Replicable functions: an introduction

John McKay[1] and Abdellah Sebbar[2]

[1] Department of Mathematics and CICMA Concordia University 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada.
mckay@cs.concordia.ca

[2] Department of Mathematics and Statistics, University of Ottawa. Ottawa, ON K1N 6N5, Canada
sebbar@mathstat.uottawa.ca

Summary. We survey the theory of replicable functions and its ramifications from number theory to physics .

1 Introduction

We survey the theory of replicable functions and matters of related interest. These functions were introduced in monstrous moonshine [4], characterizing the principal moduli attached to the conjugacy classes of the monster simple group, $\mathbb{M}$. It is not surprising that replicable functions, being related to moonshine and the monster, have a wide range of connections to other fields of mathematics and physics which remain to be fathomed. Indeed, moonshine has been described as 21st. century mathematics in the 20th. century. Having arrived, we can survey the past 25 years with some satisfaction but there is much remaining to be clarified and put into an appropriate context. The field is amazingly fertile: there are connections with several aspects of mathematical physics and number theory, and one finds classical and modern themes continually coming into play. We explain a few of these connections, some of which are presented here for the first time.

It is simplest to define replicable functions through the Faber polynomials to which the next section is devoted. We then provide examples related to classical themes such as Chebyshev polynomials and Hecke operators. Later sections will deal with the automorphic aspect of the replicable functions, links with the Schwarz derivative, the characterization of the monstrous moonshine functions, the exceptional affine correspondences, class numbers, and the soliton equations and their $\tau-$function from the 2D Toda hierarchy.

2 Faber polynomials

The Faber polynomials [9] originated in approximation theory in 1903 and are central to the theory of replicable functions. We define them in a formal way, leaving complex analysis and Riemann mappings for later in section 3 and section 12.

Let f be a function given by the expansion

$$f(q) \;=\; \frac{1}{q} \;+\; \sum_{n=1}^{\infty} a_n q^n, \tag{1}$$

where we take $q = \exp(2\pi i z), z \in \mathfrak{H}$, the upper half–plane. Throughout, we interpret derivatives of f with respect to its argument. We initially assume that the coefficients $a_n \in \mathbb{C}$, and we choose the constant term to be zero. For each $n \in \mathbb{N}$, there exists a unique monic polynomial F_n such that

$$F_n(f(q)) = \frac{1}{q^n} + \mathrm{O}(q) \quad \text{as} \quad q \to 0.$$

In fact $F_n = F_{n,f}$ depends on the coefficients of f, but we denote it simply by F_n when there is no confusion. It can be shown that the Faber polynomials are given by the generating series

$$\frac{qf'(q)}{z-f(q)} = \sum_{n=0}^{\infty} F_n(z)q^n,$$

with $F_0(z)=1$, $F_1(z)=z$, $F_2(z)=z^2-2a_1$, $F_3(z)=z^3-3a_1z-3a_2$,

and more generally:

$$F_n(z) = \det(zI - A_n),$$

where

$$A_n = \begin{pmatrix} a_0 & 1 & & & & \\ 2a_1 & a_0 & 1 & & & \\ \vdots & \vdots & \vdots & & & \\ (n-2)a_{n-3} & a_{n-4} & a_{n-5} & \cdots & 1 & \\ (n-1)a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_0 & 1 \\ na_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \end{pmatrix}.$$

It is useful to note that the Faber polynomials satisfy a Newton type of recurrence relation of the form

$$\text{for all } n \geq 1, \quad F_{n+1}(z) = zF_n(z) - \sum_{k=1}^{n-1} a_{n-k}F_k(z) - (n+1)a_n. \tag{2}$$

3 Example 1: The case of the ellipse

In this and the next two sections we look at instances for which the Faber polynomials are related to classical objects. This makes it easier to compute them as seen in section 6.

