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Subalgebras of diagonalizable algebras of theories containing arithmetic

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0. Introduction

The study of the diagonalizable algebras of theories containing arithmetic was initiated in the early seventies by Macintyre & Simmons [28] and Magari [29]. In 1976 Solovay [50] characterized the equational theory of the diagonalizable algebra of Peano arithmetic. This theory was shown to be adequately described by the well-known modal logic **L**. Later on, Montagna [32], Artemov [2], Visser [51] and Boolos [12] strengthened this result somewhat by independently demonstrating that the free diagonalizable algebra on countably many generators is (isomorphic to) a subalgebra of the diagonalizable algebra of Peano arithmetic. The equational theories of diagonalizable algebras of other strong enough theories were calculated by Visser [52] (cf. also Artemov [3]). These are given by the series $\mathbf{L}$, $\mathbf{L} + \Box\bot$, $\mathbf{L}+\Box\Box\bot, \ldots$ Among the recent investigations of the subject we should also mention Montagna's paper [33] which undertakes a systematic inquiry into generalizations of Goldfarb's Principle.

Nonetheless, the information on diagonalizable algebras of theories currently available is dejectingly scarce and therefore leaves ample scope to further research. Thus, for instance, it would be natural to attempt a closer look at subalgebras of these algebras. This is the theme of the present paper. It is predominantly devoted to the question which diagonalizable algebras can be embedded into the diagonalizable algebra of a theory. For the easier case of embeddings with r.e. range we obtain a complete solution. It turns out that a short list of most obvious restrictions constitutes a characterization of r.e. subalgebras of the diagonalizable algebra of a theory. Partial results in this (or at least in a closely parallel) direction were obtained by Jumelet [27]. In fact, the work of Jumelet was my main source of ideas and inspiration.

The plan of the paper is as follows. §1 recollects the necessary definitions and earlier results. It also contains a result on the length of proofs which, in view of a construction in §11, does not look absolutely out of place here. In §§2, 3 and 5 we carry out some modal-logical work relevant for subsequent progress. As a by-product of this we obtain a uniform version of the Craig Interpolation Lemma for **L**. The main result of the paper is to be found in §§4, 6 and 7 where r.e. subalgebras of diagonalizable algebras of a wide class of theories are characterized. This takes us three §§ because we use three slightly different approaches to handle particular kinds of theories. Here we employ extensions of techniques developed by Solovay [50], Artemov [2], Boolos [12], Jumelet [27] and Beklemishev [5]. §§8–11 are of marginal interest. In §8 we apply the result of §7 to give an alternative proof to a lemma in Simmons [43]. Unfortunately, the application will not require the full strength of our methods. A question concerning the arithmetical complexity of sentences needed to model a diagonalizable algebra in arithmetic is treated in §9. In the last two §§ we find out whether our characterization of subalgebras of diagonalizable algebras of theories extends from r.e. to arbitrary subalgebras. It is shown in §10 that for the case of Σ_1-ill theories an easy generalization is possible. As regards Σ_1-sound theories, the situation appears to be more complex and an example is given in §11 that partially justifies our failure to describe subalgebras of diagonalizable algebras of these theories.

We assume that the reader is familiar with Smoryński [49] or at least with Solovay [50]. Knowledge of (rudiments of) diagonalizable algebra theory and modal logic, especially of **L**, should also be very helpful. For these matters, good references are Magari [29] and [30], Bernardi [8] and Bellissima [7].

A few words of appreciation. I would like to thank Lev Beklemishev for numerous stimulating ideas and invaluable comments. Without his help the present paper could have hardly been written. In particular, Lev Beklemishev brought my attention to a neat trick in Beklemishev [5] which a key idea for the argument in §6 was derived from. Thanks are also due to Professors Sergei Artemov and Aleksandr Chagrov, Marc Jumelet, Andrei Muchnik and Domenico Zambella for interesting and fruitful discussions.

The present paper is a very slightly reworked version of Shavrukov [42].

1. Preliminaries

1.A. Arithmetic. We shall study r.e. consistent theories whose language comprises that of primitive recursive arithmetic. Given a set Γ of arithmetic formulae, $\Delta_0(\Gamma)$ denotes the closure of Γ under Boolean combinations and primitive recursively bounded quantification. Let

$$\Sigma_0 = \Pi_0 = \Delta_0 \ (\textit{atomic arithmetic formulae})$$

and define Σ_{n+1} to be the closure of Π_n under lattice combinations and existential quantification. The class Π_{n+1} is defined analogously. We shall say that a formula φ is Σ_n *over* a theory T if there exists a Σ_n formula which T proves φ equivalent to. Finally, φ is Δ_n *over* T if it is both Σ_n and Π_n over T.

For Γ a set of arithmetic formulae, a theory T is said to be Γ-*sound* if each theorem of T which is in Γ is true. A theory is Γ-*ill* if it is not Γ-sound.

In compliance with a recent tradition of not involving much more arithmetic than is actually needed we take $I\Sigma_1$ as our base theory. In other words, it is assumed throughout the paper that every theory under study contains induction for Σ_1 formulae as well as the basic axioms P^- (cf. Paris & Kirby [37]) and defining equations for primitive recursive function symbols. Note that our theory $I\Sigma_1$

proves the same theorems as the theory PRA of Smoryński [49]. The theory $I\Sigma_1$ of Paris & Kirby [37] formulated in the language $(\leq, 0, \mathrm{S}, +, \cdot)$ is very much the same as ours. That is not just to say that our variant of $I\Sigma_1$ is conservative over that of Paris & Kirby. What is more, every formula of the language of primitive recursive arithmetic translates easily and $I\Sigma_1$-equivalently into the smaller language and this fact is formalizable in $I\Sigma_1$ itself. (This amounts to a canonical isomorphism between the diagonalizable algebras of the two variants of the theory.)

The following facts about $I\Sigma_1$ are well worth being kept in mind: The provably recursive functions of $I\Sigma_1$ are exactly the primitive recursive ones (Mints [31]; these functions will be referred to as Δ_0 *functions*); $I\Sigma_1$ proves induction (and therefore the least number principle) for $\Delta_0(\Sigma_1)$ formulae and each $\Delta_0(\Sigma_1)$ formula is Δ_2 over $I\Sigma_1$ (Hájek & Kučera [24]); every $\Delta_0(\Sigma_n)$ *sentence* is equivalent to a Boolean combination of Σ_n sentences.

We assume that every theory comes equipped with a primitive recursive way α to recognize its axioms with which we associate a Δ_0 formula $\mathrm{Prf}_\alpha(y, x)$, the *proof predicate* (*of* T), to express that y is a (say, Hilbert-style) proof of x from the (extralogical) axioms given by α (cf. e.g. Feferman [16]). $\mathrm{Pr}_\alpha(x)$, the *provability predicate* (*of* T), is short for $\exists y\, \mathrm{Prf}_\alpha(y, x)$. In what follows we shall be omitting the subscript α since no confusion is likely.

Each formula and, in general, each syntactical object is identified with its gödelnumber. The *numeral* for n, i.e. (the gödelnumber of) a zero followed by n strokes is denoted by $\overline{n}$. Finally, if $\varphi(x_1, \ldots, x_m)$ is a formula then $\overline{\varphi(\overline{x_1}, \ldots, \overline{x_m})}$ is the primitive recursive term honestly representing the function which sends $(n_1, \ldots, n_m)$ to the numeral for $\varphi(\overline{n_1}, \ldots, \overline{n_m})$.

The least $n \in \omega$ such that T proves $\mathrm{Pr}^n(\overline{\bot})$ is called the *credibility extent of* T. (We let $\mathrm{Pr}^1(\overline{\bot}) \equiv \mathrm{Pr}(\overline{\bot})$ and $\mathrm{Pr}^{n+1}(\overline{\bot}) \equiv \mathrm{Pr}[\overline{\mathrm{Pr}^n(\overline{\bot})}]$.) If no such $n \in \omega$ exists then T is said to be of *infinite* credibility extent. Note that if T is Σ_1-sound then clearly its credibility extent is infinite. On the other hand, the credibility extent of a Σ_1-ill theory does not only depend on the set of theorems of T, but also on the primitive recursive way α which the axioms of T are presented. Thus, Beklemishev [5] shows that if a Σ_1-ill theory T contains full induction then a particular choice of α can make the credibility extent of T anything from 1 to ∞.

1.B. The modal logic L. The modal logic **L** (whose other names are **K4W** (Segerberg [41]), **G** (Solovay [50]), **GL** (Artemov [3]) and **PRL** (Smoryński [49])) was presumably first introduced by Smiley [44] whose motive for doing so was investigation of ethics rather than of provability in formal systems. The language of **L** consists of an infinite stock of propositional letters $p_0, p_1, \ldots$, the usual propositional connectives and a unary modal operator $\Box$. In addition to the axioms and rules of the classical propositional logic, **L** contains the following axiom schemata:

$$\Box(A \to B) \to . \Box A \to \Box B\,,$$

$$\Box A \to \Box\Box A\,,$$
$$\Box(\Box A \to A) \to \Box A$$

and the *necessitation rule*: from A infer $\Box A$.

For A a modal formula we write $\Diamond A$ short for $\neg\Box\neg A$ and $\Box^+ A$ short for $A \wedge \Box A$.

We write $\vdash_{\mathbf{L}} A$ to mean that the formula A is derivable in $\mathbf{L}$. $\vdash_{\mathbf{L}} \Box^+ A \to \Box^+ B$ will usually be abbreviated as $A \vdash_{\mathbf{L}} B$. Note that since $\vdash_{\mathbf{L}} A$ is equivalent to $\vdash_{\mathbf{L}} \Box^+ A$ (cf. Magari [29] and [30]), our notation is coherent in that $\vdash_{\mathbf{L}} A$ if and only if $\top \vdash_{\mathbf{L}} A$. Trivially,

$$\vdash_{\mathbf{L}} A \to B \quad \text{implies} \quad A \vdash_{\mathbf{L}} B\,;$$
$$A \vdash_{\mathbf{L}} A\,;$$
$$A \vdash_{\mathbf{L}} B \quad \text{and} \quad B \vdash_{\mathbf{L}} C \quad \text{imply} \quad A \vdash_{\mathbf{L}} C\,;$$
$$A \vdash_{\mathbf{L}} B \quad \text{implies} \quad A \vdash_{\mathbf{L}} \Box B \quad \text{and} \quad \Box A \vdash_{\mathbf{L}} \Box B \quad \text{etc.}$$

$\vec{p}$, $\vec{q}$ etc. will be treated as variables ranging over finite (possibly empty) tuples of propositional letters.

Kripke semantics has long been known as a mighty weapon in the study of modal logic. We describe a variant of it suited for our purposes. A triple $\mathcal{K} = (K, R, \Vdash)$ is a (*Kripke*) $\vec{p}$*-model* if K, the *domain* of $\mathcal{K}$, is a non-empty set (of *nodes*); R, the *accessibility relation*, is a strict partial order on K such that R^{-1} is well founded and $\Vdash$ is a *forcing relation* between nodes of K and those modal formulae all of whose propositional letters are among those in $\vec{p}$. $\Vdash$ should satisfy the usual commutativity conditions for Boolean connectives and for each $a \in K$ and each modal formula $A(\vec{p})$ one has $a \Vdash \Box A(\vec{p})$ if and only if $b \Vdash A(\vec{p})$ for all $b \in K$ such that aRb. We write $\mathcal{K} \vDash A$ (A *holds in* $\mathcal{K}$) if $a \Vdash A$ for all $a \in K$.

By a *model* we mean a $\vec{p}$-model for some tuple $\vec{p}$. A model $\mathcal{K} = (K, R, \Vdash)$ is *finite* if so is K. $\mathcal{K}$ is *rooted* if there exists a node $b \in K$ satisfying bRa for all $a \in K$ such that $a \neq b$. This b is then called the *root* of $\mathcal{K}$. A rooted model $\mathcal{K}$ is *treelike* if R is a tree on K. For $\mathcal{K}$ a rooted model, we write $\mathcal{K} \Vdash A$ ($\mathcal{K}$ *forces* A; A *is forced in* $\mathcal{K}$; $\mathcal{K}$ *is a model of* A) if the root of $\mathcal{K}$ forces A. Clearly $\mathcal{K} \vDash A$ if and only if $\mathcal{K} \Vdash \Box^+ A$.

It is well known that if a formula A is derivable in $\mathbf{L}$ then it holds in every model provided that the forcing relation is defined on A. Various specializations of the converse are also true. Thus, if a formula is forced in every finite rooted model, or even in every finite treelike model, then it is derivable in $\mathbf{L}$ (see e.g. Segerberg [41] or Solovay [50]; we shall be referring to this fact as the *Completeness Theorem for* $\mathbf{L}$). The decidability of $\mathbf{L}$ follows (cf. also Bernardi [8]).

1.C. Diagonalizable algebras. A *diagonalizable algebra* (Magari [29]) is a pair $(\mathfrak{A}, \Box)$ where $\mathfrak{A}$ is a Boolean algebra with the usual operations $\wedge, \vee, \neg, \to, \top$

and $\bot$ endowed with an operator $\Box$ (alias τ) satisfying the following identities:

$$\begin{aligned} \Box(x \to y) \to . \Box x \to \Box y &= \\ \Box x \to \Box\Box x &= \\ \Box(\Box x \to x) \to \Box x &= \\ \Box\top = \top \,. \end{aligned}$$

The confusion between modal-logical and algebraic notation is meant to stress the fact that a diagonalizable equation is an identity of the variety of diagonalizable algebras if and only if the corresponding modal formula is derivable in $\mathbf{L}$ (see Montagna [32]).

A Boolean filter $\mathfrak{f}$ of a diagonalizable algebra $\mathfrak{D}$ is a *τ-filter* if $x \in \mathfrak{f}$ implies $\Box x \in \mathfrak{f}$ for each element x of $\mathfrak{D}$. If $\mathfrak{f}$ is a τ-filter then there exists the quotient algebra $\mathfrak{D}/\mathfrak{f}$. Conversely, the elements that are sent to $\top$ by a homomorphism of diagonalizable algebras constitute a τ-filter (cf. Magari [29] and [30] or Bernardi [8]). For each subset X of a diagonalizable algebra $\mathfrak{D}$ there exists the smallest τ-filter $\mathfrak{t}(X)$ containing X. Thus we can define $\mathfrak{D}/X$, the *quotient* (*algebra*) *of* $\mathfrak{D}$ *modulo* X, to be $\mathfrak{D}/\mathfrak{t}(X)$.

Whenever we shall need to construct a particular example of diagonalizable algebra we shall produce an algebra of the form $\mathbf{F}/\mathcal{E}$ where $\mathbf{F}$ is the free diagonalizable algebra on an appropriate set of generators $\{p_i\}_{i\in I}$ (this latter algebra may be identified with the set of modal formulas using the generators as propositional letters modulo $\mathbf{L}$-provable equivalence) and $\mathcal{E}$ is a set of elements of $\mathbf{F}$, that is, of formulas in $\{p_i\}_{i\in I}$. Note that for a formula A in $\{p_i\}_{i\in I}$ one has $A = \top$ in $\mathbf{F}/\mathcal{E}$ if and only if there exists a finite subset $\mathcal{F}$ of $\mathcal{E}$ such that $\bigwedge\!\!\!\bigwedge \mathcal{F} \vdash_{\mathbf{L}} A$.

The *height* of a diagonalizable algebra $\mathfrak{D}$ is defined as the least $n \in \omega$ such that $\Box^n\bot = \top$. If for all $n \in \omega$ one has $\Box^n\bot \neq \top$ then the height of $\mathfrak{D}$ is *infinite*. $\mathfrak{D}$ is *ω-consistent* if $\bot \neq \top$ and $x = \top$ whenever $\Box x = \top$ for each element x of $\mathfrak{D}$. ω-consistency obviously implies infinite height. If $\Box x \vee \Box y = \top$ implies $\Box x = \top$ or $\Box y = \top$ then $\mathfrak{D}$ is said to possess the *disjunction property*. Clearly the height, ω-consistency and the disjunction property are inherited by subalgebras. One can show that among homomorphic images of a diagonalizable algebra of infinite height there always are ω-consistent diagonalizable algebras with the disjunction property.

A $\bot$-generated diagonalizable algebra is determined by its height up to isomorphism. Note that the disjunction property is shared by all the $\bot$-generated diagonalizable algebras whereas the only ω-consistent $\bot$-generated diagonalizable algebra is the free $\bot$-generated diagonalizable algebra.

A mapping $\nu : \omega \to \mathfrak{D}$ such that $\operatorname{rng}\nu$ generates the (denumerable) diagonalizable algebra $\mathfrak{D}$ is called a *numeration* of $\mathfrak{D}$. A numeration ν is *positive* if the set of diagonalizable polynomials $A(p_0, p_1, \ldots)$ satisfying $A(\nu 0, \nu 1, \ldots) = \top$ is r.e. A numeration ν is *locally positive* if for each $n \in \omega$ the set of diagonalizable polynomials $A(p_0, \ldots, p_n)$ satisfying $A(\nu 0, \ldots, \nu n) = \top$ is r.e. An algebra $\mathfrak{D}$ is (*locally*) *positive* if a (locally) positive numeration of it exists. Clearly $\mathfrak{D}$ is locally positive if and only if each of its finitely generated subalgebras is positive; any numeration of a locally positive algebra is a locally positive numeration; a finitely generated diagonalizable algebra is positive if and only if it is locally positive. Since any finitely generated algebra of finite height is finite (cf. Bernardi [8]), we also have that any denumerable diagonalizable algebra of finite height is locally positive.

1.D. Diagonalizable algebras and arithmetic. The example of a diagonalizable algebra which motivates the definition is constructed from a theory T of the kind described in 1.A. The Boolean algebra $\mathfrak{A}$ is taken to be the *Lindenbaum Sentence Algebra* of T, i.e. the set of sentences of T modulo T-provable equivalence, and for the mapping $\Box$ one takes the provability predicate of T, that is, for φ a sentence, $\Box\varphi \equiv \mathrm{Pr}(\overline{\varphi})$. The well-known properties of $\mathrm{Pr}(\cdot)$ guarantee that the algebra obtained in this way is a diagonalizable algebra. (In particular, the identity $\Box(\Box x \to x) \to \Box x = \top$ disguises a formalized version of Löb's Theorem.) This diagonalizable algebra is called the *diagonalizable algebra of* T and is denoted by $\mathfrak{D}_{\mathrm{T}}$. The concept was originally introduced by Macintyre & Simmons [28] without a name. The name "diagonalizable algebra" was supplied later by Magari [29].

If Γ is a set of arithmetic sentences closed under Boolean operations and $\Box$ then $\mathfrak{D}_{\mathrm{T}}^{\Gamma}$ is the corresponding subalgebra of $\mathfrak{D}_{\mathrm{T}}$. The recursive enumerability of T guarantees that $\mathfrak{D}_{\mathrm{T}}$ is locally positive. A subalgebra of $\mathfrak{D}_{\mathrm{T}}$ is *r.e.* if the underlying set of sentences is. The usual gödelnumbering of sentences gives rise to a positive numeration of each r.e. subalgebra of $\mathfrak{D}_{\mathrm{T}}$ including $\mathfrak{D}_{\mathrm{T}}$ itself.

Clearly the height of $\mathfrak{D}_{\mathrm{T}}$ is equal to the credibility extent of T.

In diagonalizable algebras (and even in diagonalizable algebras of infinite height) neither of ω-consistency and the disjunction property implies the other. The situation in diagonalizable algebras of theories is different. In fact, the following are equivalent:

(i) T *is* Σ_1-*sound.*

(ii) $\mathrm{T} \vdash \sigma \vee \tau$ *implies* $\mathrm{T} \vdash \sigma$ *or* $\mathrm{T} \vdash \tau$ *for each pair of* Σ_1 *sentences* σ *and* τ.

(iii) T *decides every sentence which is* Δ_1 *over* T.

(iv) $\mathfrak{D}_{\mathrm{T}}$ *is* ω-*consistent.*

(v) *The credibility extent of* T *is greater than* 1 *and* $\mathfrak{D}_{\mathrm{T}}$ *possesses the disjunction property.*

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is proved in Jensen & Ehrenfeucht [25] and Guaspari [23] (cf. also Friedman [19] and Smoryński [47]–[49]). The remaining equivalences are folklore and are typical applications of *Goldfarb's Principle*:

Let σ *be a* Σ_1 *sentence and let* $\mathrm{T} \vdash \mathrm{Pr}(\overline{\bot}) \to \sigma$. *Then there exists a sentence* τ (*which can be chosen either* Σ_1 *or* Π_1) *such that* $\mathrm{T} \vdash \sigma \leftrightarrow \mathrm{Pr}(\overline{\tau})$ (cf. Visser [52], Bernardi & Mirolli [9], Montagna [33] or Montagna & Sommaruga [35]).

As evidenced by (iv) $\Leftrightarrow$ (v) it will, for the purposes of our paper, be convenient to conjunct ω-consistency and the disjunction property under the name of the *strong disjunction property* which is clearly equivalent to

$$\bot \neq \top, \quad \text{and} \quad \Box x \vee \Box y = \top \text{ implies } x = \top \text{ or } y = \top\,.$$

Before doing so, however, we shall take a final look at each one of the former separately.

For the remainder of this section we shall be confusing modal and arithmetic notation.

In 1971 Parikh [36] proved that the implication of the statement (iv) for Σ_1-sound theories ($\mathrm{T} \vdash \Box\varphi \Rightarrow \mathrm{T} \vdash \varphi$) may take rather long to materialize. That is, for each provably recursive function g of T there exists a sentence φ and a proof p of $\Box\varphi$ in T such that no number $\leq \mathrm{g}(p)$ is a proof of φ in T.

We shall prove the same for the disjunction property. Our proof leans heavily on techniques of de Jongh & Montagna [26] and Carbone [13] and an idea in Carbone & Montagna [14].

1.1. Proposition. *Let* g *be a provably recursive function of a* Σ_1*-sound theory* T.

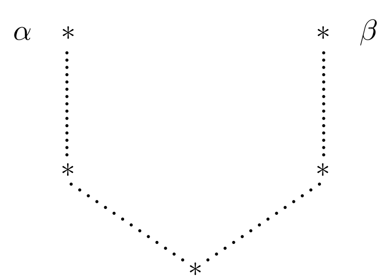
(a) *There exist* (Σ_1) *sentences* σ_1 *and* σ_2 *and a proof* p_0 *of* $\Box\sigma_1 \vee \Box\sigma_2$ *in* T *such that* $\mathrm{T} \vdash \sigma_1$, $\mathrm{T} \vdash \sigma_2$ *and no* $p_1 \leq \mathrm{g}(p_0)$ *is a proof of* $\Box\sigma_1$ *or of* $\Box\sigma_2$ *in* T.

(b) *There exist* (Σ_1) *sentences* τ_1 *and* τ_2 *and a proof* q_0 *of* $\Box\tau_1 \vee \Box\tau_2$ *in* T *such that* $\mathrm{T} \vdash \tau_1$, $\mathrm{T} \nvdash \tau_2$ *and no* $q_1 \leq \mathrm{g}(q_0)$ *is a proof of* $\Box\tau_1$ *in* T.

Proof. First we fix a pair of (Σ_1) sentences α and β such that

$$\begin{aligned} \mathrm{T} \vdash (\Box\alpha \vee \Box\beta) &\leftrightarrow \\ \Box\Box\alpha &\leftrightarrow \\ \Box\Box\beta &\leftrightarrow \Box\Box\bot\,. \end{aligned}$$

Sentences α and β satisfying these conditions could be produced with the help of Solovay's [50] Second Theorem applied to the following Kripke model (at each node, only the letters forced are shown):



(This model also appeared in Visser [52] to accomplish a similar task.)

Now let $\Box\varphi \prec_{\mathrm{g}} \Box\psi$ denote the formula saying that there exists a proof p of φ in T such that no $q \leq \mathrm{g}(p)$ is a proof of ψ in T.

(a) By self-reference find a sentence ϱ_1 such that

$$(1)\quad \mathrm{T} \vdash \varrho_1 \leftrightarrow. [\Box[\Box(\varrho_1 \vee \alpha) \vee \Box(\varrho_1 \vee \beta)] \prec_{\mathrm{g}} \Box\Box(\varrho_1 \vee \alpha)] \\ \wedge [\Box[\Box(\varrho_1 \vee \alpha) \vee \Box(\varrho_1 \vee \beta)] \prec_{\mathrm{g}} \Box\Box(\varrho_1 \vee \beta)]\,.$$

We have

$$(2)\quad \mathrm{T} \vdash \Box[\Box(\varrho_1 \vee \alpha) \vee \Box(\varrho_1 \vee \beta)] \rightarrow. \Box^+\varrho_1 \vee \Box^+\neg\varrho_1$$

(the antecedent implies that the r.h.s. of (1) is decidable and so ϱ_1 is decidable),

$$(3)\quad \mathrm{T} \vdash \neg\varrho_1 \rightarrow. \Box[\Box(\varrho_1 \vee \alpha) \vee \Box(\varrho_1 \vee \beta)] \rightarrow [\Box\Box(\varrho_1 \vee \alpha) \vee \Box\Box(\varrho_1 \vee \beta)] \quad \text{(by (1))}\,,$$

$$(4)\quad \mathrm{T} \vdash \Box^+\neg\varrho_1 \\ \rightarrow \Box^+[\Box[\Box(\varrho_1 \vee \alpha) \vee \Box(\varrho_1 \vee \beta)] \rightarrow. \Box\Box(\varrho_1 \vee \alpha) \vee \Box\Box(\varrho_1 \vee \beta)] \quad \text{(by (3))} \\ \rightarrow \Box^+[\Box(\Box\alpha \vee \Box\beta) \rightarrow. \Box\Box\alpha \vee \Box\Box\beta] \\ \rightarrow \Box^+(\Box\Box\Box\bot \rightarrow \Box\Box\bot) \quad \text{(by the choice of } \alpha \text{ and } \beta)\,,$$

$$(5)\quad \mathrm{T} \vdash \Box^+\neg\varrho_1 \rightarrow \Box\Box\Box\bot \quad \text{(from (4) by Löb's Theorem)} \\ \rightarrow \Box\Box\bot \quad \text{(by (4))}\,,$$

$$(6)\quad \mathrm{T} \vdash \Box\Box\bot \rightarrow. \Box\alpha \vee \Box\beta \quad \text{(by the choice of } \alpha \text{ and } \beta) \\ \rightarrow. \Box(\varrho_1 \vee \alpha) \vee \Box(\varrho_1 \vee \beta)\,,$$

$$(7)\quad \mathrm{T} \vdash \Box^+\neg\varrho_1 \rightarrow. \Box(\varrho_1 \vee \alpha) \vee \Box(\varrho_1 \vee \beta) \quad \text{(by (5) and (6))}\,,$$

$$(8)\quad \mathrm{T} \vdash \Box\Box\Box\varrho_1 \rightarrow \Box\Box[\Box(\varrho_1 \vee \alpha) \vee \Box(\varrho_1 \vee \beta)] \\ \rightarrow \Box(\Box\varrho_1 \vee \Box^+\neg\varrho_1) \quad \text{(by (2))} \\ \rightarrow \Box[\Box(\varrho_1 \vee \alpha) \vee \Box(\varrho_1 \vee \beta)] \quad \text{(by (7))} \\ \rightarrow. \Box\varrho_1 \vee \Box^+\neg\varrho_1 \quad \text{(by (2))} \\ \rightarrow. \Box\varrho_1 \vee \Box\Box\bot \quad \text{(by (5))} \\ \rightarrow \Box\Box\varrho_1\,,$$

$$(9)\quad \mathrm{T} \vdash \Box\Box\varrho_1 \quad \text{(from (8) by Löb's Theorem)}\,,$$

$$(10)\quad \mathrm{T} \vdash \varrho_1 \quad \text{(from (9) by } \Sigma_1\text{-soundness)}\,.$$

By (10) also the r.h.s. of (1) is provable and hence by Σ_1-soundness true. Now let $\sigma_1 \equiv \varrho_1 \vee \alpha$, $\sigma_2 \equiv \varrho_1 \vee \beta$ and note that (a) is proved.

