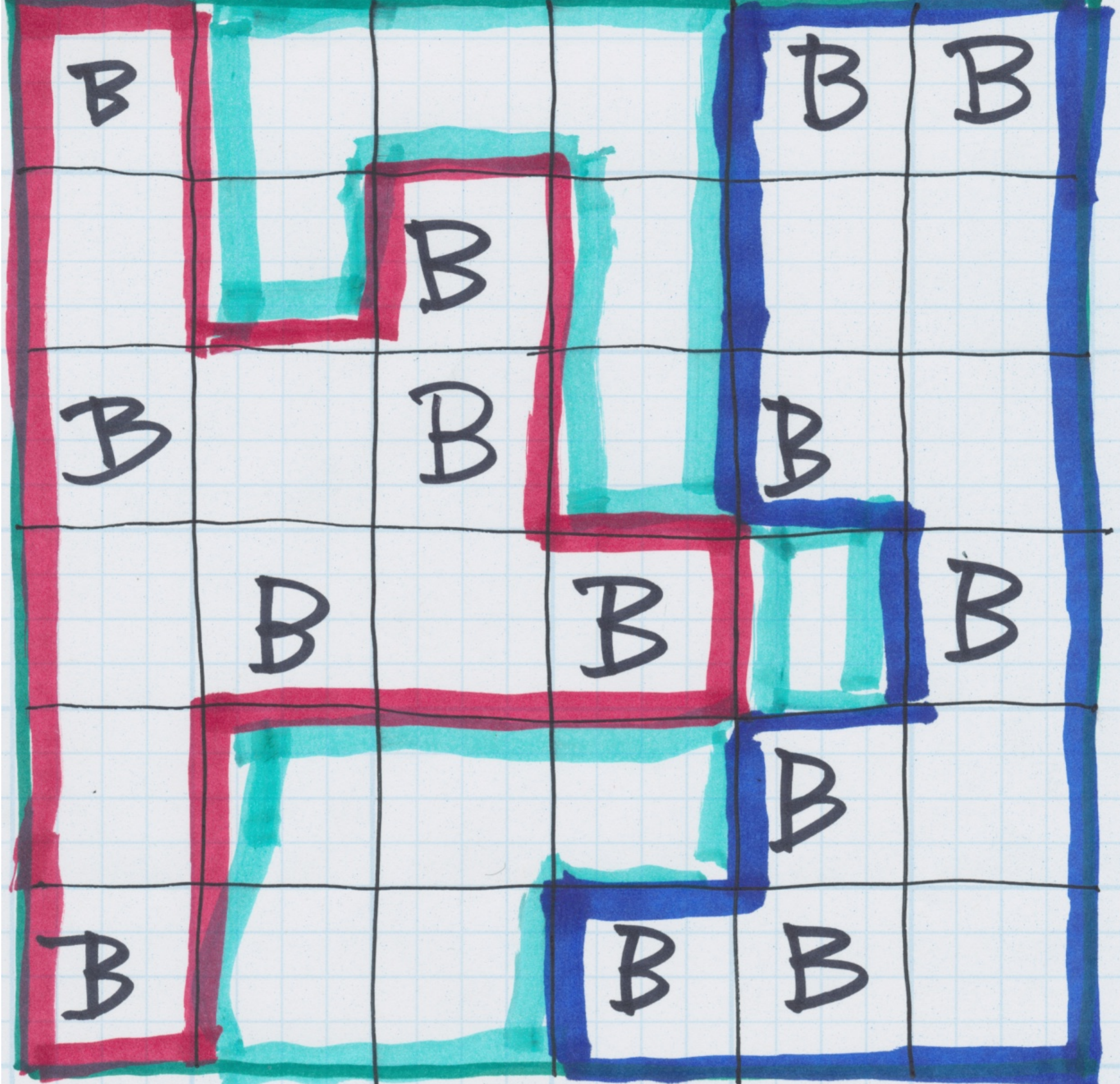
A341578
Sean Chorney
Min. no. to win
if n^2 voters

a(6)=14

A341721
Same, if V votes
b(36)=14

EMPTY SQUARES ARE 22 A VOTES

Open Q.
What if the regions must be edge-connected?



36 VOTERS
ONLY 14 B VOTES

Answer: 3 districts of size 12

Suggestion for a research project

Moon Duchin heads a study group at Tufts University (the Metric Geometry and Gerrymandering Group) which has produced many papers.

For instance, how can you detect, or prove, that Gerrymandering has taken place?

How to measure Gerrymandering?

There should be new sequences (as a function of the number of voters, or number of districts) that arise from this work!

Do a Google Scholar search for Moon Duchin, Redistricting, to see many articles.

An old paper: Moon Duchin, Gerrymandering metrics: How to measure? What's the baseline? arXiv:1801.02004, Jan 06 2018

With thanks to:

People who contributed to the sequences mentioned in this talk
(a partial list)

Lars Blomberg, Arnauld Chevallier, Sean Chorney, Harvey Dale,
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Thomas Ladouceur, Michel Marcus, Peter Munn, Grant Olson,
Hugo Pfoertner, Rob Pratt, Leroy Quet, Jeremy Rebenstock, Don Reble,
Joseph Rozhenko, Scott Shannon, Rémy Sigrist ,
Jan Ritsema van Eck, Stan Wagon, Chai Wah Wu, Piotr Zielinski,
Richard Zumkeller

A New Theorem That Exists Because of the OEIS (September 5 2021)

Theorem:
Let f(n) = number of partitions of n into distinct parts such that (greatest part) - (least part) = (number of parts)
= A238005(n),
let g(n) = (maximal number of parts in any partition of n into distinct parts) minus (number of ways to partition n into consecutive parts)
= A003056(n) - A001227(n).
Then f(n) = g(n) for all n.

Conjectured by Omar Pol, proved independently by William Keith and Roland Bacher, proof simplified by Don Reble.

See A238005 for proof (including bijection and generating functions).

Search for “It appears that …” , “It seems that …”, “Conjecture: …”, or “Empirical: …” for a large number of other potential theorems!