Let us recall the definition of the Chebyshev polynomials. For a positive integer n, and interval $[a,b]$, the nth. Chebyshev polynomial is the monic polynomial of degree n of least max norm as an element of the space of continuous functions on $[a,b]$. For the interval $[-1,1]$, the unique solution is given by

$$\begin{aligned} P_n(z) &= 2^{-n}\left([z+(z^2-1)^{\frac{1}{2}}]^n + [z-(z^2-1)^{\frac{1}{2}}]^n\right) \\ &= 2^{1-n}\cos(n \arccos z). \end{aligned} \tag{3}$$

The max norm of P_n is given by $||P_n|| = 2^{1-n}$. The general case $[a,b]$ is easily reduced to this particular case. The term Chebyshev polynomial is also given to the general class of solutions of similar extremal problems. Indeed, If E is a compact set in the complex plane, then there exists a polynomial of least norm, as a continuous function on the compact E, among all monic polynomials of degree n. The polynomial is unique if the set contains at least n points.

We initially assume $c \in \mathbb{R}$, $c \neq 0$. The simplest example of the series (1) is the function

$$f_c(q) = \frac{1}{q} + cq. \tag{4}$$

The image by f_c of the circle centred at 0 of radius $\alpha = |c|^{-\frac{1}{2}}$ is a real segment $[-\frac{2}{\alpha}, \frac{2}{\alpha}]$ for $c > 0$ or a purely imaginary vertical segment $[-\frac{2}{\alpha}i, \frac{2}{\alpha}i]$ for $c < 0$.

When $c > 0$ and $z = \frac{2}{\alpha}e^{i\theta}$, then $f_c(z) = \frac{2}{\alpha}\cos(\theta)$ and we find the Chebyshev polynomial for the interval $[-\frac{2}{\alpha}, \frac{2}{\alpha}]$ is given by

$$P_n(z) = \frac{2}{\alpha^n}\cos n \arccos \frac{\alpha}{2} z.$$

It follows that

$$P_n(f_c(q)) = \frac{1}{q^n} + c^n q^n.$$

It is easy to see that the same formula is valid for $c < 0$, hence the Faber polynomials and the Chebyshev polynomials coincide.

This identity is just the tip of the iceberg. Indeed, rather than taking the circle of radius α one could take a more uniform approach by considering the image by f_c of the unit circle U. The mapping f_c realizes the Riemann mapping from the exterior of U to the exterior of its image which is the ellipse

$$\frac{x^2}{(c+1)^2} + \frac{y^2}{(c-1)^2} = 1.$$

Here we take $c \neq \pm 1$. The transformation $z \mapsto (2c)^{-1/2}z$ sends the above ellipse onto the ellipse

$$\frac{4cx^2}{(c+1)^2} + \frac{4cy^2}{(c-1)^2} = 1$$

which has the property of having foci at $z = \pm 1$. It can be shown that for an ellipse having this property, the Chebyshev polynomial is given by (3), see Hille [11] page 267, thus we can recover the above results. This approach can be generalized to other Riemann mappings. For the degenerate cases $c = \pm 1$, the images of U by f_c are just the ordinary segments $[-2, 2]$ or $[-2i, 2i]$. As for $c = 0$, the image of U is simply U, and it is clear that both the Faber polynomials for f_0 and the Chebyshev polynomials for U are given by $F_n(z) = z^n$. Notice that cases $c = 0, -1, 1$ correspond respectively to f_c being exp, sin and cos. Although these "modular fictions" are the simplest of replicable functions, they find use by Takahashi [19] in topological Landau-Ginsburg field theory.

4 Example 2: Hecke operators

Let $f(z)$ be a modular form of weight k on $\mathrm{SL}_2(\mathbb{Z})$. The Hecke operators T_n, $n \geq 1$, act on f as

$$\text{for all } n \geq 1\,, \quad T_n(f)(z) \;=\; n^{k-1} \sum_{\substack{ad=n \\ 0\leq b<d}} d^{-k} f\left(\frac{az+b}{d}\right). \tag{5}$$

See the first five chapters of [20], and especially Zagier's article, for background details. When $k = 0$ and $f(z)$ is $j(z)$, the classical elliptic modular function, we have

$$\text{for all } n \geq 1\,, \quad T_n(j)(z) = \frac{1}{n} \sum_{\substack{ad=n \\ 0\leq b<d}} j\left(\frac{az+b}{d}\right). \tag{6}$$