(b) Construct a sentence ϱ_2 such that

$$\mathrm{T} \vdash \varrho_2 \leftrightarrow \Box[\Box(\varrho_2 \vee \alpha) \vee \Box\beta] \prec_{\mathrm{g}} \Box\Box(\varrho_2 \vee \alpha)$$

and show that $\mathrm{T} \vdash \varrho_2$ in perfect analogy with the proof of (a). Then take $\tau_1 \equiv \varrho_2 \vee \alpha$ and $\tau_2 \equiv \beta$. ∎

After the research underlying the present paper had been essentially completed I learnt that Proposition 1.1 fell corollary to very general recent results of Montagna [34].

Leaving alone the problem of actually constructing from a proof of $\Box\varphi \vee \Box\psi$ that of one of the disjuncts, one might at least ask which one of those is true.

The next proposition shows that this generally also is a very difficult question.

1.2. PROPOSITION. *If* T *is a* Σ_1*-sound theory then there is no provably recursive function of* T *which, given a proof in* T *of a sentence of the form* $\Box\varphi \vee \Box\psi$, *picks a true disjunct* (*even if one restricts the task to* Σ_1 *sentences* φ *and* ψ).

Proof. Suppose g were such a function. That is, if p is a proof of a sentence of the form $\Box\varphi \vee \Box\psi$ with φ and ψ in Σ_1 then

$$\mathrm{g}(p) = 0 \Rightarrow \mathrm{T} \vdash \varphi \quad \text{and} \quad \mathrm{g}(p) = 1 \Rightarrow \mathrm{T} \vdash \psi\,.$$

Clearly we can assume without loss of generality that

$$\mathrm{T} \vdash \forall x\,[\mathrm{g}(x) = 0 \vee \mathrm{g}(x) = 1]\,.$$

We introduce two ad hoc "modal" operators:

$$\Box_0\varphi \equiv \Box\varphi \wedge \mathrm{g}\,(\textit{the least proof of } \varphi) = 0\,,$$
$$\Box_1\varphi \equiv \Box\varphi \wedge \mathrm{g}\,(\textit{the least proof of } \varphi) = 1\,.$$

Next we define by parallel self-reference:

$$\mathrm{T} \vdash \sigma \leftrightarrow \Box_1(\Box\sigma \vee \Box\tau)\,,$$
$$\mathrm{T} \vdash \tau \leftrightarrow \Box_0(\Box\sigma \vee \Box\tau)\,.$$

We have

$$\begin{aligned}\mathrm{T} \vdash \Box(\Box\sigma \vee \Box\tau) &\to.\ (\Box_0 \vee \Box_1)(\Box\sigma \vee \Box\tau)\\ &\to.\ \sigma \vee \tau\\ &\to.\ \Box\sigma \vee \Box\tau \qquad (\sigma \text{ and } \tau \text{ are } \Sigma_1)\end{aligned}$$

and hence

$$\mathrm{T} \vdash \Box\sigma \vee \Box\tau$$

by Löb's Theorem. Now if $\mathrm{g}\,(\textit{the least proof of } \Box\sigma \vee \Box\tau) = 0$ then σ is false and therefore by Σ_1-soundness $\mathrm{T} \nvdash \sigma$ contrary to the assumptions on g; $\mathrm{g}\,(\textit{the least proof of } \Box\sigma \vee \Box\tau) = 1$ contradicts the assumptions in the symmetric manner. ∎

2. On conservativity in L

2.1. DEFINITION. The *degree* $d(A)$ of a modal formula A is defined inductively:

$$d(p_i) = d(\bot) = d(\top) = 0\,, \qquad d(\neg A) = d(A)\,,$$
$$d(A \wedge B) = d(A \vee B) = d(A \to B) = \max[d(A), d(B)]\,,$$
$$d(\Box A) = 1 + d(A)\,.$$

Thus, formulae of degree 0 are precisely the $\Box$-free formulae.

Let $\vec{p}$ be a finite tuple of propositional letters. Formulae of degree $\leq n$ containing no letters other than in $\vec{p}$ constitute (modulo **L**-equivalence) a finite Boolean algebra which we denote by $\mathbf{F}^n(\vec{p})$. Elements of $\mathbf{F}^n(\vec{p})$ will be persistently confused with modal formulas representing these elements. We also let $\mathbf{A}^n(\vec{p})$ denote the set of atoms of $\mathbf{F}^n(\vec{p})$. Clearly $\mathbf{F}^n(\vec{p})$ is a subalgebra of $\mathbf{F}^m(\vec{q})$ whenever $n \leq m$ and $\vec{p} \subseteq \vec{q}$. It is convenient to think of the modal operator $\Box$ as sending elements of $\mathbf{F}^n(\vec{p})$ to those of $\mathbf{F}^{n+1}(\vec{p})$.

$\mathbf{F}$ and $\mathbf{F}(\vec{p})$ denote the diagonalizable algebras of all formulae and of all formulae whose propositional letters are in $\vec{p}$ respectively.

2.2. LEMMA. *Consider elements of* $\mathbf{F}^{n+1}(\vec{p})$ *of the form*

$$\alpha \wedge \Box \bigvee\!\!\!\bigvee \gamma \wedge \bigwedge\!\!\!\bigwedge \Diamond\gamma$$

with α *ranging over* $\mathbf{A}^0(\vec{p})$ *and* γ *ranging over subsets of* $\mathbf{A}^n(\vec{p})$. *Call such formulas* types. (*Here* $\Diamond\gamma \equiv \{\Diamond C \mid C \in \gamma\}$.)

(a) *The conjunction of two distinct types is* ($\mathbf{L}$-*equivalent to*) $\bot$.
(b) *Each formula in* $\mathbf{F}^{n+1}(\vec{p})$ *is* ($\mathbf{L}$-*equivalent to*) *a disjunction of types.*
(c) *Each formula in* $\mathbf{A}^{n+1}(\vec{p})$ *is* ($\mathbf{L}$-*equivalent to*) *a type.*
(d) *Each type either belongs to* $\mathbf{A}^{n+1}(\vec{p})$ *or is* ($\mathbf{L}$-*equivalent to*) $\bot$.

Proof. (a) It is straightforward to show that

$$\begin{aligned}\vdash_{\mathbf{L}} \Big(\alpha_1 \wedge \Box \bigvee\!\!\!\bigvee \gamma_1 \wedge \bigwedge\!\!\!\bigwedge \Diamond\gamma_1\Big) \wedge \Big(\alpha_2 \wedge \Box \bigvee\!\!\!\bigvee \gamma_2 \wedge \bigwedge\!\!\!\bigwedge \Diamond\gamma_2\Big)\\ \leftrightarrow. (\alpha_1 \wedge \alpha_2) \wedge \Big(\Box \bigvee\!\!\!\bigvee \gamma_1 \wedge \Box \bigvee\!\!\!\bigvee \gamma_2\Big) \wedge \Big(\bigwedge\!\!\!\bigwedge \Diamond\gamma_1 \wedge \bigwedge\!\!\!\bigwedge \Diamond\gamma_2\Big)\\ \leftrightarrow. (\alpha_1 \wedge \alpha_2) \wedge \Box \bigvee\!\!\!\bigvee (\gamma_1 \cap \gamma_2) \wedge \bigwedge\!\!\!\bigwedge \Diamond(\gamma_1 \cup \gamma_2)\end{aligned}$$

and the claim follows by an easy Kripke model argument.

(b) By the definition of $\mathbf{F}^{n+1}(\vec{p})$ every formula therein can be thought of as a lattice combination of elements α of $\mathbf{A}^0(\vec{p})$ and formulas of the form $\Box C$ and $\neg\Box C$ with $C \in \mathbf{F}^n(\vec{p})$ or, equivalently, $\Box \bigvee\!\!\!\bigvee \gamma$ and $\neg\Box \bigvee\!\!\!\bigvee \gamma$ with $\gamma \subseteq \mathbf{A}^n(\vec{p})$. Thus to prove the claim it will suffice to show that α, $\Box \bigvee\!\!\!\bigvee \gamma$ and $\neg\Box \bigvee\!\!\!\bigvee \gamma$ are $\mathbf{L}$-equivalent to appropriate disjunctions of types and that the conjunction of two disjunctions of types can be $\mathbf{L}$-equivalently brought into the form of a disjunction of types. This is unproblematic:

$$\vdash_{\mathbf{L}} \alpha \leftrightarrow \bigvee\!\!\!\bigvee \Big\{\alpha \wedge \Box \bigvee\!\!\!\bigvee \delta \wedge \bigwedge\!\!\!\bigwedge \Diamond\delta \,\Big|\, \delta \subseteq \mathbf{A}^n(\vec{p})\Big\},$$

$$\vdash_{\mathbf{L}} \Box \bigvee\!\!\!\bigvee \gamma \leftrightarrow \bigvee\!\!\!\bigvee \Big\{\beta \wedge \Box \bigvee\!\!\!\bigvee \delta \wedge \bigwedge\!\!\!\bigwedge \Diamond\delta \,\Big|\, \beta \in \mathbf{A}^0(\vec{p}),\ \delta \subseteq \gamma\Big\},$$

$$\vdash_{\mathbf{L}} \neg\Box \bigvee\!\!\!\bigvee \gamma \leftrightarrow \bigvee\!\!\!\bigvee \Big\{\beta \wedge \Box \bigvee\!\!\!\bigvee \delta \wedge \bigwedge\!\!\!\bigwedge \Diamond\delta \,\Big|\, \beta \in \mathbf{A}^0(\vec{p}),\ \delta \subseteq \mathbf{A}^n(\vec{p}),\ \delta \not\subseteq \gamma\Big\}$$

and, finally,

$$\begin{aligned}\vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee_i \Big(\alpha_i \wedge \Box \bigvee\!\!\!\bigvee \gamma_i \wedge \bigwedge\!\!\!\bigwedge \Diamond\gamma_i\Big) \wedge \bigvee\!\!\!\bigvee_j \Big(\alpha_j \wedge \Box \bigvee\!\!\!\bigvee \gamma_j \wedge \bigwedge\!\!\!\bigwedge \Diamond\gamma_j\Big)\\ \leftrightarrow. \bigvee\!\!\!\bigvee_{i,j} \Big[\Big(\alpha_i \wedge \Box \bigvee\!\!\!\bigvee \gamma_i \wedge \bigwedge\!\!\!\bigwedge \Diamond\gamma_i\Big) \wedge \Big(\alpha_j \wedge \Box \bigvee\!\!\!\bigvee \gamma_j \wedge \bigwedge\!\!\!\bigwedge \Diamond\gamma_j\Big)\Big].\end{aligned}$$

Since by (a) the conjunction of two types is **L**-equivalent to $\bot$ and/or to a type we are done.

(c) and (d) follow easily from (a) and (b). ∎

The types of Lemma 2.2 are essentially the same as the normal form formulas of Fine [17] and the n-S-characters of Gleit & Goldfarb [20]. The satisfiable n-S-characters of the latter paper bear the same relation to elements of our $\mathbf{A}^n(\vec{p})$.

2.3. DEFINITION. Let $\mathcal{K}$ be a rooted model. The unique element of $\mathbf{A}^n(\vec{p})$ forced in $\mathcal{K}$ is called the $(n,\vec{p})$-*character of* $\mathcal{K}$. If the $(n,\vec{p})$-characters of two rooted models coincide then these models are said to be $(n,\vec{p})$-*twins*.

2.4. DEFINITION. If $\mathcal{K} = (K, R, \Vdash)$ is a Kripke model and $a \in K$ then $\mathcal{K}[a]$, the *a-cone of* $\mathcal{K}$, is the rooted model whose domain is the set $\{a\} \cup \{b \in K \mid aRb\}$ and the accessibility and forcing relations are R and $\Vdash$ restricted to this set respectively. A *proper cone of* $\mathcal{K}$ is the a-cone of $\mathcal{K}$ for some $a \in K$ which is not the root of $\mathcal{K}$.

The following lemma, although simple, will render us a number of valuable services. It should be compared with Theorem 1 of Fine [17].

2.5. LEMMA (Fine Lemma). (a) *Two rooted* $\vec{p}$-*models are* $(n+1,\vec{p})$-*twins iff*

(i) *they are* $(0,\vec{p})$-*twins and*

(ii) *each proper cone of one of these models has an* $(n,\vec{p})$-*twin among the proper cones of the other model and vice versa.*

(b) $(n+m,\vec{p})$-*twins are* $(n,\vec{p})$-*twins.*

Proof. (a) is easy.

(b) is proved by induction on m using (a). ∎

2.6. DEFINITION. Let $\vec{p}$ be a finite tuple of propositional letters. A formula A is said to be $\vec{p}$-*conservative over* a formula B if for each $C \in \mathbf{F}(\vec{p})$ one has $\vdash_{\mathbf{L}} B \to C$ whenever $\vdash_{\mathbf{L}} A \to C$. A is *conservative over* B if it is Λ-conservative over B where Λ is the empty tuple. A is ($\vec{p}$-) *conservative* if it is ($\vec{p}$-) conservative over $\top$.

Our aim is to show that conservativity is decidable as a ternary relation. In fact, we shall obtain stronger results.

2.7. DEFINITION. Let $\mathcal{K}_1 = (K_1, R_1, \Vdash_1)$ and $\mathcal{K}_2 = (K_2, R_2, \Vdash_2)$ be rooted models, $a \in K_1$ and assume K_1 and K_2 disjoint. By saying that we *graft* $\mathcal{K}_2$ *above* a (*in* $\mathcal{K}_1$) we mean that a new model is constructed whose domain is $K_1 \cup K_2$, the forcing relation coincides with $\Vdash_1 \cup \Vdash_2$ on propositional letters and the accessibility relation R is defined by putting

$$bRc \Leftrightarrow bR_1c \text{ or } bR_2c \text{ or } [(bR_1a \text{ or } b = a) \text{ and } c \in K_2]\,.$$

Let $\mathcal{K} = (K, R, \Vdash)$ be a rooted $\vec{p}$-model and $a \in K$. Suppose one grafts an isomorphic copy of the a-cone of $\mathcal{K}$ above $b \in K$ in $\mathcal{K}$ with bRa. Then the "old" nodes can be easily shown to force precisely the same modal formulae in the resulting model as they did in $\mathcal{K}$ (cf. Artemov [3]). Suppose $\mathcal{K}' = (K', R', \Vdash')$ is a $\vec{p}$-model obtained from $\mathcal{K}$ by a finite number of graftings of the sort described and let there exist a forcing relation $\Vdash^+$ extending $\Vdash'$ such that $\mathcal{K}^+ = (K', R', \Vdash^+)$ is a $\vec{q}$-model that forces a formula $A \in \mathbf{F}(\vec{q})$ where $\vec{p} \subseteq \vec{q}$. Then we shall say that $\mathcal{K}$ is *expandable to* (*a model of*) A and that $\mathcal{K}^+$ is an *expansion of* $\mathcal{K}$ *to* (*a model of*) A.

Instead of 2.7 we could have given a much smoother looking definition of expansion using the notion of p-morphism. There seems however to be less than no use counting twenty-five steps if the activity we are getting ready for is a fist fight as will be the case in 2.10.

2.8. LEMMA. *Let $\vec{p} \subseteq \vec{q}$. If every finite rooted* (*treelike*) *$\vec{p}$-model of a formula $A \in \mathbf{F}(\vec{p})$ is expandable to a model of $B \in \mathbf{F}(\vec{q})$ then B is $\vec{p}$-conservative over A.*

Proof. Easy. ∎

2.9. DEFINITION. If $A \equiv \alpha \wedge \Box \bigvee\!\!\!\bigvee \gamma \wedge \bigwedge\!\!\!\bigwedge \Diamond\gamma$ with $\alpha \in \mathbf{A}^0(\vec{p})$ and $\gamma \subseteq \mathbf{A}^n(\vec{p})$ and $\mathcal{K}$ is a rooted $\vec{p}$-model forcing A then α is called the *$\vec{p}$-real world of* $\mathcal{K}$ (and *of* A) and elements of γ are the *$(n, \vec{p})$-possible worlds of* $\mathcal{K}$ (and *of* A). The number of elements in γ is the *$(n, \vec{p})$-rank of* $\mathcal{K}$ (and *of* A).

Clearly the $\vec{p}$-real world and the $(n, \vec{p})$-possible worlds of each rooted $\vec{p}$-model (and of each element of $\mathbf{A}^{n+1}(\vec{p})$) are defined uniquely up to $\mathbf{L}$-equivalence.

The following lemma may be thought of as an improvement on the Joint Satisfiability Theorem of Gleit & Goldfarb [20].

2.10. LEMMA (Expansion Lemma). *Let $\vec{p} \subseteq \vec{q}$. To every $n \in \omega$ there corresponds an $N \in \omega$ such that every finite treelike $\vec{p}$-model of $B \in \mathbf{A}^N(\vec{p})$ is expandable to a model of $C \in \mathbf{A}^n(\vec{q})$ whenever $B \wedge C$ is irrefutable in $\mathbf{L}$.*

Proof. The claim is immediate for $n = 0$ (in this case we can take $N = 0$). For the remaining $n \in \omega$ we use induction on the $(n-1, \vec{q})$-rank of C. When this rank is 0 and $N > 0$ the claim is once again obvious.

Thus, given an $r \neq 0$, we assume for induction$_0$ hypothesis that each finite treelike $\vec{p}$-model of $D \in \mathbf{A}^N(\vec{p})$ is expandable to a model of $E \in \mathbf{A}^n(\vec{q})$ once $D \wedge E$ is irrefutable in $\mathbf{L}$ and the $(n-1, \vec{q})$-rank of E is smaller than r.

Now let $C \in \mathbf{A}^n(\vec{q})$ of $(n-1, \vec{q})$-rank r be forced in a rooted model $\mathcal{H}$ along with $B \in \mathbf{A}^{N+c}(\vec{p})$ and let $\mathcal{K}$ be an arbitrary finite treelike $\vec{p}$-model of B. The constant c will be specified later. We are going to expand $\mathcal{K}$ to a model of C. To avoid heavy notation we stipulate that $\mathcal{K}$ retains its name throughout the process of expansion despite the changes it undergoes and, at intermediate stages, despite being neither a $\vec{p}$- nor a $\vec{q}$-model.

First we consider a particular case when the $(n-1, \vec{q})$-rank of $\mathcal{H}$ is greater than that of any of its proper cones. In this case we let $c = 1$.

Let $a_1, \ldots, a_m$ be the immediate successors of the root of $\mathcal{K}$. By the Fine Lemma (a) there exists a sequence $\mathcal{H}[b_1], \ldots, \mathcal{H}[b_m]$ of proper cones of $\mathcal{H}$ such that $\mathcal{H}[b_i]$ is an $(N, \vec{p})$-twin of $\mathcal{K}[a_i]$, $1 \leq i \leq m$. Since the $(n-1, \vec{q})$-rank of each of the $\mathcal{H}[b_i]$'s is smaller than r, the induction$_0$ hypothesis yields an expansion of $\mathcal{K}[a_i]$ to the $(n, \vec{q})$-character of $\mathcal{H}[b_i]$. Now replace each of the $\mathcal{K}[a_i]$'s by the corresponding expansion (this is possible because $\mathcal{K}$ is treelike). Analogously, each proper cone $\mathcal{H}[b]$ of $\mathcal{H}$ has got an $(N, \vec{p})$-twin among the proper cones of $\mathcal{K}$ which is expandable to the $(n, \vec{q})$-character of $\mathcal{H}[b]$. For each such $\mathcal{H}[b]$, graft a copy of the corresponding expansion above the root of $\mathcal{K}$. Finally, extend the forcing relation at the root of $\mathcal{K}$ in the obvious way.

We show that the resulting model is an $(n, \vec{q})$-twin of $\mathcal{H}$. Their $\vec{q}$-real worlds coincide by construction. That the proper cones of the model constructed have $(n-1, \vec{q})$-twins among the proper cones of $\mathcal{H}$ follows from the fact that every proper cone of the new model is either an $(n, \vec{q})$- (and therefore by the Fine Lemma (b) an $(n-1, \vec{q})$-) twin of a proper cone of $\mathcal{H}$ or is a proper cone of an $(n, \vec{q})$-twin of a proper cone of $\mathcal{H}$ (and hence by the Fine Lemma (a) an $(n-1, \vec{q})$-twin of a proper cone of $\mathcal{H}$). As to the opposite direction, recall that we grafted in $\mathcal{K}$ an $(n, \vec{q})$-twin to each proper cone of $\mathcal{H}$. Finally, apply the Fine Lemma (a).

Now we drop the assumption on the $(n-1, \vec{q})$-ranks of the proper cones of $\mathcal{H}$ and increase c to 3, that is, we assume $\mathcal{K}$ and $\mathcal{H}$ to be $(N+3, \vec{p})$-twins.

Our plan is as follows. We set the induction$_0$ hypothesis and the skills we acquired when treating the above particular case to work and let these expand as many proper cones of $\mathcal{K}$ as possible to the $(n, \vec{q})$- or the $(n-1, \vec{q})$-characters of the corresponding proper cones of $\mathcal{H}$. What remains unexpanded in $\mathcal{K}$ after this first attack corresponds to proper cones of $\mathcal{H}$ of $(n-1, \vec{q})$-rank r and hence the $(n-1, \vec{q})$-possible worlds of these cones have to be the same as those of $\mathcal{H}$ itself. Thus, provided we have implanted all the $(n-1, \vec{q})$-possible worlds of $\mathcal{H}$ above each of the yet unexpanded nodes of $\mathcal{K}$, we only have to care that no $(n-1, \vec{q})$-possible world alien to $\mathcal{H}$ comes into existence when the forcing relation at these nodes is being extended to $\vec{q}$.

Our first move will be to classify the proper cones of $\mathcal{K}$. Thus, we call a proper cone $\mathcal{K}[a]$ along with its root a

— *frontier* if there is an $(N+1, \vec{p})$-twin $\mathcal{H}[b]$ of $\mathcal{K}[a]$ among the proper cones of $\mathcal{H}$ such that the $(n-1, \vec{q})$-rank of $\mathcal{H}[b]$ is r but each proper cone of $\mathcal{H}[b]$ is of a smaller $(n-1, \vec{q})$-rank;

— *high* if a is not frontier and there is an $(N+1, \vec{p})$-twin of $\mathcal{K}[a]$ among the proper cones of $\mathcal{H}$ of $(n-1, \vec{q})$-rank smaller than r;

— *low* if a is not frontier and every $(N+1, \vec{p})$-twin of $\mathcal{K}[a]$ among the proper cones of $\mathcal{H}$ is of $(n-1, \vec{q})$-rank r;

— *genuinely frontier* if a is frontier and every node which a is accessible from is low;

— *just high enough* if a is high and every node which a is accessible from is low;

— *essentially low* if a is low and every node which a is accessible from is also low.

Claim 1. *Each proper cone of $\mathcal{K}$ is either frontier or high or low.*

Claim 2. *Of each node a of $\mathcal{K}$ which is not the root of $\mathcal{K}$ precisely one of the following statements is true*:

(i) *a is genuinely frontier*;
(ii) *a is just high enough*;
(iii) *a is essentially low*;
(iv) *a is accessible from a genuinely frontier or from a just high enough node.*

Indeed, Claim 1 is easy. Claim 2 follows from Claim 1 by inspection of our classification.

Informally, we have this picture: To the root of $\mathcal{K}$ clings a downward closed collection of essentially low nodes and immediately above this collection there is a one-node-thick layer of genuinely frontier and just high enough nodes which separates the essentially low nodes from the rest of the model.

Claim 3. *From each* (*essentially*) *low node a frontier node is accessible.*

The proof of Claim 3 explains why we chose c to be so abnormally large:

By the Fine Lemma (a) each low proper cone $\mathcal{K}[a]$ has at least one $(N+2,\vec{p})$-twin among the proper cones of $\mathcal{H}$. Each of these $(N+2,\vec{p})$-twins has a proper cone of $(n-1,\vec{q})$-rank r, or else a would be frontier. Pick one of these $(N+2,\vec{p})$-twins and a proper cone $\mathcal{H}[b]$ of it of $(n-1,\vec{q})$-rank r such that each proper cone of $\mathcal{H}[b]$ has a smaller $(n-1,\vec{q})$-rank. By the Fine Lemma (a) the root of an $(N+1,\vec{p})$-twin of $\mathcal{H}[b]$ should be accessible from a. This root is by definition a frontier node so Claim 3 is proven.