The generators of $\mathrm{SL}_2(\mathbb{Z})$ permute the linear fractional transformations in the sum, hence $T_n(j)$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$. Since it has no pole on the upper half–plane $\mathfrak{H}$, it follows that it is a polynomial in $j(z)$, see also Serre [18]. We find that

$$\text{for all } n \geq 1\,, \quad T_n(j)(q) \;=\; \frac{1}{q^n} \;+\; \mathrm{O}(q), \quad \text{as} \quad q \to 0.$$

and so $T_n(j) = \frac{1}{n}F_{n,j}(j)$. It is this example of the Hecke action on j that the notion of a replicable function encapsulates with greater generality.

5 Example 3: Moonshine and the Monster

Let $\mathbb{M}$ be the Monster sporadic simple group of order,

$$|\mathbb{M}| = 2^{46}3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71,$$

of which Ogg remarked [15] that the 15 prime divisors of $|\mathbb{M}|$ are exactly the supersingular primes – that is, those primes p for which – for all supersingular elliptic curves defined over the closure of $\mathbb{F}_p$ – we have $N_p = p + 1$ and the j–invariant lies in the base field, $\mathbb{F}_p$.

The central observation of monstrous moonshine [4] is that, to each conjugacy class of cyclic subgroups, $\langle g \rangle$ of $\mathbb{M}$, there corresponds an automorphic function,

$$f_g(q) \;=\; \frac{1}{q} + \sum_{n=1}^{\infty} a_n(g) q^n,$$

for some genus zero congruence group such that its (Fourier) coefficients are the rational integer traces of the infinitely many "head representations" $\{H_n\}$ of $\mathbb{M}$, thus

$$\text{for all } n \geq 1, \quad a_n(g) = Tr(H_n(g)).$$

For example, the automorphic function associated with the identity element is the classical elliptic modular function j. Here we take $j(q) = 1/q + \sum_{n=0}^{\infty} c_n q^n$, to have a "monstrous" expansion at ∞ as $j(q) = 1/q +$

$196884q + \cdots$, so that $c_0 = 0, c_1 = 196884, \ldots$ with a zero constant term as opposed to the "arithmetic" $j-$function for which $c_0 = 744$. There is also the "analytic" $j-$function which is $\frac{1}{1728}$ times the arithmetic $j-$function, and has critical values of $\{0, 1, \infty\}$. Despite these variations, there is a "natural" value of 24 for the constant value among the integer expansions. Henri Cohen points this out on page 222 [20], the reason being that Rademacher's convergent expansion for the coefficients, $c_n, n \geq 1$ produces the value of 24 at $n = 0$ [16]. See also [4] page 323.

There is a relationship between the subset of replicable functions known as monstrous moonshine functions, associated with the powers of an element $g \in \mathbb{M}$, which is as follows: Let f_{g^a} denote the automorphic function associated to the power g^a of g. For each $n \geq 1$, the sum

$$\sum_{\substack{ad=n \\ 0 \leq b < d}} f_{g^a}\left(\frac{az+b}{d}\right), \tag{7}$$

is in fact a polynomial in f_g. This polynomial is nothing else but the Faber polynomial F_n associated with f_g. When g is the identity element of $\mathbb{M}$, we recover (6):

$$\text{for all } n \geq 1\,, \quad F_n(j)(z) = \sum_{\substack{ad=n \\ 0 \leq b < d}} j\left(\frac{az+b}{d}\right).$$

6 Replicable functions

The functions in the previous sections are called replicable. We give a formal definition due to Norton, and an equivalent one in terms of a generalized Hecke operator:

For f, a function of the form (1),

$$f(q) \;=\; \frac{1}{q} + \sum_{n=1}^{\infty} a_n q^n,$$

we write its corresponding Faber polynomial, $F_n(f)$ of degree n, as

$$F_n(f(q)) \;=\; \frac{1}{q^n} + n \sum_{m=1}^{\infty} h_{m,n}\, q^m.$$

Clearly $h_{n,1} = a_n$, and, further, the double sequence $\{h_{m,n}\}$ is symmetric.