Let us now start working. The root of $\mathcal{K}$ is as usual unproblematic. Next we replace each genuinely frontier and each one of the just high enough proper cones of $\mathcal{K}$ by its expansion to the $(n,\vec{q})$-character of one of those of its $(N+1,\vec{p})$-twins in $\mathcal{H}$ which this proper cone owes its frontier or high statute to respectively. For (genuinely) frontier proper cones such expansions were carried out when treating the easy particular case with $c=1$ and for expansions of just high enough nodes we turn to induction$_0$ hypothesis. By Claim 2 and since $\mathcal{K}$ is treelike these replacements cannot conflict. To each essentially low node a of $\mathcal{K}$ we do the following: extend the forcing relation at a to that at the root of one of the $(N+1,\vec{p})$-twins of $\mathcal{K}[a]$ in $\mathcal{H}$ and graft above a an expansion of a frontier proper cone $\mathcal{K}[a_0]$ with a_0 accessible from a, which exists by Claim 3, to the $(n,\vec{q})$-character of an $(N+1,\vec{p})$-twin of $\mathcal{K}[a_0]$ in $\mathcal{H}$ which enjoys $(n-1,\vec{q})$-rank r but none of its proper cones does. Lastly, for each proper cone $\mathcal{H}[b]$ of $\mathcal{H}$ such that every one of its proper cones has $(n-1,\vec{q})$-rank smaller than r pick an $(N+1,\vec{p})$-twin in (the original copy of) $\mathcal{K}$ and graft above the root of $\mathcal{K}$ an expansion of this $(N+1,\vec{p})$-twin to the $(n,\vec{q})$-character of $\mathcal{H}[b]$.

It is now easily seen from Claim 2 that $\mathcal{K}$ has been metamorphosed into a $\vec{q}$-model. We check that the $(n-1,\vec{q})$-possible worlds of $\mathcal{H}$ and of the model constructed are the same.

If there is a proper cone of $\mathcal{H}$ of $(n-1,\vec{q})$-rank r then at least one of such cones enjoys an $(n,\vec{q})$-twin in the modified $\mathcal{K}$ grafted above the root. Since $\mathcal{H}$ is itself of $(n-1,\vec{q})$-rank r, each $(n-1,\vec{q})$-possible world of $\mathcal{H}$ is also an $(n-1,\vec{q})$-possible world of this $(n,\vec{q})$-twin and hence of the expanded $\mathcal{K}$. If there were no proper cones in $\mathcal{H}$ of this $(n-1,\vec{q})$-rank then we would have grafted in $\mathcal{K}$ an $(n,\vec{q})$-twin to each proper cone of $\mathcal{H}$ and anyway this is the easy $c=1$ case that we dealt away with earlier.

It remains to see that each $(n-1,\vec{q})$-possible world of $\mathcal{K}$ is that of $\mathcal{H}$. Expansions of genuinely frontier, just high enough and frontier proper cones of $\mathcal{K}$ grafted in $\mathcal{K}$ present, as in the $c=1$ case, no problem. We show by rootward induction$_1$ on the essentially low nodes of $\mathcal{K}$ that these only gave rise to $(n-1,\vec{q})$-possible worlds that are those of $\mathcal{H}$. Consider an essentially low node a of $\mathcal{K}$. Recall that there is an expansion of something to the $(n,\vec{q})$-character of a proper cone of $\mathcal{H}$ having $(n-1,\vec{q})$-rank r grafted above a. Hence the $(n-1,\vec{q})$-possible worlds of the a-cone of the new $\mathcal{K}$ are the same as that of $\mathcal{H}$: by induction$_1$ hypothesis no extra $(n-1,\vec{q})$-possible world could have crept in. Find now the root b of the $(N+1,\vec{p})$-twin $\mathcal{H}[b]$ of $\mathcal{K}[a]$ which the forcing relation at a was extended to. Since this $(N+1,\vec{p})$-twin also had to have $(n-1,\vec{q})$-rank r and hence the same $(n-1,\vec{q})$-possible worlds as $\mathcal{H}$, we see by the Fine Lemma that the a-cone of the modified $\mathcal{K}$ is an $(n,\vec{q})$- and hence $(n-1,\vec{q})$-twin of $\mathcal{H}[b]$ which gives us the desiderata. Thus we have executed the induction$_1$ step and the proof is complete.

Since the $(n-1,\vec{q})$-rank of a formula cannot be greater than $|\mathbf{F}^{n-1}(\vec{q})|$ our proof yields $N = 1 + 3 \cdot |\mathbf{F}^{n-1}(\vec{q})|$. ∎

2.11. Lemma. *Let $\vec{p} \subseteq \vec{q}$. For each formula $B \in \mathbf{F}(\vec{q})$ there exists a formula $C \in \mathbf{F}(\vec{p})$ such that $\vdash_{\mathbf{L}} B \to C$ and any finite treelike $\vec{p}$-model is expandable to a model of B iff this model forces C.*

Proof. Let $B \in \mathbf{F}^n(\vec{q})$ and let N be the number which corresponds to n by the Expansion Lemma. Take C to be the disjunction of those elements D of $\mathbf{A}^N(\vec{p})$ whose conjunction with B is irrefutable in $\mathbf{L}$ and use the Expansion Lemma. ∎

We are now able to prove the converse to Lemma 2.8.

2.12. Lemma. *Suppose that $\vec{p} \subseteq \vec{q}$ and $B \in \mathbf{F}(\vec{q})$ is $\vec{p}$-conservative over $A \in \mathbf{F}(\vec{p})$. Then each finite treelike $\vec{p}$-model of A is expandable to a model of B.*

Proof. By Lemma 2.11 there exists a formula $C \in \mathbf{F}(\vec{p})$ such that $\vdash_{\mathbf{L}} B \to C$ and each $\vec{p}$-model of C is expandable to a model of B. Since B is $\vec{p}$-conservative over A we have $\vdash_{\mathbf{L}} A \to C$ and so each finite treelike $\vec{p}$-model of A is expandable to a model of B. ∎

Smoryński [46] establishes the Craig Interpolation Property for the modal logic $\mathbf{L}$: If $\vdash_{\mathbf{L}} A \to B$ then there exists a formula C such that $\vdash_{\mathbf{L}} A \to C$ and $\vdash_{\mathbf{L}} C \to B$ and C only contains propositional letters common to A and B (cf. also Boolos [11] and Gleit & Goldfarb [20]). The following corollary shows that all we need know of B to construct C is what propositional letters A and B have in common.

2.13. Corollary (Uniform Craig Interpolation Lemma for $\mathbf{L}$). *Let $\vec{p} \subseteq \vec{q}$. Given a formula $B \in \mathbf{F}(\vec{q})$ we can construct a formula $C \in \mathbf{F}(\vec{p})$ such that $\vdash_{\mathbf{L}} B \to C$ and $\vdash_{\mathbf{L}} C \to D$ whenever $\vec{r}$ is a tuple of propositional letters disjoint from $\vec{q}$ and $D \in \mathbf{F}(\vec{p}, \vec{r})$ is such that $\vdash_{\mathbf{L}} B \to D$. Moreover, this formula C is unique up to $\mathbf{L}$-equivalence.*

Proof. Let C be as in Lemma 2.11. Take a formula D meeting the requirements of the present corollary and let $E \in \mathbf{F}(\vec{p})$ be the interpolant between B and D provided by the usual Craig Interpolation Lemma. We show $\vdash_{\mathbf{L}} C \to E$, whence $\vdash_{\mathbf{L}} C \to D$ follows by modus ponens. For if this were not the case then we would have a finite treelike model forcing $C \wedge \neg E$. By Lemma 2.11 this model would expand to a model of B and thus $B \wedge \neg E$ would be irrefutable in $\mathbf{L}$, contradicting the assumption that E is the interpolant.

Uniqueness is left to the reader. ∎

Thus if $\vec{p} \subseteq \vec{q}$ and $B \in \mathbf{F}(\vec{q})$ then among the formulas in $\mathbf{F}(\vec{p})$ implied by B exists a strongest one.

For the case of $\vec{p}$ an empty tuple Corollary 2.13 is essentially proved in Artemov [2] and [3]. The full strength of this corollary will not be needed until §10.

2.14. Corollary. ($\vec{p}$-) *conservativity is decidable.*

Proof. To decide whether a formula A is $\vec{p}$-conservative over a formula B construct the formula C provided by the Uniform Craig Interpolation Lemma such that $\vdash_{\mathbf{L}} A \to C$ and $\vdash_{\mathbf{L}} C \to D$ whenever $\vdash_{\mathbf{L}} A \to D$ and A and D do not have common propositional letters other than those in $\vec{p}$. Use the same lemma to see that A is $\vec{p}$-conservative over B if and only if $\vdash_{\mathbf{L}} B \to C$. ∎

In what follows formalized versions of certain lemmas of the present section will appear within $\mathrm{I}\Sigma_1$ without special notice. In each case the verification that such formalizations are possible is unproblematic and therefore left to the reader.

3. A family of Kripke models

3.1. Definition. Let $\mathcal{K}$ be a finite $\vec{p}$-model. $\mathcal{K}$ is said to be *differentiated* if for each node a of $\mathcal{K}$ there exists a formula $A \in \mathbf{F}(\vec{p})$ such that a is the only node in $\mathcal{K}$ forcing A.

Note that for finite models our definition of differentiated is equivalent to that of Fine [17].

3.2. DEFINITION. Let $\mathcal{K}$ be a finite model. The least such $n \in \omega$ that $\mathcal{K} \models \Box^n \bot$ is called the *height* of $\mathcal{K}$. Thus the height of $\mathcal{K}$ is equal to the number of elements in the largest subset of the domain of $\mathcal{K}$ linearly ordered by the accessibility relation. Clearly if $\mathcal{K}$ is rooted then the height of $\mathcal{K}$ exceeds that of any one of its proper cones.

3.3. LEMMA. *Let $\mathcal{K}$ be a finite rooted differentiated $\vec{p}$-model and let $A \in \mathbf{F}(\vec{p})$.*

(a) *Each cone of $\mathcal{K}$ is differentiated.*

(b) (Fine [18]) *To each finite rooted $\vec{p}$-model $\mathcal{H}$ there corresponds a finite rooted differentiated $\vec{p}$-model which forces precisely the same formulas in $\mathbf{F}(\vec{p})$ as $\mathcal{H}$ does.*

(c) *There exists a formula, which we shall denote by $\Psi_{\mathcal{K}}(\vec{p})$ (or just $\Psi_{\mathcal{K}}$), such that any rooted differentiated $\vec{p}$-model $\mathcal{H}$ is isomorphic to $\mathcal{K}$ if and only if $\mathcal{H} \Vdash \Psi_{\mathcal{K}}(\vec{p})$.*

(d) *$\mathcal{K} \Vdash A$ iff $\vdash_{\mathbf{L}} \Psi_{\mathcal{K}}(\vec{p}) \to A$.*

(e) *Either $\vdash_{\mathbf{L}} \Psi_{\mathcal{K}}(\vec{p}) \to A$ or $\vdash_{\mathbf{L}} \Psi_{\mathcal{K}}(\vec{p}) \to \neg A$.*

Proof. (a) Obvious.

(b) Let $\mathcal{H} = (H, R, \Vdash)$. Define an equivalence relation E on H:

$$aEb \Leftrightarrow a \text{ and } b \text{ force the same formulas in } \mathbf{F}(\vec{p})\,.$$

Define R/E to be the relation on H/E which holds between two E-equivalence classes $\mathfrak{a}$ and $\mathfrak{b}$ whenever for each node $a \in \mathfrak{a}$ there exists a node $b \in \mathfrak{b}$ such that aRb. Clearly R/E is transitive and irreflexive. Let an E-equivalence class $\mathfrak{a}$ force a propositional letter $p_i \in \vec{p}$ ($\mathfrak{a} \Vdash_E p_i$) if a representative of $\mathfrak{a}$ forces p_i.

We show by induction on the structure of A that if $a \in \mathfrak{a}$ then

$$a \Vdash A \text{ iff } \mathfrak{a} \Vdash_E A\,.$$

The only interesting induction step occurs when A is of the form $\Box B$. Suppose $a \Vdash \Box B$. If $\mathfrak{a} R/E \mathfrak{b}$ then for some $b \in \mathfrak{b}$ one has aRb whence $b \Vdash B$. Hence by the induction hypothesis $\mathfrak{b} \Vdash_E B$. Conclude $\mathfrak{a} \Vdash_E \Box B$. The converse direction is equally easy.

Thus, $\mathcal{H}$ and $(H/E, R/E, \Vdash_E)$ force the same modal formulas and trivially the new model is differentiated.

(c) We prove that for $\Psi_{\mathcal{K}}(\vec{p})$ one can take the $(n, \vec{p})$-character of $\mathcal{K}$ where n is the height of $\mathcal{K}$. This we do by induction on the height of $\mathcal{K}$.

So let the height of $\mathcal{K}$ be $n+1$ and let $\mathcal{H}$ and $\mathcal{K}$ be $(n+1, \vec{p})$-twins. We construct a mapping f from the domain of $\mathcal{K}$ to the domain of $\mathcal{H}$. Let f map the root of $\mathcal{K}$ to that of $\mathcal{H}$. Next let f take the root of a proper cone $\mathcal{K}[a]$ of $\mathcal{K}$ to the root of its $(n, \vec{p})$-twin among the proper cones of $\mathcal{H}$ (which exists by the Fine Lemma (a)). Note that by the induction hypothesis $\mathcal{K}[a]$ is isomorphic to $\mathcal{H}[f(a)]$. Since $\mathcal{K}$ is differentiated f is injective for else there would exist two distinct but isomorphic proper cones of $\mathcal{K}$. Moreover, f is surjective because each proper cone of $\mathcal{H}$ enjoys an $(n, \vec{p})$-twin among the proper cones of $\mathcal{K}$ and since $\mathcal{H}$ is differentiated f connects these two.

By the Fine Lemma (a), f preserves forcing of propositional letters. It remains to check that f respects the accessibility relation. Let, in $\mathcal{K}$, b be accessible from a in which case $\mathcal{K}[b]$ is isomorphic to some proper cone of $\mathcal{H}[f(a)]$ and in particular $\mathcal{K}[b]$ is an $(n, \vec{p})$-twin of a proper cone of $\mathcal{H}[f(a)]$. So f must take b to the root of that proper cone and hence $f(b)$ is accessible from $f(a)$. A symmetric argument will establish that b is accessible from a whenever $f(b)$ is accessible from $f(a)$. This shows that f is an isomorphism and completes the proof of (c).

(d) ("if") By (b) for any finite rooted $\vec{p}$-model $\mathcal{H}$ forcing $\Psi_{\mathcal{K}}(\vec{p})$ we can construct a finite rooted differentiated $\vec{p}$-model which forces the same formulas as $\mathcal{H}$. By (c) this model will be isomorphic to $\mathcal{K}$ and will therefore force A. Hence $\mathcal{H} \Vdash A$. By the Completeness Theorem for **L** we are done.

The "only if" direction is left to the reader.

(e) follows at once from (d). ∎

Thus, formulas of the form $\Psi_{\mathcal{K}}(\vec{p})$ are atoms of $\mathbf{F}(\vec{p})$. Moreover, it can be shown that each atom of $\mathbf{F}(\vec{p})$ has this form.

Lemma 3.3(c) is proved in Artemov [4] for treelike models and a suitably adjusted notion of differentiated. To get differentiated models from Artemov differentiated models one only has to identify nodes that force the same formulae. Confer also Bellissima [7] for a related result.

3.4. Definition. Let $\mathcal{M}(\vec{p}) = (M(\vec{p}), R(\vec{p}), \Vdash_{\vec{p}})$ denote the $\vec{p}$-model whose domain is constituted by all finite rooted differentiated $\vec{p}$-models (we shall henceforth denote these by lower case Roman letters) with the accessibility relation defined by

$$aR(\vec{p})b \Leftrightarrow b \text{ is isomorphic to a proper cone of } a$$

and with $a \Vdash_{\vec{p}} p_i$ iff $a \Vdash p_i$ where $p_i \in \vec{p}$.

The models $\mathcal{M}(\vec{p})$ will be our favourite playground and an important tool for our embeddability results for diagonalizable algebras. In fact, these models can be shown isomorphic to the models employed by Grigolia [21] and [22] and Rybakov [40]. We collect some facts about $\mathcal{M}(\vec{p})$.

3.5. Lemma. *Let* $a, b \in M(\vec{p})$, $A, B \in \mathbf{F}^n(\vec{p})$.

(a) $aR(\vec{p})b$ *iff* $\vdash_{\mathbf{L}} \Psi_a(\vec{p}) \to \Diamond\Psi_b(\vec{p})$.
(b) a non $R(\vec{p})b$ *iff* $\vdash_{\mathbf{L}} \Psi_a(\vec{p}) \to \Box\neg\Psi_b(\vec{p})$.
(c) $a \Vdash_{\vec{p}} A$ *iff* $a \Vdash A$.
(d) $\mathcal{M}(\vec{p})[a]$ *is isomorphic to* a.
(e) $\mathcal{M}(\vec{p})$ *is differentiated.*
(f) $A \nvdash_{\mathbf{L}} B$ *iff there exists a node* $c \in M(\vec{p})$ *such that* $c \vDash A$ *and* $c \nVdash B$.
(g) *If* $A \nvdash_{\mathbf{L}} \neg\Psi_a(\vec{p})$ *and* $A \vdash_{\mathbf{L}} B$ *then* $a \vDash B$.
(h) *If* $aR(\vec{p})b$ *and* $A \vdash_{\mathbf{L}} \neg\Psi_b(\vec{p})$ *then* $A \vdash_{\mathbf{L}} \neg\Psi_a(\vec{p})$.

Proof. (a) and (b) follow from Lemma 3.3(d).

(c) is established by downward induction on $R(\vec{p})$ (since $R(\vec{p})^{-1}$ is clearly well founded). Assume that (c) holds for all $b \in M(\vec{p})$ such that $aR(\vec{p})b$.

It will suffice to prove the claim for propositional letters and formulas of the form $\Box B$. Since the case of propositional letters is self-evident we turn to $\Box B$.

We have: $a \Vdash_{\vec{p}} \Box B$ iff for each proper cone b of a there holds $b \Vdash_{\vec{p}} B$, iff $b \Vdash B$ for each proper cone b of a (this is by the induction hypothesis), iff $a \Vdash \Box B$, q.e.d.

(d) By (c) of the present lemma one has $a \Vdash_{\vec{p}} \Psi_a$, ergo $\mathcal{M}(\vec{p})[a] \Vdash \Psi_a$ and hence by Lemma 3.3(c) $\mathcal{M}(\vec{p})[a]$ is isomorphic to a.

(e) By (c) of the present lemma Ψ_a differentiates a from all the other nodes of $\mathcal{M}(\vec{p})$.

(f) We only prove "only if". If not $\vdash_{\mathbf{L}} \Box^+ A \to \Box^+ B$ then there exists a finite rooted $\vec{p}$-model $\mathcal{K}$ such that $\mathcal{K} \models A$ and $\mathcal{K} \Vdash \neg\Box^+ B$. Thus there is a node d of $\mathcal{K}$ such that $d \Vdash \Box^+ A$ and $d \nVdash B$. Apply to $\mathcal{K}[d]$ Lemma 3.3(b) to obtain the desired $c \in M(\vec{p})$.

(g) By (f) since $a \nvdash_{\mathbf{L}} \neg\Psi_a$ there exists a node $c \in M(\vec{p})$ such that $c \Vdash \Box^+ A$ and $c \Vdash \Psi_a$, whence by Lemma 3.3(c), $c = a$ and from $A \vdash_{\mathbf{L}} B$ we get $a \models B$.

(h) Suppose $A \nvdash_{\mathbf{L}} \neg\Psi_a$. Then by (f) there is a node $c \in M(\vec{p})$ such that $c \Vdash \Box^+ A$ and $c \Vdash \Psi_a$. By Lemma 3.3(c), $c = a$, whence $cR(\vec{p})b$. So we have $b \Vdash \Box^+ A$ and $b \Vdash \Psi_b$, therefore by (f) $A \nvdash_{\mathbf{L}} \neg\Psi_b$ contrary to assumptions. ∎

Lemma 3.5(c) permits us to drop the notational distinction between $\Vdash_{\vec{p}}$ and $\Vdash$.

We shall also need to know something about the interrelations between the models $\mathcal{M}(\vec{p})$ with different $\vec{p}$.

3.6. Definition. Let $\vec{p} \subseteq \vec{q}$. We define a relation $\lhd$ between nodes of $\mathcal{M}(\vec{q})$ and those of $\mathcal{M}(\vec{p})$. For $a \in M(\vec{q})$ and $b \in M(\vec{p})$ put

$$a \lhd b \Leftrightarrow a \text{ and } b \text{ force the same formulas in } \mathbf{F}(\vec{p})\,.$$

Thus $a \lhd b$ if and only if b is expandable to a model of Ψ_a.

3.7. Lemma. *Let* $A \in \mathbf{F}$, $\vec{p} \subseteq \vec{q}$, $a, b \in M(\vec{p})$, $c \in M(\vec{q})$.

(a) $c \lhd a$ *iff* $\vdash_{\mathbf{L}} \Psi_c(\vec{q}) \to \Psi_a(\vec{p})$.

(b) *There exists a unique* $d \in M(\vec{p})$ *such that* $c \lhd d$.

(c) *There exist only finitely many* $e \in M(\vec{q})$ *such that* $e \lhd a$.

(d) $\vdash_{\mathbf{L}} \Psi_a(\vec{p}) \leftrightarrow \bigvee\{\Psi_e(\vec{q}) \mid e \in M(\vec{q}),\ e \lhd a\}$.

(e) *If* $c \lhd a$ *and* $A \vdash_{\mathbf{L}} \neg\Psi_a(\vec{p})$ *then* $A \vdash_{\mathbf{L}} \neg\Psi_c(\vec{q})$.

(f) *If* $aR(\vec{p})b$ *and* $c \lhd a$ *then there exists a node* $e \in M(\vec{q})$ *such that* $cR(\vec{q})e$ *and* $e \lhd b$.

(g) *If* $A \nvdash_{\mathbf{L}} \neg\Psi_a(\vec{p})$ *then there exists a node* $e \in M(\vec{q})$ *such that* $e \lhd a$ *and* $A \nvdash_{\mathbf{L}} \neg\Psi_e(\vec{q})$.

Proof. (a) ("only if") Since $c \lhd a$ the node c forces the same elements of $\mathbf{F}(\vec{p})$ as a does. In particular, $c \Vdash \Psi_a$. Hence by Lemma 3.3(d), $\vdash_{\mathbf{L}} \Psi_c \to \Psi_a$.

("if") Suppose a forces a formula $B \in \mathbf{F}(\vec{p})$. Then by Lemma 3.3(d) we have $\vdash_{\mathbf{L}} \Psi_a \to B$. Hence $\vdash_{\mathbf{L}} \Psi_c \to B$ and therefore c forces B by Lemma 3.3(d).

(b) For existence, apply Lemma 3.3(b) to the model obtained from c by restricting the forcing relation to $\mathbf{F}(\vec{p})$. Uniqueness follows by Lemma 3.5(e).

(c) If $e \lhd a$ then clearly the height of e is the same as that of a and there can only exist finitely many finite differentiated $\vec{q}$-models of a given height.

(d) $(\leftarrow)$ follows from (a).

$(\rightarrow)$ Let $\mathcal{K}$ be a finite rooted $\vec{q}$-model forcing Ψ_a. By Lemma 3.3(b) there exists a finite rooted differentiated $\vec{q}$-model forcing the same formulas in $\mathbf{F}(\vec{q})$ as $\mathcal{K}$ does and hence it will force Ψ_e for some $e \in M(\vec{q})$ such that $e \lhd a$. Therefore $\mathcal{K} \Vdash \Psi_e$, whence $\mathcal{K} \Vdash \bigvee\!\!\!\bigvee\{\Psi_e \mid e \in M(\vec{q}),\ e \lhd a\}$. An application of the Completeness Theorem for $\mathbf{L}$ completes the proof.

(e) follows from (a).

(f) By Lemma 3.5(a) we have $\vdash_{\mathbf{L}} \Psi_a \rightarrow \diamond\Psi_b$. Since $c \lhd a$ we also have $\vdash_{\mathbf{L}} \Psi_c \rightarrow \diamond\Psi_b$ by (a) of the present lemma. Hence $c \Vdash \diamond\Psi_b$ and there is a node $e \in M(\vec{q})$ such that $cR(\vec{q})e$ and $e \Vdash \Psi_b$. By (a) and Lemma 3.3(d) this implies $e \lhd b$.

(g) follows from (d). ∎

4. Finite credibility extent

4.1. THEOREM. *Let the credibility extent of a theory* T *be* $n \in \omega$. *A denumerable diagonalizable algebra* $\mathfrak{D}$ *is isomorphic to an r.e. subalgebra of* $\mathfrak{D}_{\mathrm{T}}$ *iff*

(i) $\mathfrak{D}$ *is positive and*

(ii) *the height of* $\mathfrak{D}$ *is* n.

The "only if" direction is straightforward. The present section is devoted to the proof of the converse implication. Thus we are given a denumerable positive diagonalizable algebra $\mathfrak{D}$ which we have to show isomorphic to an r.e. subalgebra of $\mathfrak{D}_{\mathrm{T}}$.

To this end we have to borrow some notation from §§2 and 3. But first we simplify it a little bit. The tuple $(p_1, \ldots, p_i)$ of propositional letters will usually be represented by the symbol i. So $\mathbf{F}^n(i)$ will stand for $\mathbf{F}^n(p_1, \ldots, p_i)$; M_i will stand for the domain of the model $\mathcal{M}(p_1, \ldots, p_i)$ etc. We shall allow ourselves to omit the subscripts in $\Vdash_i$ and R_i since it will always be clear which model is meant. We stipulate further that 0 is not an element of any of the M_i's, $0R_i$ (*any element of* M_i), $0 \lhd 0$ and $0 \Vdash A$ for no modal formula A (thus $a \Vdash \neg A$ is not the same as $a \nVdash A$ unless we assume $a \neq 0$). Moreover, we shall be confusing the names of sets, relations and properties introduced in §§2 and 3 such as M_i, $\vdash_{\mathbf{L}}$, "a formula A is i-conservative over a formula B" etc. with the names of their (honest) Δ_0 binumerations in arithmetic.

We are now going to apply a variant of the Solovay construction (see Solovay [50]) to each of the models $\mathcal{M}_i$. We start with $i = 0$.

By self-reference define a Δ_0 function symbol h_0 and a closed ε-term ℓ_0 such that $\mathrm{I}\Sigma_1$ proves the following clauses:

$$\mathrm{h}_0(0) = 0\,, \tag{1}$$

$$\mathrm{h}_0(x+1) = \begin{cases} a & \text{if } a \text{ satisfies (i)–(iv) below,} \\ \mathrm{h}_0(x) & \text{if no such } a \text{ exists;} \end{cases} \tag{2}$$

(i) $a \in M_0$,
(ii) $a \Vdash \Box^n\bot$,
(iii) $\mathrm{h}_0(x) R a$ and
(iv) $\mathrm{Prf}[x, \overline{\ell_0 = \overline{a} \to \exists y\ [\overline{\mathrm{h}_0(x)} R \mathrm{h}_0(y) R \overline{a}]}]$,

$$\ell_0 = \begin{cases} \lim_{x\to\infty} \mathrm{h}_0(x) & \text{if } \mathrm{h}_0 \text{ reaches a limit,} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Here $\mathrm{Prf}(\cdot,\cdot)$ is the proof predicate of the theory T under consideration.

4.2. Lemma ($\mathrm{I}\Sigma_1$).

(a) $\forall x\,\forall y\,[x \le y \to .\ \mathrm{h}_0(x) = \mathrm{h}_0(y) \lor \mathrm{h}_0(x) R \mathrm{h}_0(y)]$.
(b) $\ell_0 = \lim_{x\to\infty} \mathrm{h}_0(x)$.
(c) $\ell_0 = 0 \lor \ell_0 \Vdash \Box^{\overline{n}}\bot$.
(d) $\forall x\,[\mathrm{h}_0(x) = \ell_0 \lor \mathrm{h}_0(x) R \ell_0]$.
(e) $\forall a \in M_0\,[a \Vdash \Box^{\overline{n}}\bot \land \ell_0 R a \to .\ \neg\,\mathrm{Pr}[\overline{\ell_0 = \overline{a} \to \exists y\,[\overline{\ell_0} R \mathrm{h}_0(y) R \overline{a}]}]]$.
(f) $\forall a \in M_0\,[a \Vdash \Box^{\overline{n}}\bot \land \ell_0 R a \to .\ \neg\,\mathrm{Pr}(\overline{\ell_0 \ne \overline{a}})]$.
(g) $\ell_0 \ne 0 \to \exists x\ \mathrm{Prf}[x, \overline{\ell_0 = \overline{\ell_0} \to \exists y\,[\overline{\mathrm{h}_0(x)} R \mathrm{h}_0(y) R \overline{\ell_0}]}]$.
(h) $\ell_0 \ne 0 \to \mathrm{Pr}(\overline{\ell_0 \ne \overline{\ell_0}})$.
(i) $\ell_0 \ne 0 \to \mathrm{Pr}(\overline{\overline{\ell_0} R \ell_0})$.

Proof. (a) follows from inspection of (2) by induction on y.

(b) is immediate from (a) because h_0 can make at most n moves.

(c) By (1) and (2) for each x we either have $\mathrm{h}_0(x) = 0$ or $\mathrm{h}_0 \Vdash \Box^n\bot$ (this is established by induction on x). Now use (b).

(d) follows from (a) and (b).

(e) Consider an x such that $\mathrm{h}_0(x) = \ell_0$. By (a) and since R verifiably is a strict patrial order we have $\mathrm{h}_0(y) = \ell_0$ for each $y \ge x$. Therefore there cannot exist a proof $y \ge x$ of the formula

$$\ell_0 = \overline{a} \to \exists y\,[\overline{\ell_0} R \mathrm{h}_0(y) R \overline{a}]$$

because then by (2), $\mathrm{h}_0(y+1) = a$. Finally, recall that each provable sentence is provable by arbitrarily large proofs.

(f) is immediate from (e).

(g) Since $\mathrm{h}_0(0) = 0$ and $\mathrm{h}_0(y) = \ell_0 \neq 0$ for some y, there exists an x such that $\mathrm{h}_0(x) \neq \mathrm{h}_0(x+1) = \ell_0$. By (iv) of (2) this x has to be a proof of the formula

$$\ell_0 = \overline{\ell_0} \to \exists y\, [\overline{\mathrm{h}_0(x)} R \mathrm{h}_0(y) R \overline{\ell_0}]\,.$$

(h) Once we assume $\ell_0 \neq 0$ we see by (g) that there is a proof x of the formula

$$\ell_0 = \overline{\ell_0} \to \exists y\, [\overline{\mathrm{h}_0(x)} R \mathrm{h}_0(y) R \overline{\ell_0}]\,.$$

Clearly by (2) and (d), $\mathrm{h}_0(x+1) = \ell_0$. Moreover, since h_0 is Δ_0 and R is a strict partial order, (a) formalized implies

$$\mathrm{Pr}[\overline{\neg \exists y\, [\overline{\mathrm{h}_0(x)} R \mathrm{h}_0(y) R \overline{\ell_0}]}]$$

and therefore $\mathrm{Pr}(\overline{\ell_0 \neq \overline{\ell_0}})$.

(i) follows from (b) and (d) formalized and (h). ■

4.3. Lemma. (a) *For each m such that $0 < m \leq n$ one has*

$$\mathrm{I}\Sigma_1 \vdash \mathrm{Pr}^m(\overline{\bot}) \leftrightarrow \ell_0 \Vdash \Box^{\overline{m}} \bot\,.$$

(b) $\mathrm{I}\Sigma_1 + \mathrm{Pr}^n(\overline{\bot}) \vdash \ell_0 \neq 0$.

(c) *For no $a \in M_0$ such that $a \Vdash \Box^n \bot$ do we have* $\mathrm{T} \vdash \ell_0 \neq \overline{a}$.

(d) $\mathbb{N} \models \ell_0 = 0$.

Proof. (a) Consider $m = 1$.

If $\mathrm{Pr}(\overline{\bot})$ then by Lemma 4.2(f) we have $\ell_0 R a$ for no $a \in M_0$. Hence $\ell_0 \Vdash \Box\bot$.

Conversely, if $\ell_0 \Vdash \Box\bot$ then by Lemma 4.2(i), $\mathrm{Pr}(\overline{\overline{\ell_0} R \ell_0})$ and therefore $\mathrm{Pr}(\overline{\forall a\, \ell_0 \neq a})$, whence $\mathrm{Pr}(\overline{\bot})$.

Now use induction on m.

Suppose $m < n$ and $\mathrm{Pr}^{m+1}(\overline{\bot})$, that is, $\mathrm{Pr}[\overline{\mathrm{Pr}^m(\overline{\bot})}]$. By the induction hypothesis this is equivalent to $\mathrm{Pr}(\overline{\ell_0 \Vdash \Box^{\overline{m}}\bot})$. In other words, for all $a \in M_0$ such that $a \Vdash \Box^n\bot$ and $a \nVdash \Box^m\bot$ we have $\mathrm{Pr}(\overline{\ell_0 \neq \overline{a}})$. By Lemma 4.2(f) no such a is accessible from ℓ_0 and hence $\ell_0 \Vdash \Box^{m+1}\bot$.

As to the converse implication, $\ell_0 \Vdash \Box^{m+1}\bot$ implies $\ell_0 \neq 0$, whence by Lemma 4.2(i), $\mathrm{Pr}(\overline{\overline{\ell_0} R \ell_0})$, which entails $\mathrm{Pr}(\overline{\ell_0 \Vdash \Box^{\overline{m}}\bot})$ ergo $\mathrm{Pr}^{m+1}(\overline{\bot})$.

(b) follows immediately from (a).

(c) From $\mathrm{T} \vdash \ell_0 \neq \overline{a}$ one has by Lemma 4.2(f), ℓ_0 non Ra. In particular, $\ell_0 \neq 0$. But then by (a) and by Lemma 4.2(c), $\mathrm{T} \vdash \mathrm{Pr}^{n-1}(\overline{\bot})$, which contradicts the assumption on the credibility extent of T.

(d) follows from (c) of the present lemma and Lemma 4.2(c) and (h). ■

Now we have to do "the same" to the models $\mathcal{M}_i$ with $i > 0$. From a straightforward rewriting of (1)–(3) with i instead of 0 we could, however, only extract an embedding into $\mathfrak{D}_\mathrm{T}$ of the diagonalizable algebra on i generators which is free in the variety of diagonalizable algebras of height $\leq n$. To insure that the extra relations required by the structure of $\mathfrak{D}$ be provable in T we have to restrict the range of the Solovay function h_i travelling in $\mathcal{M}_i$ (and therefore the possible values of ℓ_i) to a set of nodes smaller than the whole of M_i. The relevant subset of M_i can generally not be singled out by a condition decidable uniformly in i. (This can be done if the algebra $\mathfrak{D}$ we are dealing with is finitely generated.) We therefore use decidable approximations to this set which can be made uniform in i so that the model $\mathcal{M}_i$ is seen by h_i as changing with the passage of time. The approximating conditions on the set of nodes accessible to h will be given the form of the requirement that these nodes be expandable to models of a certain formula. The first variant of Solovay construction employing changing models is due to Jumelet [27]. The models in Jumelet [27] grew; ours will diminish.

It will be important for the success of our enterprises that the models stop changing as soon as it becomes clear that h is going to leave 0 as was the case in Jumelet [27]. A farewell to 0, however, can only be bid at a nonstandard moment and so in the meantime we will have obtained the proofs of all the sentences needed to mimick the structure of $\mathfrak{D}$ because we only care about standard proofs.

It should also be kept in mind that we do not want to embed into $\mathfrak{D}_{\mathrm{T}}$ the finitely generated subalgebras of $\mathfrak{D}$ in arbitrary unrelated ways. In fact, we would like the embedding of the subalgebra of $\mathfrak{D}$ generated by the first $i+1$ generators (in some fixed enumeration of those) to prolong the embedding of the subalgebra generated by the first i generators. To achieve this a kind of provable coherence between Solovay functions h_i climbing up models $\mathcal{M}_i$ with different i is needed. Recall that the model $\mathcal{M}_{i+1}$ is a refinement of $\mathcal{M}_i$ in the sense made precise by Lemma 3.7. Roughly speaking, each node of $\mathcal{M}_i$ splits into several nodes of $\mathcal{M}_{i+1}$. We want ℓ_{i+1} to be a refinement of ℓ_i in the same sense. Put formally, $\ell_{i+1} \lhd \ell_i$. Actually, in our construction h_{i+1} will move step in step with h_i, that is, $\mathrm{h}_{i+1}(x) \lhd \mathrm{h}_i(x)$. It is to maintain this kind of synchronicity that the extra property 4.2(e) which the usual Solovay function does not seem to possess is used.

Since the algebra $\mathfrak{D}$ is positive, a positive numeration $\nu : \omega - \{0\} \to \mathfrak{D}$ is available (here we have only subtracted 0 from $\operatorname{dom} \nu$ for technical convenience). Let $\{A(m)\}_{m\in\omega}$ be a Δ_0 enumeration of the set of diagonalizable polynomials $A(p_1, p_2, \ldots)$ that hit $\top$ of $\mathfrak{D}$ when νi is substituted for p_i. We rearrange this sequence slightly by defining within $\mathrm{I}\Sigma_1$ a new sequence $\{D(m)\}_{m\in\omega}$ of polynomials with the help of an auxiliary Δ_0 function $\mathrm{k}(\cdot)$:

$$D(0) = \top\,, \tag{4}$$

$$\mathrm{k}(0) = 0\,, \tag{5}$$

$$D(x+1) = \begin{cases} A(x) & \text{if} \quad \text{(i) } A(x) \vdash_{\mathbf{L}} D(x)\,, \\ & \qquad \text{(ii) } A(x) \vdash_{\mathbf{L}} A[\mathrm{k}(x)]\,, \\ & \qquad \text{(iii) } \Box^{+}A(x) \text{ is conservative over } \Box^{n}\bot\,, \\ D(x) & \text{otherwise,} \end{cases} \tag{6}$$

$$\mathrm{k}(x+1) = \begin{cases} \mathrm{k}(x)+1 & \text{if } D(x+1) \vdash_{\mathbf{L}} A[\mathrm{k}(x)], \\ \mathrm{k}(x) & \text{otherwise.} \end{cases} \tag{7}$$

Thus $\{D(m)\}_{m\in\omega}$ is a sequence of polynomials growing in $\vdash_{\mathbf{L}}$-strength and $\mathrm{k}(x)$ points a finger at the element of $\{A(m)\}_{m\in\omega}$ which waits to be majorized by

$\{D(m)\}_{m\in\omega}$. If one also recalls that any relation which holds in a diagonalizable algebra of height n is conservative over $\Box^n\bot$ then the following lemma is trivial:

4.4. Lemma. (a) $I\Sigma_1 \vdash \forall x\,\forall y\,[x<y \to D(y) \vdash_{\mathbf{L}} D(x)]$.

(b) $I\Sigma_1 \vdash \forall x$ "$\Box^+ D(x)$ is conservative over $\Box^{\overline{n}}\bot$".

(c) *For each* $y\in\omega$ *there exists an* $x\in\omega$ *such that* $D(x)\vdash_{\mathbf{L}} A(y)$.

(d) *For each* $x\in\omega$ *there exists a* $y\in\omega$ *such that* $D(x)=A(y)$.

Proof. Left to the reader. ■

We now define the Solovay functions $\mathrm{h}(\cdot,\cdot)$:

$$\mathrm{h}(0,x)=\mathrm{h}_0(x)\,, \tag{8}$$

$$\mathrm{h}(i+1,0)=0\,, \tag{9}$$

$$\mathrm{h}(i+1,x+1)=\begin{cases} a & \text{if (i)–(vii) below hold,}\\ \mathrm{h}(i+1,x) & \text{if no } a \text{ satisfying (i)–(vi) exists;}\end{cases} \tag{10}$$

(i) $a\in M_{i+1}$,

(ii) $\mathrm{h}(i,x)\neq \mathrm{h}(i,x+1)$,

(iii) $\mathrm{h}(i+1,x)Ra$,

(iv) $a \lhd \mathrm{h}(i,x+1)$,

(v) if $\mathrm{h}(i+1,x)=0$ then $D[\mathrm{g}(x)]\nvdash_{\mathbf{L}} \neg\Psi_a$,

(vi) for each b satisfying (i)–(v) in place of a one has

$$\forall z\le x\,[\mathrm{Prf}[z,\overline{\ell(\overline{i+1})=\overline{b}\to\exists y\,[\overline{\mathrm{h}(i+1,x)}R\mathrm{h}(\overline{i+1},y)R\overline{b}]}]$$
$$\to \exists w\le z\ \mathrm{Prf}[w,\overline{\ell(\overline{i+1})=\overline{a}\to\exists y\,[\overline{\mathrm{h}(i+1,x)}R\mathrm{h}(\overline{i+1},y)R\overline{a}]}]]\,,$$

(vii) a is minimal among those c that satisfy (i)–(vi) in place of a (here "minimal" refers to the natural ordering of integers),

$$\ell(0)=\ell_0\,, \tag{11}$$

$$\ell(i+1)=\begin{cases} \lim\limits_{x\to\infty}\mathrm{h}(i+1,x) & \text{if } \mathrm{h}(i+1,\cdot) \text{ reaches a limit,}\\ 0 & \text{otherwise.}\end{cases} \tag{12}$$

Of course (vii) of (10) is just another way to say that we do not care what $\mathrm{h}(i+1,x+1)$ is as long as it satisfies (i)–(vi). The weakly monotonically increasing function g occurring in (v) of (10) is Δ_0 and will be defined later.

4.5. Lemma ($I\Sigma_1$).

(a) $\forall i\,\forall x\,\forall y\,[x\le y \to\!.\ \mathrm{h}(i,x)=\mathrm{h}(i,y)\vee \mathrm{h}(i,x)R\mathrm{h}(i,y)]$.

(b) $\forall i\,\forall x\,\mathrm{h}(i+1,x)\lhd \mathrm{h}(i,x)$.

(c) $\forall j\,\forall i<j\,\forall x\,\mathrm{h}(j,x)\lhd\mathrm{h}(i,x)$.

(d) $\forall i\,\ell(i)=\lim\limits_{x\to\infty}\mathrm{h}(i,x)$.

(e) $\forall i\,\forall x\,[\mathrm{h}(i,x)R\ell(i)\vee \mathrm{h}(i,x)=\ell(i)]$.

(f) $\forall j\,\forall i<j\,\ell(j)\lhd\ell(i)$.

Proof. (a) For $i = 0$, use Lemma 4.2(a) and (8), and for $i > 0$ inspect (10) and apply induction on y.

(b) Note that since $0 \lhd 0$ the claim holds for $x = 0$ by (9) and assume $\mathrm{h}(i+1, x) \lhd \mathrm{h}(i, x)$ for (Δ_0) induction hypothesis. We shall prove $\mathrm{h}(i+1, x+1) \lhd \mathrm{h}(i, x+1)$. If $\mathrm{h}(i, x) = \mathrm{h}(i, x+1)$ then the induction step is trivial (see (ii) of (10)). So assume $\mathrm{h}(i, x) R \mathrm{h}(i, x+1)$.

Case 1: $\mathrm{h}(i, x) \neq 0$. Since $\mathrm{h}(i, x) R \mathrm{h}(i, x+1)$, by Lemma 3.7(f) there exists a node $a \in M_{i+1}$ such that $\mathrm{h}(i+1, x) R a \lhd \mathrm{h}(i, x+1)$. The existence of a node a satisfying in addition (vi) and (vii) of (10) follows by the (Δ_0) least number principle applied first to proofs and then to nodes of M_{i+1}. Hence $\mathrm{h}(i+1, x+1) = a \lhd \mathrm{h}(i, x+1)$ so we are done.

Case 2: $\mathrm{h}(i, x) = 0$. For $i > 0$, proceed as in Case 1 but use Lemma 3.7(g) instead of 3.7(f). For $i = 0$, recall that by Lemma 4.4(b) we have $D(x) \nvdash_{\mathbf{L}} \neg \Psi_a$ for all x and all $a \in M_0$ such that $a \Vdash \Box^n \bot$ and hence $D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} \neg \Psi_a$. Therefore by Lemma 3.7(d), $D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} \neg \Psi_b$ for some $b \in M_1$ such that $b \lhd a$.

(c) is proved with the help of (b) by (Π_1) induction on j.

(d) By Lemma 4.2(b), (8) and (11) pick an x such that $\forall y \geq x\, \mathrm{h}(0, y) = \ell_0$. By (a) and (c), $\forall y \geq x\, \mathrm{h}(i, y) = \mathrm{h}(i, x)$ and the claim follows by (12).

(e) follows from (a) and (d).

(f) follows from (c) and (d) without any induction. ∎

By (11) and Lemma 4.5(f) we can introduce a sentence $\ell = 0$ as an abbreviation for any of the sentences

$$\ell_0 = 0, \quad \forall i\, \ell(i) = 0 \quad \text{and} \quad \exists i\, \ell(i) = 0\,.$$

4.6. Lemma ($\mathrm{I}\Sigma_1 + \ell \neq 0$).

(a) $\forall i\ \mathrm{Pr}[\overline{\overline{\ell(i)} R \ell(\bar{i})}]$.

(b) $\forall i\, \forall a \in M_i\, [a \Vdash \Box^{\overline{n}} \bot \wedge \ell(i) R a \rightarrow . \neg \mathrm{Pr}[\overline{\ell(\bar{i}) = \overline{a} \rightarrow \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i}, y) R \overline{a}]}]]$.

(c) $\forall i\, \forall a \in M_i\, [a \Vdash \Box^{\overline{n}} \bot \wedge \ell(i) R a \rightarrow . \neg \mathrm{Pr}[\overline{\ell(\bar{i}) \neq \overline{a}}]]$.

Proof. (a) By Lemma 4.5(d) and (e) we only have to prove $\mathrm{Pr}[\overline{\overline{\ell(i)} \neq \ell(\bar{i})}]$. From $\ell \neq 0$ we get by (11) and by Lemma 4.2(h), $\mathrm{Pr}[\overline{\overline{\ell(0)} \neq \ell(0)}]$, whence by Lemma 4.5(f) formalized, $\mathrm{Pr}[\overline{\overline{\ell(i)} \neq \ell(\bar{i})}]$.

(b) Note that the formula

$$\forall a \in M_i\, [a \Vdash \Box^{\overline{n}} \bot \wedge \ell(i) R a \rightarrow . \neg \mathrm{Pr}[\overline{\ell(\bar{i}) = \overline{a} \rightarrow \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i}, y) R \overline{a}]}]]$$

is $\Delta_0(\Sigma_1)$ over $\mathrm{I}\Sigma_1$ because $\ell \neq 0$ is equivalent to the Σ_1 formula $\exists x\, \mathrm{h}(i, x) \neq 0$, the formula $\ell(i) R a$ is equivalent to the Π_1 formula $\forall x\, \mathrm{h}(i, x) R a$ and the quantifier $\forall a \in M_i$ is primitive recursively bounded by the condition $a \Vdash \Box^{\overline{n}} \bot$. In view of this we shall apply induction on i.

For $i = 0$ the claim follows by Lemma 4.2(e) and (11). Assume that it holds for i and suppose a reductio that $\ell(i+1)Ra$ and

$$\mathrm{Pr}[\overline{\ell(\overline{i+1}) = \bar{a} \to \exists y\, [\overline{\ell(i+1)} R \mathrm{h}(\overline{i+1}, y) R \bar{a}]}]\,.$$

Let $c \in M_i$ be such that $a \lhd c$ (see Lemma 3.7(b)). By the (Δ_0) least number principle we obtain a node $b \in M_i$ such that $\ell(i+1)Rb \lhd c$ and

$$\mathrm{Pr}[\overline{\ell(\overline{i+1}) = \bar{b} \to \exists y\, [\overline{\ell(i+1)} R \mathrm{h}(\overline{i+1}, y) R \bar{b}]}]$$

satisfying also the conditions (vi) and (vii) of (10) for all large enough x. Note that $\ell(i)Rc$.

Now if $\mathrm{h}(i,\cdot)$ were to jump from $\ell(i)$ directly to c then $\mathrm{h}(i+1,\cdot)$ would have to jump directly to b because all the conditions (i)–(vii) of (10) would be satisfied (in particular, (v) would hold because $\ell(i+1) \neq 0$). This argument is formalizable in $\mathrm{I}\Sigma_1$ and so we obtain

$$\mathrm{Pr}[\overline{\ell(\bar{i}) = \bar{c} \to .\ \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i}, y) R \bar{c}] \vee \ell(\overline{i+1}) = \bar{b}}]\,.$$

Combining this with

$$\mathrm{Pr}[\overline{\ell(\overline{i+1}) = \bar{b} \to \exists y\, [\overline{\ell(i+1)} R \mathrm{h}(\overline{i+1}, y) R \bar{b}]}]$$

and with Lemma 4.5(b) formalized we get

$$\mathrm{Pr}[\overline{\ell(\bar{i}) = \bar{c} \to \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i}, y) R \bar{c}]}]$$

contrary to the induction hypothesis.

(c) follows immediately from (b). ■

Time is now ripe to define the primitive recursive function g:

(13) $$\mathrm{g}(x) = \begin{cases} z & \text{if } z \text{ satisfies (i)–(iii) below,} \\ x & \text{if no such } z \text{ exists;} \end{cases}$$

(i) $z < x$,

(ii) there exists an $i \in \omega$ and a node $a \in M_i$ such that $a \Vdash \Box^n \bot$,

$$\mathrm{Prf}[z, \overline{\ell(\bar{i}) = \bar{a} \to \exists y\, [0 R \mathrm{h}(\bar{i}, y) R \bar{a}]}]$$

and $D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg \Psi_a$,

(iii) z is minimal among those satisfying (i) and (ii).

4.7. Lemma ($\mathrm{I}\Sigma_1$).

(a) $\forall x\, \forall y\, [x \le y \to D[\mathrm{g}(y)] \vdash_{\mathbf{L}} D[\mathrm{g}(x)]]\,.$

(b) $\forall i\, \forall x\, \forall a \in M_i\, [a \Vdash \Box^{\overline{n}} \bot \wedge \mathrm{Prf}[x, \overline{\ell(\bar{i}) = \bar{a} \to \exists y\, [0 R \mathrm{h}(\bar{i}, y) R \bar{a}]}]$

$$\wedge D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} \neg \Psi_a \to .\ \forall y \ge x\, \mathrm{g}(y) = \mathrm{g}(x)]\,.$$

(c) $\forall y\, [\mathrm{g}(y) \neq y \to \exists i\, \exists a \in M_i\, [a \Vdash \Box^{\overline{n}} \bot$

$$\wedge\, \mathrm{Pr}[\overline{\ell(\bar{i}) = \bar{a} \to \exists y\, [0 R \mathrm{h}(\bar{i}, y) R \bar{a}]}] \wedge \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg \Psi_a]]\,.$$

P r o o f. (a) We clearly have $\mathrm{g}(x) \le \mathrm{g}(y)$. Now recall Lemma 4.4(a).

(b) Suppose the antecedent holds. By the (Δ_0) least number principle find the least $z \le x$ satisfying the antecedent in place of x. By (13) it is seen that $\forall y \ge z\, \mathrm{g}(y) = z$.