Definition 1. The function f is said to be replicable if $h_{m,n} = h_{r,s}$ whenever $\gcd(m,n) = \gcd(r,s)$ and $\mathrm{lcm}(m,n) = \mathrm{lcm}(r,s)$.

It follows immediately from this definition that the functions f_c of section 3 are replicable functions. It has been shown [6] that these are the only replicable functions with a finite number of nonzero $q-$coefficients.

To justify this terminology, we give an equivalent definition:

Definition 2. The function f is said to be replicable if for each positive integer n and each positive divisor a of n, there are functions $f^{(a)}$ of the form (1), called the replication powers of f, such that

$$\text{for all } n \geq 1\,, \quad F_n(f(q)) \;=\; \sum_{\substack{ad=n\\0\leq b<d}} f^{(a)}\left(\frac{az+b}{d}\right). \tag{8}$$

We refer to the right side of (8) as the action of a new generalized Hecke operator, $\widehat{T}_n$ and can show that these two definitions are equivalent. Moreover, the replication powers of f are given by:

$$f^{(k)}(q) \;=\; \frac{1}{q} \;+\; \sum_{i=1}^{\infty} a_i^{(k)} q^i,$$

where

$$a_i^{(k)} = k \sum_{d|k} \mu(d) h_{dki,k/d},$$

and μ is the Möbius function.

It follows immediately from the second definition that the examples of section 4 and 5 are replicable functions.

7 Automorphic aspects of replicable functions

Replicable functions have been extensively studied since the advent of moonshine. The monstrous moonshine functions associated to the conjugacy classes of the Monster are both replicable functions and automorphic functions for some genus zero congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$, and it is legitimate to ask whether all replicable functions are automorphic. In fact, Norton conjectured that the set of replicable functions f with integer coefficients coincides with the set of principal moduli for genus zero subgroups of $\mathrm{PSL}_2(\mathbb{R})$, having translations generated by $z \mapsto z+1$, and containing to finite index, the group, $\Gamma_0(N)$, of all upper triangular matrices mod some level, N. Later, Cummins and Norton [7] proved that all principal moduli, as above, with rational $q-$coefficients, are replicable. It was known that the number of replicable functions is finite, and Norton and others have computed a conjectural full list of over 600 replicable functions with rational integer coefficients which are principal moduli. A satisfactory proof of the completeness of the list remains to be found. Their determination is based on a remarkable result due to Norton which asserts that every replicable function is determined by 12 of its first 23 Fourier coefficients.

8 Links with the Schwarz derivative

The prototypical replicable function is the $j-$function, which replicates to itself. Dedekind, [8] defines the analytic $j-$function as a solution of a third order differential equation involving the Schwarz derivative which is an invariant differential operator for the action of $\mathrm{PGL}_2(\mathbb{C})$. It has the form

$$\{f,z\} \;=\; 2\left(\frac{f''}{f'}\right)' - \left(\frac{f''}{f'}\right)^2.$$

The Schwarz derivative preserves the group invariance and raises the modular weight by 4. Now by definition, a principal modulus generates the field of meromorphic functions on its defining genus zero Riemann surface. Now $f' = \frac{df}{dz}$ has weight 2, and so we deduce the Schwarz equation, $\{f,z\} + R(f){f'}^2 = 0$, with $R(f)$ a rational function of f, and $\{f,z\}$, an automorphic form everywhere holomorphic except at its elliptic fixed points where it has double poles. For Dedekind, $R(j) = \frac{36j^2-41j+32}{36(j(j-1))^2}$. This strange rational function of j is seen in a more familiar light when expanded as a partial fraction, thus:

$$R(j) = \frac{1-\frac{1}{3^2}}{j^2} + \frac{1-\frac{1}{2^2}}{(j-1)^2} - \frac{1-\frac{1}{3^2}-\frac{1}{2^2}}{j(j-1)}, \tag{9}$$