(c) On inspection of (13) one sees that if $\mathrm{g}(y) \ne y$ then for certain $i \in \omega$, $x < y$ and $a \in M_i$ the antecedent of (b) holds. Since by (b) we have $\forall z \ge x\, \mathrm{g}(z) = \mathrm{g}(x)$ it is seen with the help of (a) that $\forall z\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} D[\mathrm{g}(z)]$ and hence $\forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg\Psi_a$. ■

4.8. Lemma ($\mathrm{I}\Sigma_1$).

(a) $\ell \ne 0 \to \forall i\, \forall x\, \forall A \in \mathbf{F}(i)\, [h(i,x) = 0 \wedge D[\mathrm{g}(x)] \vdash_{\mathbf{L}} A \to\!.\ \ell(i) \Vdash A]$.

(b) $\ell = 0 \to \forall i\, \forall a \in M_i\, [a \Vdash \Box^{\overline{n}}\bot$

$$\wedge \Pr[\overline{\ell(\overline{i}) = \overline{a} \to \exists y\, [0R\mathrm{h}(\overline{i}, y)R\overline{a}]}] \to\!.\ \exists x D[\mathrm{g}(x)] \vdash_{\mathbf{L}} \neg\Psi_a]\,.$$

(c) $\ell = 0 \to \forall i\, \forall a \in M_i\, [a \Vdash \Box^{\overline{n}}\bot \wedge \Pr[\overline{\ell(\overline{i}) \ne \overline{a}}] \to\!.\ \exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} \neg\Psi_a]\,.$

Proof. (a) From $\ell \ne 0$ we have $\ell(i) \ne 0$. Consider the $z \ge x$ such that $0 = \mathrm{h}(i,z) \ne \mathrm{h}(i,z{+}1)$. By inspection of (v) of (10) one has $D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg\Psi_{\mathrm{h}(i,z+1)}$ and from Lemma 4.7(a) we get $D[\mathrm{g}(z)] \vdash_{\mathbf{L}} A$. Now $\mathrm{h}(i, z+1) \vDash A$ by Lemma 3.5(g), whence by Lemma 4.5(e), $\ell(i) \Vdash A$.

(b) The proof is much the same as that of Lemma 4.6(b). We proceed by ($\Delta_0(\Sigma_1)$) induction on i. The case when $i = 0$ is immediate by Lemma 4.2(e).

So we assume the claim to hold for i, deny it for $i+1$ and seek for a contradiction. We have $\ell = 0$ and for a node $a \in M_{i+1}$ such that $a \Vdash \Box^n\bot$ and a suitable z,

$$\mathrm{Prf}[z, \overline{\ell(\overline{i+1}) = \overline{a} \to \exists y\, [0R\mathrm{h}(\overline{i+1}, y)R\overline{a}]}] \quad \text{and}$$
$$D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg\Psi_a\,.$$

Let $c \in \mathcal{M}_i$ be such that $a \lhd c$. By Lemma 3.7(e) one also has

$$D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg\Psi_c\,.$$

Note that by Lemma 4.7(b) we have $\forall y \ge z\, D[\mathrm{g}(y)] = D[\mathrm{g}(z)]$ and therefore a satisfies conditions (i) and (iii)–(v) of (10) for all large enough x. Moreover, this fact is formalizable. By the (Δ_0) least number principle we can without loss of generality stipulate that a also satisfies (vi) and (vii) of (10). As in the proof of Lemma 4.6(b) we obtain

$$\Pr[\overline{\ell(\overline{i}) = \overline{c} \to\!.\ \exists y\, [0R\mathrm{h}(\overline{i}, y)R\overline{c}] \vee \ell(\overline{i+1}) = \overline{a}}]\,,$$

whence by Lemma 4.5(c) and since

$$\Pr[\overline{\ell(\overline{i+1}) = \overline{a} \to \exists y\, [0R\mathrm{h}(\overline{i+1}, y)R\overline{a}]}]$$

one gets

$$\Pr[\overline{\ell(\overline{i}) = \overline{c} \to \exists y\, [0R\mathrm{h}(\overline{i}, y)R\overline{c}]}]\,,$$

which along with $\forall y\, D[\mathrm{g}(y)] \nvdash_{\mathbf{L}} \neg\Psi_c$ (this follows from $D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg\Psi_c$) yields the desired contradiction with the induction hypothesis.

(c) follows straightforwardly from (b). ■

4.9. Lemma. $\mathbb{N} \vDash$ "g is the identity function".

Proof. Suppose not. Then by Lemma 4.7(c) there would exist a node $a \in M_i$ such that $a \Vdash \Box^n \bot$,

$$\mathrm{Pr}[\overline{\ell(\bar{i}) = \bar{a} \to \exists y\, [0R\mathrm{h}(\bar{i}, y)R\bar{a}]}] \quad \text{and} \quad \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg\Psi_a\,.$$

By Lemma 4.8(b) this would imply $\ell \neq 0$ contradicting Lemma 4.3(d). ■

Next we define a mapping $^\circ$ from the set of propositional letters $\{p_i\}_{i\in\omega-\{0\}}$ to $\mathfrak{D}_{\mathrm{T}}$ by putting

$$p_i^\circ \equiv \ell(\bar{i}) \Vdash \overline{p_i} \tag{14}$$

and extend it in the obvious way to every modal formula in these propositional letters.

4.10. Lemma. *For each $i \in \omega$ and each modal formula $A(p_1, \ldots, p_i)$,*

$$\mathrm{I}\Sigma_1 + \mathrm{Pr}^n(\overline{\bot}) \vdash [A(p_1, \ldots, p_i)]^\circ \leftrightarrow \ell(\bar{i}) \Vdash \overline{A(p_1, \ldots, p_i)}\,.$$

Proof. We execute induction on the structure of $A(p_1, \ldots, p_i)$. The case of propositional letters is handled by Lemma 4.5(f). The induction step is immediate for Boolean connectives.

We turn to $\Box$. Reason in $\mathrm{I}\Sigma_1 + \mathrm{Pr}^n(\overline{\bot})$:

Suppose $\ell(i) \Vdash \Box A(p_1, \ldots, p_i)$. Since $\Vdash$ is Δ_0 this can be formalized

$$\mathrm{Pr}[\overline{\overline{\ell(i)} \Vdash \overline{\Box A(p_1, \ldots, p_i)}}]\,.$$

From Lemma 4.3(b) we have $\ell \neq 0$ so with the help of Lemma 4.6(a) conclude

$$\mathrm{Pr}[\overline{\ell(\bar{i}) \Vdash \overline{A(p_1, \ldots, p_i)}}]\,.$$

Now assume $\ell(i) \Vdash \Diamond A(p_1, \ldots, p_i)$. By Lemma 4.3(b), $\ell \neq 0$ so from Lemmas 4.5(f) and 4.2(c) we get $\ell(i) \Vdash \Box^n \bot$. There exists a node $a \in M_i$ such that $a \Vdash \Box^n \bot$, $a \Vdash A(p_1, \ldots, p_i)$ and $\ell(i)Ra$, therefore with Lemma 4.6(c) one has $\neg\,\mathrm{Pr}[\overline{\ell(\bar{i}) \neq \bar{a}}]$, whence

$$\neg\,\mathrm{Pr}[\overline{\ell(\bar{i}) \nVdash \overline{A(p_1, \ldots, p_i)}}]\,. \ ■$$

4.11. Lemma. *If $A(\nu 1, \nu 2, \ldots) = \top$ for $A(x_1, x_2, \ldots)$ a diagonalizable polynomial then*

$$\mathrm{I}\Sigma_1 + \mathrm{Pr}^n(\overline{\bot}) \vdash [A(p_1, p_2, \ldots)]^\circ\,.$$

Proof. By the definition of the sequence $\{A(m)\}_{m\in\omega}$ the equality $A(\nu 1, \nu 2, \ldots) = \top$ implies that $A(k) = A(p_1, p_2, \ldots)$ for a suitable $k \in \omega$ and hence by Lemma 4.4(c) there exists an $m \in \omega$ such that $D(m) \vdash_{\mathbf{L}} A(p_1, p_2, \ldots)$ so by Lemma 4.9, $D[\mathrm{g}(m)] \vdash_{\mathbf{L}} A(p_1, p_2, \ldots)$, whence by Lemmas 4.3(b), 4.8(a) and 4.10

$$\mathrm{I}\Sigma_1 + \mathrm{Pr}^n(\overline{\bot}) \vdash \mathrm{I}\Sigma_1 + \ell \neq 0 \vdash [A(p_1, p_2, \ldots)]^\circ\,. \ ■$$

Lemma 4.11 licenses us to define a mapping $^*: \operatorname{rng}\nu \to \mathfrak{D}_{\mathrm{T}}$ by putting

$$(\nu i)^* \equiv p_i^\circ \equiv \ell(\bar{i}) \Vdash \overline{p_i} \tag{15}$$

because if $\nu i = \nu j$ then this lemma guarantees that

$$\mathrm{T} \vdash \mathrm{I}\Sigma_1 + \mathrm{Pr}^n(\overline{\bot}) \vdash p_i^\circ \leftrightarrow p_j^\circ\,.$$

We shall show that * gives rise to an embedding of $\mathfrak{D}$ into $\mathfrak{D}_{\mathrm{T}}$.

4.12. Proof of Theorem 4.1 concluded. Clearly rng^* is r.e.

Let $A(\nu 1, \ldots, \nu i)$ hit $\top$ in $\mathfrak{D}$. Then by Lemma 4.11 we have

$$\begin{aligned}\mathrm{T} \vdash \mathrm{I}\Sigma_1 + \mathrm{Pr}^n(\overline{\bot}) &\vdash [A(p_1, \ldots, p_i)]^\circ \\ &\vdash A[(\nu 1)^*, \ldots, (\nu i)^*]\,.\end{aligned}$$

Conversely, let $\mathrm{T} \vdash A[(\nu 1)^*, \ldots, (\nu i)^*]$. If it were not the case that $A(\nu 1, \ldots, \nu i) = \top$ then by Lemma 4.4(d) we would have $D[\mathrm{g}(m)] \nvdash_{\mathbf{L}} A(p_1, \ldots, p_i)$ for every $m \in \omega$. Hence for each m there would exist a node $a \in M_i$ such that $a \Vdash \Box^n\bot$, $D[\mathrm{g}(m)] \nvdash_{\mathbf{L}} \neg\Psi_a$ and $a \Vdash \neg A(p_1, \ldots, p_i)$. Since there are only finitely many nodes in M_i forcing $\Box^n\bot$, using Lemma 4.7(a) we can choose a single a for all $m \in \omega$. By Lemma 4.10 $\mathrm{T} \vdash [A(p_1, \ldots, p_i)]^\circ$ implies $\mathrm{T} \vdash \ell(\bar{i}) \neq \overline{a}$, whence by Lemma 4.8(c), $\ell \neq 0$ contrary to Lemma 4.3(d). The contradiction completes the proof of Theorem 4.1. ■

5. The strong disjunction property and steady formulae

5.1. Definition. A formula A is *steady* if $A \nvdash_{\mathbf{L}} \bot$ and for each pair B, C of formulas

$$A \vdash_{\mathbf{L}} \Box B \vee \Box C \Rightarrow A \vdash_{\mathbf{L}} B \text{ or } A \vdash_{\mathbf{L}} C\,.$$

The definition of a steady formula bears a strong resemblance to the strong disjunction property in diagonalizable algebras. An even tighter connection between these will be brought out in Lemma 5.15.

5.2. Lemma. *If A is steady and $A \vdash_{\mathbf{L}} \Box B_0 \vee \ldots \vee \Box B_n$ then for some i such that $0 \le i \le n$ we have $A \vdash_{\mathbf{L}} B_i$.*

Proof. Use induction on n. For $n = 0$ and $n = 1$ the claim holds by the definition of steady formulae. Let

$$A \vdash_{\mathbf{L}} \Box B_0 \vee \ldots \vee \Box B_n \vee \Box B_{n+1}\,.$$

Then

$$A \vdash_{\mathbf{L}} \Box(\Box B_0 \vee \ldots \vee \Box B_n) \vee \Box B_{n+1}\,,$$

whence by the steadiness of A

$$A \vdash_{\mathbf{L}} \Box B_0 \vee \ldots \vee \Box B_n \quad \text{or} \quad A \vdash_{\mathbf{L}} B_{n+1}\,.$$

Now apply the induction hypothesis to the former case. ■

5.3. Lemma. *If a formula A is steady then $\Box^+ A$ is conservative.*

Proof. If $\Box^+ A$ were not conservative then we would have $\vdash_{\mathbf{L}} \Box^+ A \to \Box^n \bot$ for some $n \in \omega$, whence $A \vdash_{\mathbf{L}} \Box^n \bot$, which would imply $A \vdash_{\mathbf{L}} \bot$ since A is steady. A contradiction. ■

We work now towards an effective description of steady formulae. To this end it will be convenient to think of each formula A in $\mathbf{F}^n(\vec{p})$ as the disjunction of a set γ of atoms in $\mathbf{A}^n(\vec{p})$. We shall introduce a preorder $\mathrm{Q}^n(\vec{p})$ on $\mathbf{A}^n(\vec{p})$. By analyzing the preordered set obtained by restriction of $\mathrm{Q}^n(\vec{p})$ to a certain subset of γ it will be decided whether A is steady.

5.4. Definition. The binary relation $\mathrm{Q}^n(\vec{p})$ on $\mathbf{A}^n(\vec{p})$ is defined by putting

$$B \mathrm{Q}^n(\vec{p}) C \Leftrightarrow \nvdash_{\mathbf{L}} B \to \Box\neg C\,.$$

5.5. Lemma. *Let $\mathcal{K}$ be a $\vec{p}$-model, let b and c be nodes of $\mathcal{K}$ such that c is accessible from b and give the $(n, \vec{p})$-characters of b and c the names B and C respectively. Then $B \mathrm{Q}^n(\vec{p}) C$.*

Proof. Obvious. ■

5.6. Definition. For A a formula we define the $(n, \vec{p})$-*shadow* of A to be the conjunction of all formulas in $\mathbf{F}^n(\vec{p})$ implied by A.

5.7. Lemma. *If $A \in \mathbf{A}^n(\vec{q})$, $m \le n$ and $\vec{p} \subseteq \vec{q}$ then the $(m, \vec{p})$-shadow of A is an element of $\mathbf{A}^m(\vec{p})$.*

Proof. Trivial. ■

5.8. Lemma. *For $B, C \in \mathbf{A}^{n+1}(\vec{p})$ one has $B \mathrm{Q}^{n+1}(\vec{p}) C$ iff*

(i) *the $(n, \vec{p})$-shadow of C is an $(n, \vec{p})$-possible world of B and*
(ii) *the $(n, \vec{p})$-possible worlds of C are among those of B.*

Proof. "only if" is established by considering a rooted model that forces $B \wedge \Diamond C$.

("if") Let $\mathcal{K}_1 \Vdash B$ and $\mathcal{K}_2 \Vdash C$. Graft $\mathcal{K}_2$ above the root of $\mathcal{K}_1$. Use (i) and (ii) to see that the resulting model forces $B \wedge \Diamond C$. ■

5.9. Corollary. *$\mathrm{Q}^n(\vec{p})$ is transitive.*

Proof. Follows immediately from Lemma 5.8. ■

5.10. Definition. If a formula A is in $\mathbf{F}^n(\vec{p})$ and $\vdash_{\mathbf{L}} A \leftrightarrow \bigvee\!\!\!\bigvee \gamma$ with $\gamma \subseteq \mathbf{A}^n(\vec{p})$ then $\bigvee\!\!\!\bigvee \gamma$ is called the $(n, \vec{p})$-*normal form* of A. A formula A in $\mathbf{F}^n(\vec{p})$ is called $(n, \vec{p})$-*trimmed* if letting $\bigvee\!\!\!\bigvee \gamma$ be its $(n, \vec{p})$-normal form one has $A \vdash_{\mathbf{L}} \neg G$ for no $G \in \gamma$.

5.11. Lemma. *To each formula A in $\mathbf{F}^n(\vec{p})$ there corresponds an $(n, \vec{p})$-trimmed formula B such that $A \vdash_{\mathbf{L}} B$ and $B \vdash_{\mathbf{L}} A$. The formula B with these properties is unique up to $\mathbf{L}$-equivalence.*

P r o o f. Take B to be the conjunction of all formulas C in $\mathbf{F}^n(\vec{p})$ such that $A \vdash_{\mathbf{L}} C$. Let $\bigvee\!\!\!\bigvee \gamma$ be the $(n,\vec{p})$-normal form of B. If $G \in \mathbf{A}^n(\vec{p})$ and $B \vdash_{\mathbf{L}} \neg G$ then $\neg G$ is a conjunct of B and hence $G \notin \gamma$.

We leave uniqueness to the reader. ∎

5.12. Definition. Let $A \in \mathbf{F}^n(\vec{p})$ be $(n,\vec{p})$-trimmed and let $\bigvee\!\!\!\bigvee \gamma$ be the $(n,\vec{p})$-normal form of A. A formula $E \in \gamma$ is called an $(n,\vec{p})$-*bottom* of A if $E\mathrm{Q}^n(\vec{p})C$ for each $C \in \gamma$. In this case A is said to be $(n,\vec{p})$-*bottomed.*

The following lemma gives us a convenient characterization of steady formulae along with an algorithm for deciding steadiness.

5.13. Lemma. *Let A be a formula in $\mathbf{F}^n(\vec{p})$ and let B be the $(n,\vec{p})$-trimmed formula which corresponds to A by Lemma* 5.11. *Then the following are equivalent*:

(i) *A is steady.*

(ii) *B is $(n,\vec{p})$-bottomed.*

(iii) *A is irrefutable in $\mathbf{L}$ and for each pair $\mathcal{K}_1, \mathcal{K}_2$ of finite rooted models in which A holds there exists a model $\mathcal{H}$ such that $\mathcal{H} \vDash A$ and $\mathcal{K}_1$ and $\mathcal{K}_2$ are isomorphic to some proper cones of $\mathcal{H}$.*

P r o o f. (i)⇒(ii). Let $\bigvee\!\!\!\bigvee \gamma$ be the $(n,\vec{p})$-normal form of B. An $(n,\vec{p})$-bottom E of B exists for otherwise we would have $A \vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee\{\Box\neg G \mid G \in \gamma\}$, which by the steadiness of A and by Lemma 5.2 implies $A \vdash_{\mathbf{L}} \neg G$ for some $G \in \gamma$, contradicting the $(n,\vec{p})$-trimmedness of B.

(ii)⇒(iii). Let $E \in \mathbf{F}^n(\vec{p})$ be an $(n,\vec{p})$-bottom of B and let $\mathcal{H}$ be a rooted $\vec{p}$-model forcing $E \wedge \Box^+ A$ (and therefore $\Box^+ B$). For $\vec{q} \supseteq \vec{p}$ take a pair of $\vec{q}$-models in which A holds and extend to $\vec{q}$ the forcing relation at the nodes of $\mathcal{H}$ in an arbitrary way. Next graft the chosen pair of $\vec{q}$-models above the root of $\mathcal{H}$. The resulting model will still force $E \wedge \Box^+ B$ (this can be seen through Lemma 5.8) and hence will also force $\Box^+ A$.

(iii)⇒(i). Let $A \vdash_{\mathbf{L}} \Box C \vee \Box D$. If it were the case that neither $A \vdash_{\mathbf{L}} C$ nor $A \vdash_{\mathbf{L}} D$ then there would exist two finite rooted models $\mathcal{K}_1$ and $\mathcal{K}_2$ in which A holds such that $\mathcal{K}_1 \nVdash C$ and $\mathcal{K}_2 \nVdash D$. Taking the model $\mathcal{H}$ which corresponds to $\mathcal{K}_1$ and $\mathcal{K}_2$ by the assumption we would have $\mathcal{H} \nVdash \Box C \vee \Box D$ and yet $\mathcal{H} \vDash A$. A contradiction. ∎

5.14. Corollary. *Steadiness is decidable.*

P r o o f. Follows from (i)⇔(ii) of Lemma 5.13. ∎

Consider the following property of modal formulas A: for each pair B, C of modal formulas

$$\vdash_{\mathbf{L}+A} \Box B \vee \Box C \quad \text{implies} \quad \vdash_{\mathbf{L}+A} B \quad \text{or} \quad \vdash_{\mathbf{L}+A} C\,.$$

(Here $\mathbf{L} + A$ means that A is added to $\mathbf{L}$ in the right of a new axiom schema.) Chagrov [15] shows it to be undecidable in contrast to Corollary 5.14.

5.15. Lemma. *Let $\mathfrak{D}$ be a diagonalizable algebra generated by $x_0, x_1, \ldots$ Then $\mathfrak{D}$ enjoys the strong disjunction property iff for each modal formula (=diagonalizable polynomial) B such that $B(x_0, x_1, \ldots) = \top$ there exists a steady formula A such that $A(x_0, x_1, \ldots) = \top$ and $\vdash_{\mathbf{L}} A \to B$.*

Proof. ("if") Suppose that for elements c and d of $\mathfrak{D}$ we have $\Box c \vee \Box d = \top$. Let $c = C(x_0, x_1, \ldots)$ and $d = D(x_0, x_1, \ldots)$ so that

$$\Box C(x_0, x_1, \ldots) \vee \Box D(x_0, x_1, \ldots) = \top\,.$$

By the assumptions on $\mathfrak{D}$ there exists a steady formula A such that $A \vdash_{\mathbf{L}} \Box C \vee \Box D$ and $A(x_0, x_1, \ldots) = \top$. We therefore have $A \vdash_{\mathbf{L}} C$ or $A \vdash_{\mathbf{L}} D$, whence $c = \top$ or $d = \top$. This proves the strong disjunction property.

("only if") Let $B \in \mathbf{F}^n(\vec{p})$ be such that $B(x_0, x_1, \ldots) = \top$ and let A be the conjunction of all formulas C in $\mathbf{F}^n(\vec{p})$ such that $C(x_0, x_1, \ldots) = \top$ (there are, up to $\mathbf{L}$-equivalence, only finitely many). Clearly $\vdash_{\mathbf{L}} A \to B$. Suppose $\bigvee\!\!\!\bigvee \gamma$ is the $(n, \vec{p})$-normal form of A. Note that A is $(n, \vec{p})$-trimmed because if $A \vdash_{\mathbf{L}} \neg C$ for some $C \in \mathbf{A}^n(\vec{p})$ then $\neg C(x_0, x_1, \ldots) = \top$, whence $\vdash_{\mathbf{L}} A \to \neg C$ and so $C \notin \gamma$. If there were no $(n, \vec{p})$-bottom to A then we would have $A \vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee \{\Box \neg G \mid G \in \gamma\}$ and hence

$$\bigvee\!\!\!\bigvee \{\Box \neg G(x_0, x_1, \ldots) \mid G \in \gamma\} = \top\,,$$

whence by the strong disjunction property we have $\neg G(x_0, x_1, \ldots) = \top$ for some $G \in \gamma$. Therefore $\vdash_{\mathbf{L}} A \to \neg G$ by the choice of A and so $G \notin \gamma$. The contradiction shows that A is $(n, \vec{p})$-bottomed and therefore steady by (ii) $\Rightarrow$ (i) of Lemma 5.13. ∎

5.16. Definition. An element a of a diagonalizable algebra $\mathfrak{D}$ with the strong disjunction property is *admissible* if $a \to \Box b = \top$ implies $b = \top$ for each element b of $\mathfrak{D}$.

One of the intended uses of the notion of an admissible element will be based on the fact that if T is a sound enough theory then the elements of $\mathfrak{D}_{\mathrm{T}}$ that correspond to true sentences of low arithmetical complexity have to be admissible.

In fact, if T is a Σ_1-sound theory then a sentence φ is an admissible element of $\mathfrak{D}_{\mathrm{T}}$ if and only if $\mathrm{T} + \varphi$ is Σ_1-sound.

5.17. Lemma. *Let $\mathfrak{D}$ be a diagonalizable algebra with the strong disjunction property.*

(a) *The element $\top$ of $\mathfrak{D}$ is admissible.*

(b) *If an element a of $\mathfrak{D}$ is admissible, $b \in \mathfrak{D}$ and $a \to b = \top$ then b is admissible.*

(c) *If an element a of $\mathfrak{D}$ is admissible and b is an arbitrary element of $\mathfrak{D}$ then at least one of the elements $a \wedge b$ and $a \wedge \neg b$ is admissible.*

Proof. (a) and (b) are straightforward.

(c) Suppose neither $a \wedge b$ nor $a \wedge \neg b$ is admissible. Then there exist elements c and d of $\mathfrak{D}$ distinct from $\top$ such that

$$a \wedge b \rightarrow . \Box c = a \wedge \neg b \rightarrow . \Box d = \top .$$

But then $a \rightarrow . \Box c \vee \Box d = \top$, whence $a \rightarrow . \Box(\Box c \vee \Box d) = \top$. Since a is admissible we have $\Box c \vee \Box d = \top$ so by the strong disjunction property $c = \top$ or $d = \top$, which contradicts the assumptions and therefore proves the lemma. ■

5.18. Definition. Let A be a steady formula. A *admits* a formula B if for each formula C one has

$$A \vdash_{\mathbf{L}} B \rightarrow \Box C \;\Rightarrow\; A \vdash_{\mathbf{L}} C .$$

The relation "to admit" parallels the notion of an admissible element of a diagonalizable algebra with the strong disjunction property in the same way that steadiness parallels the strong disjunction property itself.

5.19. Lemma. *If a steady formula A admits a formula B and $A \vdash_{\mathbf{L}} B \rightarrow (\Box C_0 \vee \ldots \vee \Box C_n)$ then for some i such that $0 \le i \le n$ we have $A \vdash_{\mathbf{L}} C_i$. Analogously, if a is an admissible element of a diagonalizable algebra with the strong disjunction property and $a \rightarrow . \Box b_0 \vee \ldots \vee \Box b_n = \top$ then $b_i = \top$ for some i such that $0 \le i \le n$.*

Proof. Similar to that of Lemma 5.2. ■

5.20. Lemma. *Let A be a steady formula.*

(a) *A admits $\top$.*

(b) *If A admits B and $\vdash_{\mathbf{L}} B \rightarrow C$ then A admits C.*

(c) *Let A admit B. Then for each formula C at least one of the formulas $B \wedge C$ and $B \wedge \neg C$ is admitted by A.*

Proof. Similar to the proof of Lemma 5.17. ■

Our investigation of the relation "A admits B" will henceforth be restricted to $\Box$-free formulas B. This involves no loss of generality because A admits B if and only if $A \wedge (q \leftrightarrow B)$ admits q where q is a new propositional letter.

To decide the relation "A admits B" we have to take a little bit closer look at the $(n, \vec{p})$-bottoms of A than we did when deciding steadiness.

5.21. Lemma. *Let $A \in \mathbf{F}^n(\vec{p})$, $\alpha \in \mathbf{F}^0$ and let B be the $(n, \vec{p})$-trimmed formula which corresponds to A by Lemma 5.11. Suppose A is steady. Then the following are equivalent:*

(i) *A admits α.*

(ii) *For some $(n, \vec{p})$-bottom E of B the formula $E \wedge \alpha$ is irrefutable in $\mathbf{L}$ (which is the same as to say that α is consistent with the $\vec{p}$-real world of E in propositional logic).*

(iii) *For each finite rooted (treelike $\vec{p}$-) model $\mathcal{K}$ in which A holds there exists a rooted model $\mathcal{H}$ such that $\mathcal{H} \Vdash \alpha \wedge \Box^+ A$ and $\mathcal{K}$ is isomorphic to a proper cone of $\mathcal{H}$.*

Proof. (i) ⇒ (ii). Let $\bigvee\!\!\bigvee \gamma$ be the $(n,\vec{p})$-normal form of B. Since B is $(n,\vec{p})$-trimmed $A \vdash_{\mathbf{L}} \neg G$ holds for no $G \in \gamma$. Therefore by Lemma 5.19

$$A \nvdash_{\mathbf{L}} \alpha \to \bigvee\!\!\bigvee \{\Box\neg G \mid G \in \gamma\}\,.$$

This implies that there exists a rooted model $\mathcal{K}$ forcing $\Box^+ A \wedge \alpha \wedge \bigwedge\!\!\bigwedge\{\Diamond G \mid G \in \gamma\}$. For E the $(n,\vec{p})$-character of $\mathcal{K}$ we conclude that $E \wedge \alpha$ is irrefutable in $\mathbf{L}$ and $E\mathrm{Q}^n(\vec{p})G$ for all $G \in \gamma$.

(ii) ⇒ (iii). Consider a $\vec{p}$-model $\mathcal{H}$ of $E \wedge \Box^+ A$. Since $E \wedge \alpha$ is irrefutable in $\mathbf{L}$ there exists an extension of the forcing relation at the nodes of $\mathcal{H}$ such that the resulting model $\mathcal{K}$ forces $\alpha \wedge E \wedge \Box^+ A$. If one grafts a model in which A (and hence $\bigvee\!\!\bigvee \gamma$) holds above the root of $\mathcal{K}$ and extends the forcing relation to new propositional letters if necessary then Lemma 5.8 guarantees that the result forces $\alpha \wedge E \wedge \Box^+ A$.

(iii) ⇒ (i). Suppose $A \nvdash_{\mathbf{L}} D$. Then there exists a finite rooted treelike model $\mathcal{K}$ forcing $\neg D \wedge \Box^+ A$. Let $\mathcal{K}^-$ be the model obtained from $\mathcal{K}$ by restricting the forcing relation of $\mathcal{K}$ to $\vec{p}$. From (iii) we obtain a rooted model $\mathcal{H}$ forcing $\alpha \wedge \Box^+ A$ of which $\mathcal{K}^-$ is a proper cone (say, $\mathcal{H}[a]$). Now change the forcing relation on the propositional letters not in $\vec{p}$ at the nodes of $\mathcal{H}[a]$ so that $\mathcal{H}[a]$ becomes isomorphic to $\mathcal{K}$ and hence forces $\neg D \wedge \Box^+ A$. Since $\mathcal{H}[a]$ is a proper cone of $\mathcal{H}$ and $\alpha \in \mathbf{F}^0$, $\mathcal{H}$ still forces α. So $\mathcal{H}$ now forces $\alpha \wedge \Diamond\neg D \wedge \Box^+ A$ and hence testifies to the fact that $A \nvdash_{\mathbf{L}} \alpha \to \Box D$. Thus we have shown that A admits α. ∎

5.22. Corollary. *The binary relation* "A admits B" *is decidable.*

Proof. Immediate from Corollary 5.14 and (i) ⇔ (ii) of Lemma 5.21. ∎

5.23. Corollary. *Let $B \in \mathbf{F}(\vec{p})$ be steady and let a steady formula A which admits $\alpha \in \mathbf{F}^0(\vec{p})$ be such that $A \vdash_{\mathbf{L}} B$ and $\Box^+ A$ is $\vec{p}$-conservative over $\Box^+ B$. Then B admits α.*

Proof. Let $\mathcal{K}$ be a rooted treelike $\vec{p}$-model in which B holds. Since $\Box^+ A$ is $\vec{p}$-conservative over $\Box^+ B$, by Lemma 2.12 there exists an expansion $\mathcal{H}$ of $\mathcal{K}$ in which A holds. Note that $\mathcal{K}$ is isomorphic to a proper cone of $\mathcal{H}$. Since A admits α, from (i) ⇒ (iii) of Lemma 5.21 we get a rooted model forcing $\alpha \wedge \Box^+ A$ of which $\mathcal{H}$ (and hence also $\mathcal{K}$) is a proper cone. By (iii) ⇒ (i) of Lemma 5.21 this implies that α is admitted by B as required. ∎

5.24. Lemma. *Let $\mathfrak{D}$ be a diagonalizable algebra with the strong disjunction property generated by $x_0, x_1, \ldots$ Suppose $\alpha \in \mathbf{F}^0$. An element $a = \alpha(x_0, x_1, \ldots)$ of $\mathfrak{D}$ is admissible iff for each modal formula* (=*diagonalizable polynomial*) *B such that $B(x_0, x_1, \ldots) = \top$ there exists a steady formula C admitting α such that $C(x_0, x_1, \ldots) = \top$ and $\vdash_{\mathbf{L}} C \to B$.*

Proof. ("if") Suppose $a \to \Box b = \top$ for b an element of $\mathfrak{D}$. Let $b = B(x_0, x_1, \ldots)$. There exists a steady formula C such that $C(x_0, x_1, \ldots) = \top$, C admits α and $C \vdash_{\mathbf{L}} \alpha \to \Box B$. By the definition of the relation "to admit" we have $C \vdash_{\mathbf{L}} B$, therefore $b = \top$ and hence a is admissible.

("only if") Let $B(x_0, x_1, \ldots) = \top$ and let n and $\vec{p}$ be such that $B \in \mathbf{F}^n(\vec{p})$. Define C to be the conjunction of all formulas D in $\mathbf{F}^n(\vec{p})$ such that $D(x_0, x_1, \ldots) = \top$. Clearly $\vdash_{\mathbf{L}} C \to B$. As in Lemma 5.15 we can show that C is steady and $(n, \vec{p})$-trimmed. Let $\bigvee\!\!\!\bigvee \gamma$ be the $(n, \vec{p})$-normal form of C. If C had no $(n, \vec{p})$-bottom E such that $\nvdash_{\mathbf{L}} \neg(E \wedge \alpha)$ then we would have

$$C \vdash_{\mathbf{L}} \alpha \to \bigvee\!\!\!\bigvee \{\Box\neg G \mid G \in \gamma\}$$

and hence

$$a \to \bigvee\!\!\!\bigvee \{\Box\neg G(x_0, x_1, \ldots) \mid G \in \gamma\} = \top\,.$$

Since a is admissible and $\mathfrak{D}$ possesses the strong disjunction property, by Lemma 5.19 there exists a $G \in \gamma$ such that $\neg G(x_0, x_1, \ldots) = \top$, thus contradicting the choice of C. Therefore an $(n, \vec{p})$-bottom E of C such that $\nvdash_{\mathbf{L}} \neg(E \wedge \alpha)$ exists and hence by Lemma 5.21, C admits α. ∎

Finally, we prove a lemma which will enable us to construct steady formulae with a number of meritorious properties.

5.25. Lemma. *Let $\vec{p} \subseteq \vec{q}$, $A \in \mathbf{F}(\vec{q})$ and $B \in \mathbf{F}(\vec{p})$. Suppose that B is steady and that $\Box^+ A$ is $\vec{p}$-conservative over $\Box^+ B$. Then there exists a steady formula $C \in \mathbf{F}(\vec{q})$ such that $C \vdash_{\mathbf{L}} A$ and $\Box^+ C$ is $\vec{p}$-conservative over $\Box^+ B$. Moreover, C can be chosen to admit every formula in $\mathbf{F}^0(\vec{p})$ that is admitted by B.*

Proof. To keep notation at bay we shall only consider the case when B admits just two elements of $\mathbf{F}^0(\vec{p})$, namely α and β. It will easily be seen how to generalize the proof to any larger number of formulas.

Let $A \in \mathbf{F}^n(\vec{q})$ and let $\bigvee\!\!\!\bigvee \delta$ be the $(n, \vec{q})$-normal form of the $(n, \vec{q})$-trimmed formula corresponding to A by Lemma 5.11. Put

$$\begin{aligned}\gamma_{\alpha,\beta} = \{D \in \delta \mid \text{there exist formulas } & E_\alpha, E_\beta \in \delta \text{ such that} \\ & E_\alpha \mathrm{Q}^n(\vec{q}) E_\beta \mathrm{Q}^n(\vec{q}) E_\alpha \mathrm{Q}^n(\vec{q}) D \\ \text{and neither } & \alpha \wedge E_\alpha \text{ nor } \beta \wedge E_\beta \text{ is refutable in } \mathbf{L}\}\,.\end{aligned}$$

Let $\Box_D G$ denote the formula $\Box(D \to G)$. Dually, $\Diamond_D G$ denotes $\Diamond(D \wedge G)$. Note that since B is steady and admits both α and β we have $B \vdash_{\mathbf{L}} F$ whenever $B \vdash_{\mathbf{L}} \Box_\alpha \Box F$ or $B \vdash_{\mathbf{L}} \Box_\beta \Box F$.

Claim 1. *For each formula $G \in \delta - \gamma_{\alpha,\beta}$ there exists an $N \in \omega$ such that $\bigvee\!\!\!\bigvee \delta \vdash_{\mathbf{L}} (\Box_\alpha \Box_\beta)^N \Box\neg G$.*

For if this were not so then there would exist models of $\Box^+ \bigvee\!\!\!\bigvee \delta \wedge (\Diamond_\alpha \Diamond_\beta)^N \Diamond G$ for arbitrarily large $N \in \omega$. Hence by Lemma 5.5 there would exist arbitrarily long sequences $D_1, E_1, \ldots, D_N, E_N$ of elements of δ such that

$$D_i \mathrm{Q}^n(\vec{q}) E_i \mathrm{Q}^n(\vec{q}) D_{i+1} \mathrm{Q}^n(\vec{q}) G\,, \quad \vdash_{\mathbf{L}} D_i \to \alpha \quad \text{and} \quad \vdash_{\mathbf{L}} E_i \to \beta$$

for each i satisfying $1 \le i < N$. But since $\mathrm{Q}^n(\vec{q})$ is transitive and δ is finite this would imply the existence of formulas $D, E \in \delta$ such that $D\mathrm{Q}^n(\vec{q}) E \mathrm{Q}^n(\vec{q}) D \mathrm{Q}^n(\vec{q}) G$,

$\vdash_{\mathbf{L}} D \to \alpha$ and $\vdash_{\mathbf{L}} E \to \beta$, and this puts G in $\gamma_{\alpha,\beta}$ and therefore contradicts the assumption on G. Thus Claim 1 is proved.

Claim 2. *$\Box^+ \bigvee\!\!\!\bigvee \gamma_{\alpha,\beta}$ is $\vec{p}$-conservative over $\Box^+ B$.*

Suppose $F \in \mathbf{F}(\vec{p})$ and $\bigvee\!\!\!\bigvee \gamma_{\alpha,\beta} \vdash_{\mathbf{L}} F$. Since $\delta - \gamma_{\alpha,\beta}$ is finite Claim 1 ensures the existence of a single $N \in \omega$ such that $\bigvee\!\!\!\bigvee \delta \vdash_{\mathbf{L}} (\Box_\alpha \Box_\beta)^N \Box \neg G$ whenever $G \in \delta - \gamma_{\alpha,\beta}$.

We have

$$\begin{aligned} \bigvee\!\!\!\bigvee \delta &\vdash_{\mathbf{L}} (\Box_\alpha \Box_\beta)^N \Box \bigwedge\!\!\!\bigwedge \{\neg G \mid G \in \delta - \gamma_{\alpha,\beta}\} \\ &\vdash_{\mathbf{L}} (\Box_\alpha \Box_\beta)^N \Box \bigvee\!\!\!\bigvee \delta \\ &\vdash_{\mathbf{L}} (\Box_\alpha \Box_\beta)^N \Box \bigvee\!\!\!\bigvee \gamma_{\alpha,\beta} \\ &\vdash_{\mathbf{L}} (\Box_\alpha \Box_\beta)^N \Box F\,. \end{aligned}$$

But since $\Box^+ \bigvee\!\!\!\bigvee \delta$ is $\vec{p}$-conservative over $\Box^+ B$ we also have $B \vdash_{\mathbf{L}} (\Box_\alpha \Box_\beta)^N \Box F$, whence $B \vdash_{\mathbf{L}} F$ for B is steady and admits both α and β. This proves Claim 2.

Next we let

$$\begin{aligned} \varepsilon_{\alpha,\beta} = \{D \in \delta \mid {} & \text{there exist formulas } E_\alpha, E_\beta \in \delta \text{ such that} \\ & D\mathrm{Q}^n(\vec{q}) E_\alpha \mathrm{Q}^n(\vec{q}) E_\beta \mathrm{Q}^n(\vec{q}) D \\ & \text{and neither } \alpha \wedge E_\alpha \text{ nor } \beta \wedge E_\beta \text{ is refutable in } \mathbf{L}\} \end{aligned}$$

and for each $E \in \varepsilon_{\alpha,\beta}$

$$\gamma_E = \{D \in \delta \mid E\mathrm{Q}^n(\vec{q}) D\}\,.$$

Claim 3.

$$\bigvee\!\!\!\bigvee \gamma_{\alpha,\beta} \vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee \left\{\Box^+ \bigvee\!\!\!\bigvee \gamma_E \,\middle|\, E \in \varepsilon_{\alpha,\beta}\right\}.$$

Let a rooted $\vec{q}$-model $\mathcal{K}$ be such that $\mathcal{K} \vDash \bigvee\!\!\!\bigvee \gamma_{\alpha,\beta}$ and let D be the $(n, \vec{q})$-character of $\mathcal{K}$. Since $D \in \gamma_{\alpha,\beta}$ there exists a formula $E \in \varepsilon_{\alpha,\beta}$ such that $E\mathrm{Q}^n(\vec{q})D$. Consider an arbitrary node a of $\mathcal{K}$. For G the $(n, \vec{q})$-character of $\mathcal{K}[a]$ one has $D\mathrm{Q}^n(\vec{q})G$ or $D = G$ by Lemma 5.5. Therefore by Corollary 5.9, $E\mathrm{Q}^n(\vec{q})G$ and hence $G \in \gamma_E$. So $\mathcal{K} \vDash \bigvee\!\!\!\bigvee \gamma_E$ and $\mathcal{K} \vDash \bigvee\!\!\!\bigvee \{\Box^+ \bigvee\!\!\!\bigvee \gamma_E \mid E \in \varepsilon_{\alpha,\beta}\}$, q.e.d.

Claim 4. *For each $E \in \varepsilon_{\alpha,\beta}$ the formula $\bigvee\!\!\!\bigvee \gamma_E$ is $(n, \vec{q})$-trimmed.*

Take a formula $G \in \gamma_E$. Let $\mathcal{H}$ be a rooted $\vec{q}$-model forcing $E \wedge \Box^+ \bigvee\!\!\!\bigvee \delta$ (such a model exists because $\bigvee\!\!\!\bigvee \delta$ is $(n, \vec{q})$-trimmed). By Lemma 5.8, $\bigvee\!\!\!\bigvee \gamma_E$ holds in $\mathcal{H}$. Next pick a rooted $\vec{q}$-model $\mathcal{K}$ such that $\mathcal{K} \Vdash G \wedge \Box^+ \bigvee\!\!\!\bigvee \delta$. Since $E\mathrm{Q}^n(\vec{q})G$ we have $\mathcal{K} \vDash \bigvee\!\!\!\bigvee \gamma_E$. Now graft $\mathcal{K}$ above the root of $\mathcal{H}$. By Lemma 5.8 it is easily seen that the resulting model forces $E \wedge \Box^+ \bigvee\!\!\!\bigvee \gamma_E$ and it clearly forces $\Diamond G$. So $\bigvee\!\!\!\bigvee \gamma_E \nvdash_{\mathbf{L}} \neg G$. Thus $\bigvee\!\!\!\bigvee \gamma_E$ is $(n, \vec{q})$-trimmed.

Claim 5. *There exists an $E \in \varepsilon_{\alpha,\beta}$ such that $\Box^+ \bigvee\!\!\!\bigvee \gamma_E$ is $\vec{p}$-conservative over $\Box^+ B$.*

Deny this. For each $E \in \varepsilon_{\alpha,\beta}$ let $G_E \in \mathbf{F}(\vec{p})$ be a formula such that $\bigvee\!\!\!\bigvee \gamma_E \vdash_{\mathbf{L}} G_E$ and $B \nvdash_{\mathbf{L}} G_E$. Now use Claim 3:

$$\begin{aligned}\bigvee\!\!\!\bigvee \gamma_{\alpha,\beta} &\vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee \left\{\Box^{+}\bigvee\!\!\!\bigvee \gamma_E \,\middle|\, E \in \varepsilon_{\alpha,\beta}\right\} \\ &\vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee \{\Box^{+}G_E \mid E \in \varepsilon_{\alpha,\beta}\} \\ &\vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee \{\Box G_E \mid E \in \varepsilon_{\alpha,\beta}\}\,.\end{aligned}$$

By Claim 2 this gives

$$B \vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee \{\Box G_E \mid E \in \varepsilon_{\alpha,\beta}\}\,,$$

whence for some $E \in \varepsilon_{\alpha,\beta}$ we have $B \vdash_{\mathbf{L}} G_E$ by Lemma 5.2 for B is steady. This contradiction settles Claim 5.

Now note that we have actually proved that at least one of the formulas $\bigvee\!\!\!\bigvee \gamma_E$ meets the requirements on C in the statement of the present lemma. For $\bigvee\!\!\!\bigvee \gamma_E \vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee \gamma_{\alpha,\beta} \vdash_{\mathbf{L}} \bigvee\!\!\!\bigvee \delta \vdash_{\mathbf{L}} A$ because $\gamma_E \subseteq \gamma_{\alpha,\beta} \subseteq \delta$; for each $E \in \varepsilon_{\alpha,\beta}$ the formula $\bigvee\!\!\!\bigvee \gamma_E$ is clearly $(n, \vec{q})$-bottomed by E and hence steady by Claim 4 and Lemma 5.13 and at least one of the formulas $\Box^{+}\bigvee\!\!\!\bigvee \gamma_E$ is $\vec{p}$-conservative over $\Box^{+}B$. Finally, $\bigvee\!\!\!\bigvee \gamma_E$ clearly admits both α and β for each $E \in \varepsilon_{\alpha,\beta}$. ■

As usual we shall not hesitate to use appropriate versions of certain results of this section formalized in $\mathrm{I}\Sigma_1$. Thus Lemma 5.25 is meant to be applied to nonstandard modal formulae.

6. Σ_1-ill theories of infinite credibility extent

6.1. THEOREM. *Let* T *be a* Σ_1*-ill theory of infinite credibility extent. A denumerable diagonalizable algebra* $\mathfrak{D}$ *is isomorphic to an r.e. subalgebra of* $\mathfrak{D}_{\mathrm{T}}$ *iff*

(i) $\mathfrak{D}$ *is positive and*

(ii) *the height of* $\mathfrak{D}$ *is infinite.*

Many details of the proof of this theorem are similar to those of that of Theorem 4.1. Therefore the proofs of certain lemmas are omitted whenever they exhibit no considerable deviation from the proofs of the corresponding lemmas in §4. Also the conventions of §4 on formalized modal logic and Kripke models are still valid.

First we define the Solovay function for $\mathcal{M}_0$ along with its limit value:

(1) $$\mathrm{h}_0(0) = 0\,,$$

(2) $$\mathrm{h}_0(x+1) = \begin{cases} a & \text{if } a \text{ satisfies (i)–(iii) below,} \\ \mathrm{h}_0(x) & \text{if no such } a \text{ exists;}\end{cases}$$

(i) $a \in M_0$,

(ii) $\mathrm{h}_0(x) R a$ and

(iii) $\mathrm{Prf}[x, \overline{\ell_0 = \overline{a} \to \exists y\, [\overline{\mathrm{h}_0(x)} R \mathrm{h}_0(y) R \overline{a}]}]$,

$$\ell_0 = \begin{cases} \lim_{x\to\infty} \mathrm{h}_0(x) & \text{if } \mathrm{h}_0 \text{ reaches a limit,} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

6.2. Lemma ($\mathrm{I}\Sigma_1$).

(a) $\forall x \forall y\, [x \le y \to .\ \mathrm{h}_0(x) = \mathrm{h}_0(y) \vee \mathrm{h}_0(x) R \mathrm{h}_0(y)]$.

(b) $\ell_0 = \lim_{x\to\infty} \mathrm{h}_0(x)$.

(c) $\forall x\, [\mathrm{h}_0(x) = \ell_0 \vee \mathrm{h}_0(x) R \ell_0]$.

(d) $\forall a \in M_0\, [\ell_0 R a \to \neg\, \mathrm{Pr}[\overline{\ell_0 = \overline{a} \to \exists y\, [\overline{\ell_0} R \mathrm{h}_0(y) R \overline{a}]}]]$.

(e) $\forall a \in M_0\, [\ell_0 R a \to \neg\, \mathrm{Pr}(\overline{\ell_0 \neq \overline{a}})]$.

(f) $\ell_0 \neq 0 \to \exists x\ \mathrm{Prf}[x, \overline{\ell_0 = \overline{\ell_0} \to \exists y\, [\overline{\mathrm{h}_0(x)} R \mathrm{h}_0(y) R \overline{\ell_0}]}]$.

(g) $\ell_0 \neq 0 \to \mathrm{Pr}(\overline{\ell_0 \neq \overline{\ell_0}})$.

(h) $\ell_0 \neq 0 \to \mathrm{Pr}(\overline{\overline{\ell_0} R \ell_0})$.

Proof. The only twist new to §4 occurs in (b). In the present situation we have to apply the least number principle on n to the formula $\exists x\, \mathrm{h}_0(x) \Vdash \Box^n \bot$. ∎

6.3. Lemma. (a) $\mathrm{I}\Sigma_1 \vdash \forall x\, [x \neq 0 \to .\ \mathrm{Pr}^x(\overline{\bot}) \leftrightarrow \ell_0 \Vdash \Box^x \bot]$.
(b) *For no* $a \in M_0$ *do we have* $\mathrm{T} \vdash \ell_0 \neq \overline{a}$.
(c) $\mathbb{N} \vDash \ell_0 = 0$. ∎

The theory T will prove $\ell_0 \neq 0$ as likely as not. In the former case the constructions below could be considerably simplified along the lines of §4.

As in §4, let $\nu : \omega - \{0\} \to \mathfrak{D}$ be a positive numeration of $\mathfrak{D}$ and let $\{A(m)\}_{m\in\omega}$ be a Δ_0 enumeration of diagonalizable polynomials in propositional letters $\{p_i\}_{i\in\omega-\{0\}}$ that turn to $\top$ of $\mathfrak{D}$ on substituting νi for p_i. We construct a better behaved and a slightly longer Δ_0 sequence $\{D(m)\}_{m\in\omega\cdot 2}$. The domain of the Δ_0 function $\mathrm{k}(\cdot)$ is however just ω.

$$D(0) = \top, \tag{4}$$

$$\mathrm{k}(0) = 0, \tag{5}$$

$$D(x+1) = \begin{cases} A(x) & \text{if (i) } A(x) \vdash_{\mathbf{L}} D(x), \\ & \quad\ \text{(ii) } A(x) \vdash_{\mathbf{L}} A[\mathrm{k}(x)] \text{ and} \\ & \quad\ \text{(iii) } \Box^+ A(x) \text{ is conservative,} \\ D(x) & \text{otherwise,} \end{cases} \tag{6}$$

$$\mathrm{k}(x+1) = \begin{cases} \mathrm{k}(x) + 1 & \text{if } D(x+1) \vdash_{\mathbf{L}} A[\mathrm{k}(x)], \\ \mathrm{k}(x) & \text{otherwise.} \end{cases} \tag{7}$$

(8) Let $D(\omega + x)$ be the formula manufactured by Lemma 5.25 such that
(i) $D(\omega + x) \vdash_{\mathbf{L}} D(x)$,
(ii) $D(\omega + x)$ is steady and
(iii) $\Box^+ D(\omega + x)$ is conservative.