where the residues at the double poles give the ramification at the $n-1=2$ finite critical values, $j=0, j=1$. Finding the terms other than the double poles is the problem of the $n-3$ accessory parameters and is known to be hard in general but here for j, since there are only 3 critical values, there is no problem, however, for replicable functions, the largest number of critical values (including ∞) is $n=26$. As we remark in section 11, these critical values have the symmetry of a "generalized dihedral group" and this awakens memories of Poincaré's remark to the effect that "if one has sufficient symmetry, the accessory parameter problem may be solved". Effectively $R(f)$ describes the ramification occurring at intersections of consecutive arcs of a sequence of (possibly degenerate) circles centred on the real axis and it is these circles that bound a natural fundamental domain in $\mathfrak{H}$ which is completely determined once edge identifications are made. We see that solving the Schwarzian differential equation for f is equivalent to finding f from its natural fundamental domain.

The Schwarz derivative relates to the Faber polynomials when it exhibits the double sequence $\{h_{m,n}\}$ in an elegant way. Namely, we have the identity

$$\frac{1}{4\pi^2}\{f,z\} = 1 + 12 \sum_{m,n\geq 1} mn\, h_{m,n}\, q^{m+n}, \tag{10}$$

or in an alternative form:

$$\zeta(-1)\{f,q\} \;=\; \sum_{m,n\geq 1} mn\, h_{m,n}\, q^{m+n-2}. \tag{11}$$

As an illustration of our identity, we have:

$$\{\lambda, z\} \;=\; \pi^2\, E_4(z),$$

where λ is the classical level 2 modular elliptic function (principal modulus for $\Gamma(2)$), and E_4 is the weight 4 Eisenstein series. The coefficient 4 disappears because $\Gamma(2)$ has cusp width of 2 at ∞. On j, our identity is the differential limit derived from Borcherds' renowned product:

$$p(j(p) - j(q)) = \prod_{m\in\mathbb{N}, n\in\mathbb{Z}} (1 - p^m q^n)^{c_{mn}}.$$

9 The characterization of monstrous moonshine

Recently [3], a purely group-theoretic characterization of the 171 replicable functions that occur in the Monster has been obtained. We first introduce some notation:

Let N be a positive integer and h be the largest integer such that $h^2 \mid N$ and $h \mid 24$ and set $N = nh$. Let $\Gamma_0(n|h)$ be the group of matrices of the form

$$\begin{pmatrix} a & b/h \\ cn & d \end{pmatrix}, \quad ad - bcn/h = 1.$$

The group $\Gamma_0(n|h)$ is conjugate to $\Gamma_0(n/h)$ by $z \mapsto hz$.

For each exact divisor e of N (we write $e \,||\, N$), the Atkin-Lehner involution W_e is the set of matrices $\begin{pmatrix} ae & b \\ cN & de \end{pmatrix}$ with determinant e.

Each W_e is a single coset of $\Gamma_0(N)$. The full normalizer of $\Gamma_0(N)$ in $\mathrm{PSL}_2(\mathbb{R})$ is obtained by adjoining to the group $\Gamma_0(n|h)$ its Atkin-Lehner involutions w_e which are the conjugates by $z \mapsto hz$ of the Atkin-Lehner involutions W_e of $\Gamma_0(n/h)$.

The set of the exact divisors of N, $\mathrm{Ex}(N)$, is a group of exponent 2, where the group operation is given by $e * f = ef/\gcd(e,f)^2$. For each subgroup $\langle e, f, g, \ldots \rangle$ of $\mathrm{Ex}(n/h)$, we use the notation $\Gamma_0(n|h) + e, f, g, \ldots$ for the extension of $\Gamma_0(n|h)$ by its Atkin-Lehner involutions $w_e, w_f, w_g, \ldots$. Each of these extended groups has a subgroup of index h, denoted by $\Gamma_0(n||h) + e, f, g, \ldots$ and is defined as the kernel of the homomorphism $\lambda : \Gamma_0(n|h) + e, f, g, \ldots \to \mathbb{C}^\times$ defined by:

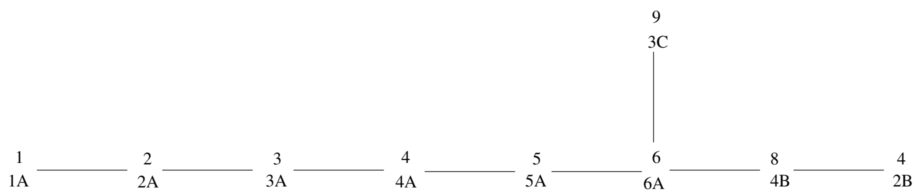
1. $\lambda = 1$ for elements of $\Gamma_0(N)$,
2. $\lambda = 1$ for all Atkin-Lehner involutions of $\Gamma_0(N)$
3. $\lambda = \exp(-2\pi i/h)$ for cosets containing $z \mapsto z + 1/h$,
4. $\lambda = \exp(\pm 2\pi i/h)$ for cosets containing $z \mapsto 1/(nz + 1)$, where the sign is $+$ if $-1/(Nz)$ is present and $-$ if not.

The main result of [3] is that a replicable function occurs in moonshine if and only if it is an automorphic function for a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ which

1. is genus zero,
2. has the form $\Gamma_0(n||h) + e, f, g \cdots$,
3. its quotient by $\Gamma_0(nh)$ is a group of exponent 2.
4. each cusp can be mapped to ∞ by an element of $\mathrm{PSL}_2(\mathbb{R})$ which conjugates the group to one containing $\Gamma_0(nh)$.

10 Affine Dynkin diagrams and sporadic correspondences

There is a distinguished set of nine conjugacy classes of $\mathbb{M}$ in which the product of any pair of Fischer involutions lies. These classes are described in the Atlas [2] on page 230. By monstrous moonshine, see [10] for details, each class corresponds to a unique modular function and thus we have a replicable function attached to each node of the affine E_8 diagram. Here class names lie below the modular levels.



Connections between simple sporadic groups and Lie groups are ample motivation for investigating this. We have further similar relations existing between the central extensions $2 \cdot B$, and $3 \cdot F'_{24}$ but we now work modulo the centres.

An alternative approach is to fold the E_7 and E_6 Dynkin diagrams to form F_4 and G_2 diagrams making the correspondence direct.

It is noteworthy that the Schur multipliers of these three sporadic groups $\mathbb{M}, B$, and F_{24}' are the fundamental groups of type E_8, E_7, and E_6 respectively. The very existence of many sporadic groups is dependent on a larger than expected Schur multiplier in some (possibly non-sporadic) centralizer subgroup.

Although the geometry of the Weyl groups of type E_6 (the 27 lines on a cubic surface) and of type E_7 (the 28 bitangents on a quartic curve) are well studied, the same cannot be said for E_8 and the 120 tritangent planes of a sextic curve of genus 4. The correspondence appears not to extend beyond the three exceptionals and one may speculate that del Pezzo surfaces are involved.

We have the potential of finding an analogue of the operator $R\otimes$ which occurs in defining the classical McKay correspondence of [13]. This conjectured analogue would act on a replicable function identified with a Dynkin node and would generate the set of functions on adjacent nodes. Note that functions adjacent to a given node may have different modular levels from the function at the node. This suggests lifting the modular functions to Jacobi forms. Currently we have no interpretation of adjacency.

11 Class numbers

There are two notions of class number treated in this section. The first is part of classical number theory, the second is less studied and comes from finite group theory.

The study of the critical values of principal moduli has been initiated in [5]. These are the values of $f(z_i)$ such that $f'(z_i) = 0$. The interest here is in finding the field extension,$K = K_f$, of $\mathbb{Q}$, generated by these critical values. The finite critical values of f are the roots of $D(f)$ where $R(f) = N(f)/D(f)^2$ for the rational function $R(f)$ of the Schwarz equation. This is, in general, reducible over $\mathbb{Q}$. Examination of computations of Gal(K/$\mathbb{Q}$) suggests that it is of “generalized dihedral type” in that Gal(K/$\mathbb{Q}$) almost always has a cyclic subgroup of index two, thus generalizing what one might expect from Hilbert class fields.

Using Shimura reciprocity, the number fields generated by the values of principal moduli at their elliptic points have been determined in [5], and the ring class fields are identified for the principal moduli of $\Gamma_0(n)$ (n square-free) and its normal extension by the Fricke involution.