6.4. Lemma. (a) $I\Sigma_1 \vdash \forall x \in \omega\, \forall y \in \omega\, [x \le y \to D(y) \vdash_{\mathbf{L}} D(x)]$.
(b) $I\Sigma_1 \vdash \forall x \in \omega\, D(\omega + x) \vdash_{\mathbf{L}} D(x)$.
(c) $I\Sigma_1 \vdash \forall x \in \omega$ "$D(\omega + x)$ is steady".
(d) $I\Sigma_1 \vdash \forall x \in \omega \cdot 2$ "$\Box^+ D(x)$ is conservative".
(e) *For each* $y \in \omega$ *there exists an* $x \in \omega$ *such that* $D(x) \vdash_{\mathbf{L}} A(y)$.
(f) *For each* $x \in \omega$ *there exists a* $y \in \omega$ *such that* $D(x) = A(y)$. ∎

The definition of the Solovay functions $\mathrm{h}(\cdot,\cdot)$ is typographically identical with (8)–(12) of §4. Of course the provability predicate employed here is that of the theory T of Theorem 6.1 and the Δ_0 function symbol $\mathrm{g}(\cdot)$ will be defined later in a way different from that of §4.

$$\mathrm{h}(0,x) = \mathrm{h}_0(x)\,, \tag{9}$$

$$\mathrm{h}(i+1,0) = 0\,, \tag{10}$$

$$\mathrm{h}(i+1,x+1) = \begin{cases} a & \text{if } a \text{ satisfies (i)–(vii) below,} \\ \mathrm{h}(i+1,x) & \text{if no } a \text{ satisfying (i)–(vi) exists;} \end{cases} \tag{11}$$

(i) $a \in M_{i+1}$,
(ii) $\mathrm{h}(i,x) \ne \mathrm{h}(i,x+1)$,
(iii) $\mathrm{h}(i+1,x) R a$,
(iv) $a \lhd \mathrm{h}(i,x+1)$,
(v) if $\mathrm{h}(i+1,x) = 0$ then $D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} \neg\Psi_a$,
(vi) for each b satisfying (i)–(v) in place of a one has

$$\forall z \le x\, [\mathrm{Prf}[z, \overline{\ell(\overline{i+1}) = \bar{b} \to \exists y\, [\overline{\mathrm{h}(i+1,x)} R \mathrm{h}(\overline{i+1}, y) R \bar{b}]}] \\ \to \exists w \le z\ \mathrm{Prf}[w, \overline{\ell(\overline{i+1}) = \bar{a} \to \exists y\, [\overline{\mathrm{h}(i+1,x)} R \mathrm{h}(\overline{i+1}, y) R \bar{a}]}]]\,,$$

(vii) a is minimal among those c that satisfy (i)–(vi) in place of a,

$$\ell(0) = \ell_0\,, \tag{12}$$

$$\ell(i+1) = \begin{cases} \lim\limits_{x\to\infty} \mathrm{h}(i+1,x) & \text{if } \mathrm{h}(i+1,\cdot) \text{ reaches a limit,} \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

6.5. Lemma ($I\Sigma_1$).

(a) $\forall i\, \forall x\, \forall y\, [x \le y \to .\ \mathrm{h}(i,x) = \mathrm{h}(i,y) \lor \mathrm{h}(i,x) R \mathrm{h}(i,y)]$.

(b) $\forall i\, \forall x\, \mathrm{h}(i+1,x) \lhd \mathrm{h}(i,x)$.

(c) $\forall j\, \forall i < j\, \forall x\, \mathrm{h}(j,x) \lhd \mathrm{h}(i,x)$.

(d) $\forall i\, \ell(i) = \lim\limits_{x\to\infty} \mathrm{h}(i,x)$.

(e) $\forall i\, \forall x\, [\mathrm{h}(i,x) R \ell(i) \lor \mathrm{h}(i,x) = \ell(i)]$.

(f) $\forall j\, \forall i < j\, \ell(j) \lhd \ell(i)$. ∎

As in §4 let $\ell = 0$ abbreviate $\ell(\bar{i}) = 0$ for each i.

6.6. Lemma ($I\Sigma_1 + \ell \ne 0$).

(a) $\forall i\, \mathrm{Pr}[\overline{\overline{\ell(i)} R \ell(\bar{i})}]$.

(b) $\forall i\, \forall a \in M_i\, [\ell(i) R a \to \neg \Pr[\overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i}, y) R \overline{a}]}]]$.

(c) $\forall i\, \forall a \in M_i\, [\ell(i) R a \to \neg \Pr[\overline{\ell(\bar{i}) \neq \overline{a}}]]$.

P r o o f is essentially the same as that of Lemma 4.6. The only trouble happens with (b) and it is that the quantifier $\forall a \in M_i$ in the formula

$$\forall a \in M_i\, [\ell(i) R a \to \neg \Pr[\overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i}, y) R \overline{a}]}]]$$

is no longer bounded and so we cannot claim that this formula is $\Delta_0(\Sigma_1)$. However, this quantifier does not, in a sense, "mind" being bounded. That is, we apply induction on i just as in the proof of Lemma 4.6(b) to the formula

$$\forall a \in M_i\, [a \Vdash \Box^n \bot \wedge \ell(i) R a \to .\, \neg \Pr[\overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i}, y) R \overline{a}]}]]$$

which is $\Delta_0(\Sigma_1)$ with n a free variable to obtain

$$\forall i\, \forall a \in M_i\, [a \Vdash \Box^n \bot \wedge \ell(i) R a \to .\, \neg \Pr[\overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i}, y) R \overline{a}]}]]$$

and this formula after being prefixed by $\forall n$ turns equivalent to

$$\forall i\, \forall a \in M_i\, [\ell(i) R a \to \neg \Pr[\overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i}, y) R \overline{a}]}]]\,. \ \blacksquare$$

Let σ be a false Σ_1 sentence proved by T. We assume that σ is of the form $\exists x\, \sigma_0(x)$ where $\sigma_0(x)$ is Δ_0 and introduce in $\mathrm{I}\Sigma_1 + \sigma$ a closed ε-term s such that

$$\mathrm{I}\Sigma_1 + \sigma \vdash \sigma_0(\mathsf{s}) \wedge \forall y < \mathsf{s}\, \neg \sigma_0(y)\,. \tag{14}$$

(This is clearly possible by the (Δ_0) least number principle.) When working in $\mathrm{I}\Sigma_1$ we can treat expressions

$$x \leq \mathsf{s}, \quad x = \mathsf{s} \quad \text{and} \quad x \geq \mathsf{s}$$

as abbreviations for the Δ_0 expressions

$$\forall y < x\, \neg \sigma_0(y), \quad \sigma_0(x) \wedge \forall y < x\, \neg \sigma_0(y) \quad \text{and} \quad \exists y \leq x\, \sigma_0(y)$$

respectively.

Here is the definition of g:

$$\mathrm{g}(x) = \begin{cases} z & \text{if } z \text{ satisfies (i)–(iv) below,} \\ x & \text{if } x < \mathsf{s} \text{ and no } z \text{ satisfying (i)–(iii) exists,} \\ \omega + \mathsf{s} & \text{if } x \geq \mathsf{s} \text{ and no } z \text{ satisfying (i)–(iii) exists;} \end{cases} \tag{15}$$

(i) $z < x$,

(ii) $z < \mathsf{s}$,

(iii) there exists an $i \in \omega$ and a node $a \in M_i$ such that

$$\mathrm{Prf}[z, \overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [0 R \mathrm{h}(\bar{i}, y) R \overline{a}]}] \quad \text{and} \quad D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg \Psi_a\,,$$

or there exists an $i \in \omega$ and a formula A such that

$$\mathrm{Prf}[z, \overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{A}}] \quad \text{and} \quad D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} A\,,$$

(iv) z is minimal among those satisfying (i)–(iii).

The reason why we introduce the second disjunct in (iii) of (15) is that when finishing the proof of Theorem 6.1 we shall need to know that T does not prove sentences of the form $\ell(i) \Vdash B$ unless $\exists x\, A(x) \vdash_{\mathbf{L}} B$. (iii) is a way to make sure that once an unwanted sentence $\ell(i) \Vdash B$ is proved it results in the proof of an equally unwanted sentence of the form $\ell(i) \neq a$ which we are able to bring to a contradiction. In §4 an analogous situation was handled by a compactness argument which depended crucially on the number of nodes accessible to each of $\mathrm{h}(i, \cdot)$ being finite. In the present section this is not the case and so the construction has to be more alert. This twist also comes from Jumelet [27].

6.7. Lemma ($\mathrm{I}\Sigma_1$).

(a) $\forall x\, \forall y\, [x \le y \to D[\mathrm{g}(y)] \vdash_{\mathbf{L}} D[\mathrm{g}(x)]]$.

(b) $\forall x = \mathsf{s}\, \forall y\, D(\omega + x) \vdash_{\mathbf{L}} D[\mathrm{g}(y)]$.

(c) $\forall i\, \forall x\, \forall a \in M_i\, [\mathrm{Prf}[x, \overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [0 R \mathrm{h}(\bar{i}, y) R \overline{a}]}]$
$\qquad \wedge D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} \neg \Psi_a \to .\ \forall y \ge x\, \mathrm{g}(y) = \mathrm{g}(x)]$.

(d) $\forall i\, \forall x\, \forall A \in \mathbf{F}(i)\, [\mathrm{Prf}[x, \overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{A}}]$
$\qquad \wedge D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} A \to .\ \forall y \ge x\, \mathrm{g}(y) = \mathrm{g}(x)]$.

(e) $\forall x < \mathsf{s}\, [\mathrm{g}(x) \neq x \to \exists i\, [\exists a \in M_i\, [\mathrm{Pr}[\overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [0 R \mathrm{h}(\bar{i}, y) R \overline{a}]}]$
$\wedge \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg \Psi_a] \vee \exists A \in \mathbf{F}(i)\, [\mathrm{Pr}[\overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{A}}] \wedge \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} A]]]$.

(f) $\forall x = \mathsf{s}\, [\mathrm{g}(x) \neq \omega + x \to \exists i\, [\exists a \in M_i\, [\mathrm{Pr}[\overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [0 R \mathrm{h}(\bar{i}, y) R \overline{a}]}]$
$\wedge \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg \Psi_a] \vee \exists A \in \mathbf{F}(i)\, [\mathrm{Pr}[\overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{A}}] \wedge \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} A]]]$.

Proof. By inspection of (15) we see that either for some $z < \mathsf{s}$ the conditions (ii)–(iv) of (15) hold in which case $\mathrm{g}(x) = z$ for all $x \ge z$, or s exists and we have $\mathrm{g}(y) = y$ for all $y < \mathsf{s}$ and $\mathrm{g}(x) = \omega + \mathsf{s}$ for all $x \ge \mathsf{s}$.

In the former case (a) is a direct consequence of Lemma 6.4(a) and in the latter case (a) follows from Lemma 6.4(a) and (b).

Clause (b) is also immediate.

(c) and (d) are proved in perfect analogy with Lemma 4.7(b).

(e) and (f) enjoy proofs similar to that of Lemma 4.7(c). ∎

6.8. Lemma ($\mathrm{I}\Sigma_1$).

(a) $\ell \neq 0 \to \forall i\, \forall x \forall A \in \mathbf{F}(i)\, [\mathrm{h}(i, x) = 0 \wedge D[\mathrm{g}(x)] \vdash_{\mathbf{L}} A \to .\ \ell(i) \Vdash A]$.

(b) $\ell = 0 \to \forall i\, \forall a \in M_i\, [\mathrm{Pr}[\overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [0 R \mathrm{h}(\bar{i}, y) R \overline{a}]}] \to \exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} \neg \Psi_a]$.

(c) $\ell = 0 \to \forall i\, \forall a \in M_i\, [\mathrm{Pr}[\overline{\ell(\bar{i}) \neq \overline{a}}] \to \exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} \neg \Psi_a]$.

(d) $\ell = 0 \to \forall A \in \mathbf{F}\, [\mathrm{Pr}[\overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{A}}] \to \exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} A]$.

Proof. Clauses (a)–(c) are proved in the same way as those of Lemma 4.8. When handling (b), however, we have to execute a trick similar to that in the proof of Lemma 6.6(b), that is, before applying induction we impose on $\forall a \in M_i$ the dummy bound $a \Vdash \Box^n \bot$.

(d) Assume $\ell = 0$ and $\mathrm{Prf}[x, \overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{A}}]$. If $D[\mathrm{g}(x)] \vdash_{\mathbf{L}} A$ then we are done.

If $D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} A$ then by Lemma 6.7(d), $\forall y \geq x\, \mathrm{g}(y) = \mathrm{g}(x)$ and so Lemma 3.5(g) provides a node $a \in M_i$ such that

$$\forall y\, D[\mathrm{g}(y)] \nvdash_{\mathbf{L}} \neg \Psi_a \quad \text{and} \quad a \Vdash \neg A\,.$$

We have $\mathrm{Pr}[\overline{\ell \neq 0 \to \ell(\bar{i}) \neq \overline{a}}]$, whence $\mathrm{Pr}[\overline{\ell(\bar{i}) \neq \overline{a}}]$. Now (c) brings us to a contradiction. ■

In order to construct the desired embedding we have to define an analogue of forcing relation at 0. This analogue will be denoted by $\mathrm{T}(\cdot)$. And even before we construct $\mathrm{T}(\cdot)$ we have to introduce some notation.

In $\mathrm{I}\Sigma_1$, we shall think of lower case Greek letters from the beginning of the alphabet as variables ranging over finite strings of $\bot$'s and $\top$'s. Λ is the empty string. The Δ_0 function $\mathrm{lh}(\cdot)$ tells the number of "digits" in a string. The ith digit in α is written as $(\alpha)_i$. We shall always be careful enough not to use $(\alpha)_i$ when $\mathrm{lh}(\alpha) > i$. Stipulate also that each string begins with its first digit so that the expression $(\alpha)_0$ is meaningless. We write $\alpha \prec \beta$ if α and β are strings of equal length and α lexicographically precedes β. If one adopts the first coding of strings that comes to mind then $\prec$ can be taken to coincide with the usual ordering of integers. Finally, $\alpha \subseteq \beta$ means that α is an initial segment of β.

In fact, we shall identify strings of length i with elements of $\mathbf{A}^0(i)$ so that when we say "A admits α" where $\mathrm{lh}(\alpha) = i$ we actually mean that the formula A admits the formula $\bigwedge\!\!\!\bigwedge \{p_j \leftrightarrow (\alpha)_j \mid 1 \leq j \leq i\}$.

Define in $\mathrm{I}\Sigma_1 + \sigma$:

$$\mathrm{Adm}(\alpha) \equiv \text{“}D(\omega + \mathsf{s}) \text{ admits } \alpha\text{”}\,. \tag{16}$$

6.9. Lemma ($\mathrm{I}\Sigma_1 + \sigma$).

(a) $\mathrm{Adm}(\overline{\Lambda})$.

(b) $\forall \alpha\, \forall \beta\, [\mathrm{Adm}(\beta) \wedge \alpha \subseteq \beta \to\!.\ \mathrm{Adm}(\alpha)]$.

(c) $\forall i\, \forall \alpha\, [\mathrm{lh}(\alpha) \leq i \wedge \mathrm{Adm}(\alpha) \to\!.\ \exists \beta\, [\mathrm{lh}(\beta) = i \wedge \alpha \subseteq \beta \wedge \mathrm{Adm}(\beta)]]$.

(d) $\forall i\, \exists \alpha\, [\mathrm{lh}(\alpha) = i \wedge \mathrm{Adm}(\alpha)]$.

Proof. (a) follows at once from Lemmas 5.20(a) and 6.4(c).

(b) follows from Lemma 5.20(b).

(c) By Lemma 5.20(c) if β is a string of length i admitted by $D(\omega + \mathsf{s})$ then there exists a string of length $i + 1$ which $D(\omega + \mathsf{s})$ also admits and of which β is an initial segment. Applying induction on i we establish the claim. Induction is applicable because the formula

$$\forall\alpha\,[\mathrm{lh}(\alpha) \le i \wedge \mathrm{Adm}(\alpha) \to.\ \exists\beta\,[\mathrm{lh}(\beta) = i \wedge \alpha \subseteq \beta \wedge \mathrm{Adm}(\beta)]]$$

is Π_1 over $\mathrm{I}\Sigma_1 + \sigma$. Indeed, $\mathrm{Adm}(\beta)$ is Δ_1 over $\mathrm{I}\Sigma_1 + \sigma$ and the condition $\mathrm{lh}(\beta) = i$ is a primitive recursive bound on the quantifier $\exists\beta$ so taking $\mathrm{Adm}(\beta)$ in its Π_1 form we can by an instance of the Σ_1 collection schema available in $\mathrm{I}\Sigma_1$ also bring the formula $\exists\beta\,[\mathrm{lh}(\beta) = i \wedge \alpha \subseteq \beta \wedge \mathrm{Adm}(\beta)]$ to Π_1 form. $\mathrm{Adm}(\alpha)$ is to be rewritten as a Σ_1 formula.

(d) follows from (a) and (c). ∎

Still in $\mathrm{I}\Sigma_1 + \sigma$, put

(17) $$\mathrm{Adm}^+(\alpha) \equiv \mathrm{Adm}(\alpha) \wedge \forall\beta \prec \alpha\,[\mathrm{lh}(\alpha) = \mathrm{lh}(\beta) \to \neg\,\mathrm{Adm}(\beta)]\,.$$

6.10. Lemma ($\mathrm{I}\Sigma_1 + \sigma$).

(a) $\forall i\,\exists!\alpha\,[\mathrm{lh}(\alpha) = i \wedge \mathrm{Adm}^+(\alpha)]\,.$

(b) $\forall i\,\forall\alpha\,[\mathrm{lh}(\alpha) \le i \wedge \mathrm{Adm}^+(\alpha) \to.\ \exists\beta\,[\mathrm{lh}(\beta) = i \wedge \alpha \subseteq \beta \wedge \mathrm{Adm}^+(\beta)]]\,.$

(c) $\forall\alpha\,\forall\beta\,[\mathrm{Adm}^+(\alpha) \wedge \mathrm{Adm}^+(\beta) \wedge \mathrm{lh}(\alpha) \le \mathrm{lh}(\beta) \to.\ \alpha \subseteq \beta]\,.$

In other words, $\mathrm{Adm}^+(\cdot)$ *singles out an infinite branch in the tree of finite* $\bot$-$\top$-*strings.*

Proof. (a) follows from Lemma 6.9(d) by the (Δ_1) least number principle.

(b) Suppose $\mathrm{Adm}^+(\alpha)$ for a string α of length $\le i$. By (a) there exists a string β of length i such that $\mathrm{Adm}^+(\beta)$. Consider the initial segment $\gamma \subseteq \beta$ of length equal to that of α. We claim $\alpha = \gamma$. For if $\alpha \prec \gamma$ then by Lemma 6.9(c) there exists a string δ of length i such that $\alpha \subseteq \delta$ and $\mathrm{Adm}(\delta)$. Since $\alpha \prec \gamma$ implies $\delta \prec \beta$ this contradicts $\mathrm{Adm}^+(\beta)$. Finally, it cannot be the case that $\gamma \prec \alpha$ because then $\mathrm{Adm}^+(\alpha)$ would not hold.

(c) Let α and β satisfy $\mathrm{Adm}^+(\cdot)$ and let the length of β be greater than the length of α. By (b) there is a string γ of length i prolonging α and such that $\mathrm{Adm}^+(\gamma)$. Conclude by (a) that $\beta = \gamma$. ∎

At last we can define $\mathrm{T}(\cdot)$:

(18) $$\begin{aligned}\mathrm{T}(A) \equiv \exists i\,\exists k\,\exists\lambda \in \mathbf{F}^0(i+k)\,\exists B_1,\ldots,B_k \in \mathbf{F}(i)\,\exists\alpha &\\ [\mathrm{lh}(\alpha) = i + k \wedge \lambda[(\alpha)_1,\ldots,(\alpha)_{i+k}] = \top &\\ \wedge A = \lambda(p_1,\ldots,p_i,\Box B_1,\ldots,\Box B_k) &\\ \wedge\forall j \le i\,[j \ne 0 \to.\ (\alpha)_j = \top \leftrightarrow \exists\beta\,[\mathrm{lh}(\beta) = i \wedge \mathrm{Adm}^+(\beta) \wedge (\beta)_j = \top]] &\\ \wedge\forall j \le k\,[j \ne 0 \to.\ (\alpha)_{i+j} = \top \leftrightarrow D(\omega + \mathrm{s}) \vdash_{\mathbf{L}} B_j]]\,. &\end{aligned}$$

6.11. Lemma ($\mathrm{I}\Sigma_1 + \sigma$).

(a) $\forall i\,[\mathrm{T}(p_i) \leftrightarrow \exists\beta\,[\mathrm{lh}(\beta) = i \wedge \mathrm{Adm}^+(\beta) \wedge (\beta)_i = \top]]\,.$

(b) $\forall A \in \mathbf{F}\,[\mathrm{T}(\Box A) \leftrightarrow D(\omega + \mathrm{s}) \vdash_{\mathbf{L}} A]\,.$

(c) $\forall i\,\forall\lambda\in\mathbf{F}^0(i)\,\forall B_1,\dots,B_i\in\mathbf{F}\,[\mathrm{T}[\lambda(B_1,\dots,B_i)]$
$$\leftrightarrow \exists\alpha\,[\mathrm{lh}(\alpha)=i\wedge\lambda[(\alpha)_1,\dots,(\alpha)_i]=\top \wedge\forall j\le i\,[j\ne 0\rightarrow.\,(\alpha)_j=\top\leftrightarrow\mathrm{T}(B_j)]]]\,.$$

That is, $\mathrm{T}(\cdot)$ *distributes over Boolean connectives.*

Proof. A routine inspection of (18). ∎

Lemma 6.11 may be viewed as an alternative definition of $\mathrm{T}(\cdot)$. In fact, we only wrote down (18) in order to be able to directly estimate the arithmetical complexity of $\mathrm{T}(\cdot)$.

6.12. Lemma ($\mathrm{I}\Sigma_1+\sigma$).

(a) $\forall\alpha\Big[\mathrm{T}\Big[\bigwedge\!\!\!\bigwedge_{1\le j\le \mathrm{lh}(\alpha)}[p_j\leftrightarrow(\alpha)_j]\Big]\leftrightarrow\mathrm{Adm}^+(\alpha)\Big]\,.$

(b) $\forall A\in\mathbf{F}\,\forall B\in\mathbf{F}\,[\mathrm{T}(\diamond A\wedge\Box B)\rightarrow\mathrm{T}[\diamond(A\wedge\Box B)]]\,.$

(c) $\forall A\in\mathbf{F}\,[\mathrm{T}(\Box A)\rightarrow\mathrm{T}(A)]\,.$

Proof. (a) Let $i=\mathrm{lh}(\alpha)$. Then

$$\begin{aligned}
\mathrm{T}\Big[\bigwedge\!\!\!\bigwedge_{1\le j\le i}[p_j\leftrightarrow(\alpha)_j]\Big]&\leftrightarrow\forall j\le i\,[j\ne0\rightarrow.\,\mathrm{T}(p_j)\leftrightarrow(\alpha)_j=\top]\\
&\qquad\qquad\text{(by Lemma 6.11(c))}\\
&\leftrightarrow\forall j\le i\,[j\ne0\rightarrow.\,\exists\beta\,[\mathrm{lh}(\beta)=j\\
&\qquad\wedge\mathrm{Adm}^+(\beta)\wedge(\beta)_j=\top]\leftrightarrow(\alpha)_j=\top]\\
&\qquad\qquad\text{(by Lemma 6.11(a))}\\
&\leftrightarrow\forall j\le i\,[j\ne0\rightarrow.\,\exists\gamma\,[\mathrm{lh}(\gamma)=i\\
&\qquad\wedge\mathrm{Adm}^+(\gamma)\wedge(\gamma)_j=\top]\leftrightarrow(\alpha)_j=\top]\\
&\qquad\qquad\text{(by Lemma 6.10(b) and (c))}\\
&\leftrightarrow\forall\gamma\,[\mathrm{lh}(\gamma)=i\wedge\mathrm{Adm}^+(\gamma)\\
&\qquad\rightarrow.\,\forall j\le i\,[j\ne0\rightarrow.\,(\gamma)_j=(\alpha)_j]]\\
&\qquad\qquad\text{(by Lemma 6.10(a))}\\
&\leftrightarrow\forall\gamma\,(\mathrm{lh}(\gamma)=i\wedge\mathrm{Adm}^+(\gamma)\rightarrow.\,\gamma=\alpha)\\
&\leftrightarrow\mathrm{Adm}^+(\alpha)\ \text{(by Lemma 6.10(a))}.
\end{aligned}$$

(b) If $\mathrm{T}(\diamond A\wedge\Box B)$ then by Lemma 6.11(b) and (c) we have $D(\omega+\mathsf{s})\vdash_{\mathbf{L}} B$ and hence $D(\omega+\mathsf{s})\vdash_{\mathbf{L}}\Box B$. Suppose $\mathrm{T}[\diamond(A\wedge\Box B)]$ is not the case, that is, by Lemma 6.11(b), $D(\omega+\mathsf{s})\vdash_{\mathbf{L}}\Box B\rightarrow\neg A$. But then $D(\omega+\mathsf{s})\vdash_{\mathbf{L}}\neg A$, which contradicts $\mathrm{T}(\diamond A)$.

(c) We shall prove that $\mathrm{T}(A)$ implies $\mathrm{T}(\diamond A)$.

Let A be $\lambda(p_1,\dots,p_i,\Box B_1,\dots,\Box B_k)$ for λ a Boolean formula. Bring λ into the disjunctive normal form:

$$\bigvee_j \Big[\bigwedge_n [p_n \leftrightarrow (\alpha)^j_n] \wedge \bigwedge_m [\Box B_m \leftrightarrow (\beta)^j_m] \Big]$$

for α and β appropriate matrices of $\bot$'s and $\top$'s. Since we have $\mathrm{T}(A)$ by Lemma 6.11(c) this implies the existence of a j_0 such that

$$\mathrm{T}\Big[\bigwedge_n [p_n \leftrightarrow (\alpha)^{j_0}_n] \wedge \bigwedge_m [\Box B_m \leftrightarrow (\beta)^{j_0}_m] \Big].$$

Clearly the above is equivalent to

$$\mathrm{T}\Big[\bigwedge_n [p_n \leftrightarrow (\alpha)_n] \wedge \bigwedge_m \Box C_m \wedge \bigwedge_l \Diamond E_l \Big]$$

for the obvious choice of C_m's and E_l's (we let $(\cdot)_i$ stand for $(\cdot)^{j_0}_i$). From this it follows by (a) that $D(\omega + \mathrm{s})$ admits $(\alpha)^{j_0}$ and so by Lemma 5.19

$$D(\omega + \mathrm{s}) \nvdash_{\mathbf{L}} \bigwedge_n [p_n \leftrightarrow (\alpha)_n] \rightarrow \bigvee_l \Box \neg E_l\,,$$

therefore

$$\mathrm{T}\Big[\Diamond \Big[\bigwedge_n [p_n \leftrightarrow (\alpha)_n] \wedge \bigwedge_l \Diamond E_l \Big]\Big].$$

With the help of (b) this yields

$$\mathrm{T}\Big[\Diamond \Big[\bigwedge_n [p_n \leftrightarrow (\alpha)_n] \wedge \bigwedge_m \Box C_m \wedge \bigwedge_l \Diamond E_l \Big]\Big],$$

that is, $\mathrm{T}(\Diamond A)$. ∎

As in §4 we define a mapping $^\circ \colon \{p_i\}_{i \in \omega - \{0\}} \rightarrow \mathfrak{D}_{\mathrm{T}}$:

(19) $$p_i^\circ \equiv \ell \neq 0 \wedge \ell(\bar{i}) \Vdash \overline{p_i} \vee\!. \ \ell = 0 \wedge \mathrm{T}(\overline{p_i})\,.$$

$^\circ$ is prolonged to all modal formulae.