One more classical arena of number theory is suggested by the fact that when the $q-$coefficients of replicable functions are replaced by their signs in $\{0, 1, -1\}$, we find that they form an ultimately periodic sequence related to the modular level. Hardy's circle dissection method should prove itself capable of proving this observation.

We end with some recent speculations stimulated by reading the last section of Aspinwall, Katz and Morrison, [1] entitled "Numerical Oddities".

They consider an ellipticly fibred Calabi-Yau threefold, the elliptic fibration being $\pi : X \to \Sigma$ with a section. Let $\rho(\Sigma)$ be the Picard number of Σ. then anomaly cancellation leads to numerical values for $\rho(\Sigma)$. The extreme case of 24 point-like E_8 instantons on a binary icosahedral singularity in the heterotic string leads to $\rho(\Sigma) = 194$, a number recognizable as the class number (number of conjugacy classes) of the sporadic group, $\mathbb{M}$.

In [17], Miles Reid describes the Vafa, Hirzebruch-Höfer stringy Euler number $c_{string}(M, G)$, for a finite group G acting on a manifold M, and it turns out that this is the class number of G. (Recall $|\{(g, h) \in G \times G : gh = hg\}| = |G| \times$ the class number of G.) He goes on to conjecture the "physicist's Euler number conjecture" that in appropriate circumstances, c_{string} = Euler number of the minimal resolution of M/G, and continues: "If $M = \mathbb{C}^n$, then for any reasonable resolution of singularities $Y \to X = \mathbb{C}^n/G$, the cohomology is spanned by algebraic cycles, so that $c(Y)$ = the number of algebraic cycles of Y and further, it seems unlikely that we could prove the numerical $c(Y)$ = class number of G without setting up a bijection between the two sets."

It is significant that these facts were unknown to one of us (J.McK) at the time of his observation in April 2001. Further computations of Anca Degeratu and Katrin Wendland, aided by Harald Skarke, in the analogous situations of E_7, and E_6, also lead to class numbers, although not quite as expected!

A group class number is the dimension of the centre of its group algebra. It is unknown whether these are significant in physics, but, if so, it may be worth noting the values of the class numbers for the Mathieu groups on whose unique combinatorics the existence of the monster is based. We find that the class number of M_{24} and of $2.M_{12}$ is 26, and for both M_{11} and $M_{21} \cong \mathrm{PSL}_3(4)$ it is 10.

12 Solitons and the $\tau-$ function

In this section we proceed with a change of variables $z = 1/q$ so that our functions $w(z) = f(q)$ have the form

$$w(z) = z + \sum_{n=1}^{\infty} \frac{a_n}{z^n}.$$

More precisely we are interested in those functions which conformally map from the exterior of the unit disc U to the exterior of a closed analytic curve γ.

The Faber polynomials are defined as in section 2. Moreover, we have the integral representation

$$F_n(z) = \int_{|t|=1} \frac{t^n w'(t)}{w(t) - z} \, dt \, .$$

The conformal maps under study provide a solution to the dispersionless limit of the 2D Toda hierarchy [12] and it is natural to investigate the consequences of introducing the replication property in this context.

It is known in soliton theory that the $\tau-$function represent solutions of integrable hierarchies. This $\tau-$function depends on variables $\{t_1, t_2, \cdots\}$ which are moments coming from the expansion around the origin of the potential created in the interior of the curve γ when filled homogeneously by electric charge of density 1 . In fact, the moments determine uniquely the curve γ in our setting and thus determine $w(z)$.

The $\tau-$function for the hierarchy in question satisfies the dispersionless Hirota equation which is shown to have the form [12]

$$\{w, z\} = 12z^{-2} \sum_{m,n \geq 1} z^{-m-n} \frac{\partial \log \tau}{\partial t_m \partial t_n}.$$

This is exactly our formula (10) when $\{w, z\}$ replaces the Schwarz derivative with respect to the modular variable in $\mathfrak{H}$, recalling that $z = 1/q$ or simply (11). It follows that the $h_{m,n}$, which are used to define replicability of $w(z)$, are given by the second derivatives of $\log \tau$. In particular, the relation $h_{m,n} = h_{n,m}$ is simply the identity $\partial t_m \partial t_n \log \tau = \partial t_n \partial t_m \log \tau$. Also, the coefficients of $w(z)$ are given by $\partial t_n \partial t_1 \log \tau$. Moreover, the recurrence relations (2) provide an infinite number of relations among the second derivatives of $\log \tau$.

It is natural to ask what does it mean for a solution to soliton equations to be replicable? or in a geometric context, since the moments $\{t_1, t_2, \cdots\}$ provide local coordinates for the space of analytic curves γ, what role do the curves attached to replicable functions play in this space?

References

1. Aspinwall, P.S., Katz, S., Morrison, D.R. Lie groups, Calabi-Yau manifolds and F-theory Adv. Theor. Math. Phys 4 (2000) 95-126.
2. Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A., Wilson, R. A., ATLAS of finite groups, Clarendon Press, Oxford, 1985.
3. Conway, J.H., McKay, J., Sebbar, A., The discrete groups of moonshine, *To appear.*
4. Conway, J. H., Norton, S. P., Monstrous Moonshine, Bull. London Math. Soc. 11 (1979) 308-339.

5. Cox, D., McKay, J., Stevenhagen, P., Principal moduli and their class fields. Bull. London Math. Soc. *To appear*
6. Cummins, C., Some comments on replicable functions. Modern trends in Lie algebra representation theory (Kingston, ON, 1993), 48–55, Queen's Papers in Pure and Appl. Math., 94, Queen's Univ., Kingston, ON, 1994.
7. Cummins, C., Norton, S. P., Rational Hauptmoduls are replicable, Can. J. Math., 47 (1995), 1201-1218.
8. Dedekind, R., Schreiben an Hern Borchardt Über die Theorie der elliptischen Modulfunktionen, Crelle, 83, (1878) 265-292.
9. Faber, G., Über polynomische Entwicklungen, Math. Ann. 57 (1903), 389-408.
10. Glauberman, G., Norton, S. P. On McKay's connection between the affine E_8 diagram and the Monster. Proceedings on Moonshine and Related Topics (Montréal, 1999), 37–42, CRM Proc. Lecture Notes, 30, Amer. Math. Soc., Providence, RI, 2001.
11. Hille, E., Analytic function theory. Vol. II. Introduction to Higher Mathematics Ginn and Co., 1962.
12. Kostov, I.K., Krichever, I., Mineev-Weinstein, M., Wiegmann, P. B., Zabrodin, A. The τ-function for analytic curves. Random matrix models and their applications, 285–299, Math. Sci. Res. Inst. Publ., 40, Cambridge Univ. Press, Cambridge, 2001.
13. McKay, J. Graphs, singularities, and finite groups, in The Santa Cruz Conference on Finite Groups, Proc. Symp. Pure Math. 37, ed. by B. Cooperstein and G. Mason, AMS, Providence, 1980.
14. McKay, J., Sebbar, A., Fuchsian groups, automorphic forms and Schwarzians, Math. Ann. 318 (2000), no. 2, 255-275.
15. Ogg, A. P. Modular functions. The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), pp. 521–532, Proc. Sympos. Pure Math., 37, Amer. Math. Soc., Providence, R.I., 1980.
16. Rademacher, H., The Fourier series and the functional equation of the absolute modular invariant $J(\tau)$, Am. J. Math, 61, 237-248, 1939.
17. Reid, Miles. McKay correspondence, alg-geom/9702016.
18. Serre, J-P., A Course in Arithmetic, Springer-Verlag, 1973.
19. Takahashi, A. Primitive Forms, Topological LG models coupled to Gravity and Mirror Symmetry, preprint AG/9802059.
20. From Number Theory to Physics. Papers from the Meeting on Number Theory and Physics held in Les Houches, March 7–16, 1989. Edited by M. Waldschmidt, P. Moussa, J. M. Luck and C. Itzykson. Springer-Verlag, Berlin, 1992.

Index