6.13. Lemma. (a) *For each $i \in \omega$ and for each modal formula $A(p_1, \ldots, p_i)$*

$$\mathrm{I}\Sigma_1 \vdash \ell \neq 0 \rightarrow\!. \ [A(p_1, \ldots, p_i)]^\circ \leftrightarrow \ell(\bar{i}) \Vdash \overline{A(p_1, \ldots, p_i)}\,.$$

(b) $\mathrm{I}\Sigma_1 \vdash \forall i \forall A \in \mathbf{F}(i) \ \mathrm{Pr}[\overline{\ell \neq 0 \rightarrow\!. \ A^\circ \leftrightarrow \ell(\bar{i}) \Vdash \overline{A}}]$. ($^\circ$ *is representable by a Δ_0 function*).

(c) $\mathrm{I}\Sigma_1 \vdash \ell = 0 \rightarrow \forall A \in \mathbf{F}\, [\mathrm{Pr}(\overline{A^\circ}) \rightarrow \exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} A]$.

(d) $\mathrm{I}\Sigma_1 + \sigma \vdash \ell = 0 \rightarrow \mathrm{g}(\mathrm{s}) = \omega + \mathrm{s}$.

(e) *For each $i \in \omega$ and for each modal formula $A(p_1, \ldots, p_i)$*

$$\mathrm{I}\Sigma_1 + \sigma \vdash [A(p_1, \ldots, p_i)]^\circ$$
$$\leftrightarrow [\ell \neq 0 \wedge \ell(\bar{i}) \Vdash \overline{A(p_1, \ldots, p_i)} \vee\!. \ \ell = 0 \wedge \mathrm{T}[\overline{A(p_1, \ldots, p_i)}]]\,.$$

Proof. (a) Analogous to the proof of Lemma 4.10. The assumption $\ell \neq 0$ being a Σ_1 sentence finds its way inside the provability predicate and therefore validates the induction step for $\Box$.

(b) is proved by (Σ_1) induction on the structure of A. For the induction step, formalize that of (a).

(c) follows at once from (b) and Lemma 6.8(d).

(d) Suppose $\mathrm{g}(\mathsf{s}) \neq \omega + \mathsf{s}$. By Lemma 6.7(f) this means that either

(i) there exists an $i \in \omega$ and a node $a \in M_i$ such that

$$\Pr[\overline{\ell(\bar{i}) = \bar{a} \to \exists y\,[0R\mathrm{h}(\bar{i}, y)R\bar{a}]}] \quad \text{and} \quad \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg\Psi_a$$

or

(ii) there exists an $i \in \omega$ and a formula $A \in \mathbf{F}(i)$ such that

$$\Pr[\overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{A}}] \quad \text{and} \quad \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} A\,.$$

But (i) contradicts Lemma 6.8(b) and (ii) contradicts Lemma 6.8(d). Thus $\mathrm{g}(\mathsf{s}) = \omega + \mathsf{s}$.

(e) We use induction on A. Suppose $A \in \mathbf{F}(i)$. The only interesting case is $\Box$. So assume A is $\Box B$ and go inside $\mathrm{I}\Sigma_1 + \sigma$. By (a) we may also assume $\ell = 0$.

($\to$) We have $\Pr(\overline{B^\circ})$, whence by (b), $\Pr[\overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{A}}]$ and so from Lemma 6.8(d) we have $\exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} B$. Hence by Lemma 6.7(b), $D(\omega{+}\mathsf{s}) \vdash_{\mathbf{L}} B$. Lemma 6.11(b) yields then $\mathrm{T}(\overline{\Box B})$.

($\leftarrow$) Since $\mathrm{T}(\overline{\Box B})$ is Σ_1 over $\mathrm{I}\Sigma_1 + \sigma$ it implies $\Pr[\overline{\mathrm{T}(\overline{\Box B})}]$ and hence $\Pr[\overline{\mathrm{T}(\overline{B})}]$ by Lemma 6.12(c) formalized. Therefore $\Pr[\overline{\ell = 0 \to \mathrm{T}(\overline{B})}]$.

On the other hand, $\mathrm{T}(\Box B)$ is equivalent to $D(\omega + \mathsf{s}) \vdash_{\mathbf{L}} B$ and hence by (d) to $D[\mathrm{g}(\mathsf{s})] \vdash_{\mathbf{L}} B$. Since clearly $\mathrm{h}(i, \mathsf{s}) = 0$ for each $i \in \omega$ it is seen through Lemma 6.8(a) formalized that $\Pr[\overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{B}}]$.

Thus $\mathrm{T}(\overline{\Box B})$ implies $\Pr(\overline{B^\circ})$. ∎

For proofs of lemmata of the kind represented by Lemmas 4.10 and 6.13(e) (i.e. lemmas of the form $\vdash A^\circ \leftrightarrow \ell \Vdash A$) it is typical to use some property like $\Pr(\overline{\bar{\ell} R \ell})$ which is usually enjoyed by all nodes of the model but the root 0. Therefore these lemmas usually need the assumption that the function h leaves 0 unless the node 0 is *reflexive*, that is, $0 \Vdash \Box A$ implies $0 \Vdash A$ for all the relevant formulas A. In the latter case the proof goes through equally well (cf. Solovay [50]). Another way one can use this observation is to let h jump to a reflexive node the moment some Δ_0 event happens. A clever choice of this Δ_0 event can help to obtain an h with some extra desirable properties. This idea flowered in Beklemishev [5] and [6].

In most applications the number of formulas for which it is important that $0 \Vdash \Box A \to A$ is finite. Our construction, on the contrary, purports to take care of all the infinite collection of modal formulae. Moreover, our Kripke models do not stay the same and since the diagonalizable algebras we deal with are not generally strongly disjunctive (nor even ω-consistent) we cannot generally do with models whose root comes close to being reflexive. Recall, however, that the theory T of the present section believes that there exists a nonstandardly large recursive number s. So the way out is to fool T into thinking that after the moment s the model stops changing and the root of this frozen model *is* reflexive. This is the content of Lemma 6.12(c). In fact, to achieve reflexivity of 0 at the moment s we have to delete a nonstandardly sophisticated collection of nodes which is specified by Lemma 5.25.

It should be noted that the construction in §6 of Beklemishev [5] also may be thought of as chopping off certain parts of the Kripke model at a nonstandard moment so as to make the root eventually reflexive, and that that construction led us to the one presented in this section.

6.14. LEMMA. $\mathbb{N} \vDash$ "g is the identity function".

Proof. If $\mathrm{g}(x) \neq x$ for some $x \in \omega$ then it must be for one of the two reasons given in Lemma 6.7(e). Lemma 6.8(b) and (d) shows that either of the two reasons implies $\ell \neq 0$. Quod non. ■

6.15. LEMMA. *If* $A(\nu 1, \nu 2, \ldots) = \top$ *for* $A(x_1, x_2, \ldots)$ *a diagonalizable polynomial then*

$$\mathrm{I}\Sigma_1 + \sigma \vdash [A(p_1, p_2, \ldots)]^\circ .$$

Proof. Let $A(p_1, p_2, \ldots) \in \mathbf{F}(i)$. As in Lemma 4.11 we have $D[\mathrm{g}(m)] \vdash_{\mathbf{L}} D(m) \vdash_{\mathbf{L}} A(p_1, \ldots, p_i)$ for some $m \in \omega$ by Lemma 6.4(e). Note that $\mathrm{h}(i, m) = 0$.

Reason in $\mathrm{I}\Sigma_1 + \sigma$. By Lemmas 6.11(b) and 6.7(b) we have $\mathrm{T}[\overline{\Box A(p_1, \ldots, p_i)}]$ and hence by Lemma 6.12(c), $\mathrm{T}[\overline{A(p_1, \ldots, p_i)}]$. If $\ell \neq 0$ then by Lemma 6.8(a), $\ell(i)$ forces $A(p_1, \ldots, p_i)$.

In view of Lemma 6.13(e) this amounts to $[A(p_1, \ldots, p_i)]^\circ$. ■

In full analogy with §4 we define $^*\colon \operatorname{rng} \nu \to \mathfrak{D}_{\mathrm{T}}$:

$$(\nu i)^* \equiv p_i^\circ \tag{20}$$

and show that * embeds $\mathfrak{D}$ into $\mathfrak{D}_{\mathrm{T}}$:

6.16. Proof of Theorem 6.1 is concluded in nearly the same manner as the proof of Theorem 4.1 (see 4.12). The only difference is that instead of the compactness argument in 4.12 we use Lemma 6.8(d) to see that $\mathrm{T} \vdash [A(\nu 1, \nu 2, \ldots)]^*$ implies $A(\nu 1, \nu 2, \ldots) = \top$. ■

The reasons why the proof of Theorem 6.1 requires the use of an infinite sequence of increasingly restrictive conditions on the range of the Solovay function h (cf. (v) of (11)) to carry out the embedding of $\mathfrak{D}$ into $\mathfrak{D}_{\mathrm{T}}$ are somewhat deeper than those for the proof of Theorem 4.1. Even if one is going to model in $\mathfrak{D}_{\mathrm{T}}$ a finitely generated diagonalizable algebra it will not generally do to impose on h the constant condition

$$\text{if} \quad \mathrm{h}(x) = 0, \quad \text{then} \quad \mathrm{h}(x+1) = 0 \quad \text{or} \quad \forall m\, \mathrm{h}(x+1) \vDash D(m)$$

even in the case when this condition is recursive. To see that, think of the diagonalizable algebra of infinite height on just one generator a which satisfies the relation

$$a \to \Diamond^n \top = \top$$

for each $n \in \omega$ and yet $a \neq \bot$.

7. Σ_1-sound theories

7.1. Theorem. *Let* T *be a* Σ_1*-sound theory. A denumerable diagonalizable algebra* $\mathfrak{D}$ *is isomorphic to an r.e. subalgebra of* $\mathfrak{D}_{\mathrm{T}}$ *iff*

(i) $\mathfrak{D}$ *is positive and*

(ii) $\mathfrak{D}$ *enjoys the strong disjunction property.*

The scheme of the proof of Theorem 7.1 coincides with that of Theorem 6.1. We employ here much the same objects as we did in §6 and prove lemmas very similar to those of §6. Therefore we shall be very sketchy about the proofs which will usually be modifications of proofs of corresponding lemmas in §6.

We proceed to list the necessary definitions.

$$\mathrm{h}_0(0) = 0\,, \tag{1}$$

$$\mathrm{h}_0(x+1) = \begin{cases} a & \text{if } a \text{ satisfies (i)–(iii) below,} \\ \mathrm{h}_0(x) & \text{if no such } a \text{ exists;} \end{cases} \tag{2}$$

(i) $a \in M_0$,

(ii) $\mathrm{h}_0(x) R a$ and

(iii) $\mathrm{Prf}[x, \overline{\ell_0 = \overline{a} \to \exists y\,[\overline{\mathrm{h}_0(x)} R \mathrm{h}_0(y) R \overline{a}]}]$,

$$\ell_0 = \begin{cases} \lim\limits_{x\to\infty} \mathrm{h}_0(x) & \text{if } \mathrm{h}_0 \text{ reaches a limit,} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

7.2. Lemma ($\mathrm{I}\Sigma_1$).

(a) $\forall x\,\forall y\,[x \le y \to.\ \mathrm{h}_0(x) = \mathrm{h}_0(y) \vee \mathrm{h}_0(x) R \mathrm{h}_0(y)]$.

(b) $\ell_0 = \lim\limits_{x\to\infty} \mathrm{h}_0(x)$.

(c) $\forall x\,[\mathrm{h}_0(x) = \ell_0 \vee \mathrm{h}_0(x) R \ell_0]$.

(d) $\forall a \in M_0\,[\ell_0 R a \to \neg\,\mathrm{Pr}[\overline{\ell_0 = \overline{a} \to \exists y\,[\overline{\ell_0} R \mathrm{h}_0(y) R \overline{a}]}]]$.

(e) $\forall a \in M_0\,[\ell_0 R a \to \neg\,\mathrm{Pr}(\overline{\ell_0 \neq \overline{a}})]$.

(f) $\ell_0 \neq 0 \to \exists x\ \mathrm{Prf}[x, \overline{\ell_0 = \overline{\ell_0} \to \exists y[\overline{\mathrm{h}_0(x)} R \mathrm{h}_0(y) R \overline{\ell_0}]}]$.

(g) $\ell_0 \neq 0 \to \mathrm{Pr}(\overline{\ell_0 \neq \overline{\ell_0}})$.

(h) $\ell_0 \neq 0 \to \mathrm{Pr}(\overline{\overline{\ell_0} R \ell_0})$. ∎

7.3. Lemma. (a) $\mathrm{I}\Sigma_1 \vdash \forall x\,[x \neq 0 \to.\ \mathrm{Pr}^x(\overline{\bot}) \leftrightarrow \ell_0 \Vdash \Box^x \bot]$.

(b) *For no* $a \in M_0$ *do we have* $\mathrm{T} \vdash \ell_0 \neq \overline{a}$.

(c) $\mathbb{N} \vDash \ell_0 = 0$. ∎

As usual we fix a positive numeration $\nu\colon \omega - \{0\} \to \mathfrak{D}$ and a Δ_0 enumeration $\{A(m)\}_{m\in\omega}$ of the set of modal formulae that ν brings to $\top$. $\{A(m)\}_{m\in\omega}$ gives rise to a better manageable sequence $\{D(m)\}_{m\in\omega}$. As in §6 our main concern is to guarantee that 0 is reflexive, that is, $0 \Vdash \Box A$ implies $0 \Vdash A$ for each formula

A. This turns out to be possible once we secure that each of the formulas in $\{D(m)\}_{m\in\omega}$ is steady.

$$D(0) = \top\,, \tag{4}$$

$$\mathrm{k}(0) = 0\,, \tag{5}$$

$$D(x+1) = \begin{cases} A(x) & \text{if (i) } A(x) \vdash_{\mathbf{L}} D(x), \\ & \quad\text{(ii) } A(x) \vdash_{\mathbf{L}} A[\mathrm{k}(x)] \text{ and} \\ & \quad\text{(iii) } A(x) \text{ is steady,} \\ D(x) & \text{otherwise,} \end{cases} \tag{6}$$

$$\mathrm{k}(x+1) = \begin{cases} \mathrm{k}(x)+1 & \text{if } D(x+1) \vdash_{\mathbf{L}} A[\mathrm{k}(x)], \\ \mathrm{k}(x) & \text{otherwise.} \end{cases} \tag{7}$$

7.4. Lemma. (a) $\mathrm{I}\Sigma_1 \vdash \forall x\,\forall y\,[x \le y \to D(y) \vdash_{\mathbf{L}} D(x)]$.

(b) $\mathrm{I}\Sigma_1 \vdash \forall i\,\forall x$ "$D(x)$ is steady".

(c) $\mathrm{I}\Sigma_1 \vdash \forall x$ "$\Box^+ D(x)$ is conservative".

(d) *For each* $y \in \omega$ *there exists an* $x \in \omega$ *such that* $D(x) \vdash_{\mathbf{L}} A(y)$.

(e) *For each* $x \in \omega$ *there exists a* $y \in \omega$ *such that* $D(x) = A(y)$.

Proof. (a), (b) and (e) are unproblematic.

(c) follows from (b) by Lemma 5.3.

(d) Suppose y is the minimal such that $D(x) \vdash_{\mathbf{L}} A(y)$ for no $x \in \omega$. Then $\lim_{x\to\infty} \mathrm{k}(x) = y$ and $\lim_{x\to\infty} D(x) = A(z)$ for some $z < y$. By Lemma 5.15 there exists a steady formula B such that $B(\nu 1, \nu 2, \ldots) = \top$ and $B \vdash_{\mathbf{L}} A(z) \wedge A(y)$. Since the height of $\mathfrak{D}$ is infinite $\Box^+ B$ is conservative. Note that the number of formulas $\mathbf{L}$-equivalent to B is infinite and therefore for some $w > y$ one has $\vdash_{\mathbf{L}} A(w) \leftrightarrow B$. But then clearly $D(w+1) = A(w) \vdash_{\mathbf{L}} B \vdash_{\mathbf{L}} A(y)$. ■

The following definitions are cited verbatim from §6.

$$\mathrm{h}(0,x) = \mathrm{h}_0(x)\,, \tag{8}$$

$$\mathrm{h}(i+1,0) = 0\,, \tag{9}$$

$$\mathrm{h}(i+1,x+1) = \begin{cases} a & \text{if } a \text{ satisfies (i)–(vii) below,} \\ \mathrm{h}(i+1,x) & \text{if no } a \text{ satisfying (i)–(vi) exists;} \end{cases} \tag{10}$$

(i) $a \in M_{i+1}$,

(ii) $\mathrm{h}(i,x) \ne \mathrm{h}(i,x+1)$,

(iii) $\mathrm{h}(i+1,x) R a$,

(iv) $a \lhd \mathrm{h}(i,x+1)$,

(v) if $\mathrm{h}(i+1,x) = 0$ then $D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} \neg\Psi_a$,

(vi) for each b satisfying (i)–(v) in place of a one has

$$\forall z \le x\,[\mathrm{Prf}[z, \overline{\ell(\overline{i+1}) = \overline{b} \to \exists y\,[\overline{\mathrm{h}(i+1,x)} R \mathrm{h}(\overline{i+1},y) R \overline{b}]}]$$
$$\to \exists w \le z\ \mathrm{Prf}[w, \overline{\ell(\overline{i+1}) = \overline{a} \to \exists y\,[\overline{\mathrm{h}(i+1,x)} R \mathrm{h}(\overline{i+1},y) R \overline{a}]}]]\,,$$

(vii) a is minimal among those c that satisfy (i)–(vi) in place of a,

$$\ell(0) = \ell_0\,, \tag{11}$$

(12) $$\ell(i+1) = \begin{cases} \lim\limits_{x\to\infty} \mathrm{h}(i+1,x) & \text{if } \mathrm{h}(i+1,\cdot) \text{ reaches a limit,} \\ 0 & \text{otherwise.} \end{cases}$$

7.5. Lemma ($I\Sigma_1$).

(a) $\forall i\, \forall x\, \forall y\, [x \le y \to\!. \; \mathrm{h}(i,x) = \mathrm{h}(i,y) \vee \mathrm{h}(i,x) R \mathrm{h}(i,y)]$.

(b) $\forall i\, \forall x\, \mathrm{h}(i+1,x) \lhd \mathrm{h}(i,x)$.

(c) $\forall j\, \forall i < j\, \forall x\, \mathrm{h}(j,x) \lhd \mathrm{h}(i,x)$.

(d) $\forall i\, \ell(i) = \lim\limits_{x\to\infty} \mathrm{h}(i,x)$.

(e) $\forall i\, \forall x\, [\mathrm{h}(i,x) R \ell(i) \vee \mathrm{h}(i,x) = \ell(i)]$.

(f) $\forall j\, \forall i < j\, \ell(j) \lhd \ell(i)$. ■

As usual $\ell = 0$ means $\ell(\bar{i}) = 0$ for each i.

7.6. Lemma ($I\Sigma_1 + \ell \neq 0$).

(a) $\forall i\, \mathrm{Pr}[\overline{\overline{\ell(i)} R \ell(\bar{i})}]$.

(b) $\forall i\, \forall a \in M_i\, [\ell(i) R a \to \neg\, \mathrm{Pr}[\overline{\ell(\bar{i}) = \bar{a} \to \exists y\, [\overline{\ell(i)} R \mathrm{h}(\bar{i},y) R \bar{a}]}]]$.

(c) $\forall i\, \forall a \in M_i\, [\ell(i) R a \to \neg\, \mathrm{Pr}[\overline{\ell(\bar{i}) \neq \bar{a}}]]$. ■

(13) $$\mathrm{g}(x) = \begin{cases} z & \text{if } z \text{ satisfies (i)–(iii) below,} \\ x & \text{if no } z \text{ satisfying (i)–(ii) exists;} \end{cases}$$

(i) $z < x$,

(ii) there exists an $i \in \omega$ and a node $a \in M_i$ such that

$$\mathrm{Prf}[z, \overline{\ell(\bar{i}) = \bar{a} \to \exists y\, [0 R \mathrm{h}(\bar{i},y) R \bar{a}]}] \quad \text{and} \quad D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg \Psi_a\,,$$

or there exists an $i \in \omega$ and a formula A such that

$$\mathrm{Prf}[z, \overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \bar{A}}] \quad \text{and} \quad D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} A\,,$$

(iii) z is minimal among those satisfying (i) and (ii).

7.7. Lemma ($I\Sigma_1$).

(a) $\forall x\, \forall y\, [x \le y \to D[\mathrm{g}(y)] \vdash_{\mathbf{L}} D[\mathrm{g}(x)]]$.

(b) $\forall i\, \forall x\, \forall a \in M_i\, [\mathrm{Prf}[x, \overline{\ell(\bar{i}) = \bar{a} \to \exists y\, [0 R \mathrm{h}(\bar{i},y) R \bar{a}]}]$
$\wedge D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} \neg \Psi_a \to\!. \; \forall y \ge x\, \mathrm{g}(y) = \mathrm{g}(x)]$.

(c) $\forall i\, \forall x\, \forall A \in \mathbf{F}(i)\, [\mathrm{Prf}[x, \overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \bar{A}}]$
$\wedge D[\mathrm{g}(x)] \nvdash_{\mathbf{L}} A \to\!. \; \forall y \ge x\, \mathrm{g}(y) = \mathrm{g}(x)]$.

(d) $\forall x\, [\mathrm{g}(x) \neq x \to \exists i\, [\exists a \in M_i\, [\mathrm{Pr}[\overline{\ell(\bar{i}) = \bar{a} \to \exists y\, [0 R \mathrm{h}(\bar{i},y) R \bar{a}]}]$
$\wedge \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} \neg \Psi_a] \vee \exists A \in \mathbf{F}(i)\, [\mathrm{Pr}[\overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \bar{A}}] \wedge \forall z\, D[\mathrm{g}(z)] \nvdash_{\mathbf{L}} A]]]$. ■

7.8. Lemma ($I\Sigma_1$).

(a) $\ell \neq 0 \to \forall i\, \forall x \forall A \in \mathbf{F}(i)\, [\mathrm{h}(i,x) = 0 \wedge D[\mathrm{g}(x)] \vdash_{\mathbf{L}} A \to\!. \; \ell(i) \Vdash A]$.

(b) $\ell = 0 \to \forall i\, \forall a{\in}M_i\, [\mathrm{Pr}[\overline{\ell(\bar{i}) = \overline{a} \to \exists y\, [0R\mathrm{h}(\bar{i}, y)R\overline{a}]}] \to \exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} \neg\Psi_a]$.

(c) $\ell = 0 \to \forall i\, \forall a \in M_i\, [\mathrm{Pr}[\overline{\ell(\bar{i}) \neq \overline{a}}] \to \exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} \neg\Psi_a]$.

(d) $\ell = 0 \to \forall A \in \mathbf{F}\, [\mathrm{Pr}[\overline{\ell \neq 0 \to \ell(\bar{i}) \Vdash \overline{A}}] \to \exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} A]$. ∎

The definition of the formula Adm(·) is slightly different from the one in §6. (All notation concerning ⊥-⊤-strings not explained here comes from §6.)

$$\mathrm{Adm}(\alpha) \equiv \forall x\, \exists y > x \text{ “}D[\mathrm{g}(y)] \text{ admits } \alpha\text{”}. \tag{14}$$

An alternative way to define Adm(·) which works equally well is

$$\mathrm{Adm}'(\alpha) \equiv \forall x\, \exists y > x\, \exists A \in \mathbf{F}\, [D[\mathrm{g}(y)] \vdash_{\mathbf{L}} A \vdash_{\mathbf{L}} D[\mathrm{g}(x)] \wedge \text{“}A \text{ is steady”} \wedge \text{“}A \text{ admits } \alpha\text{”}]. \tag{14$'$}$$

The advantage of (14) is that it is somewhat easier to deal with. The advantage of (14′) is that in this case we have for each string α

$$\mathbb{N} \vDash \mathrm{Adm}'(\alpha) \text{ if and only if } \bigwedge\!\!\!\!\bigwedge_{1 \le i \le \mathrm{lh}(\alpha)} [\nu i \leftrightarrow (\alpha)_i] \text{ is an admissible element of } \mathfrak{D},$$

whereas this is not true of Adm(·). A line of attack equivalent to (14′) was followed in Shavrukov [42].

7.9. Lemma. (a) $\mathrm{I}\Sigma_1 \vdash \mathrm{Adm}(\overline{\Lambda})$.

(b) $\mathrm{I}\Sigma_1 \vdash \forall\alpha\, \forall\beta\, [\mathrm{Adm}(\beta) \wedge \alpha \subseteq \beta \to .\ \mathrm{Adm}(\alpha)]$.

(c) *For each* $i \in \omega$

$$\mathrm{I}\Sigma_1 \vdash \forall\alpha\, [\mathrm{lh}(\alpha) \le \bar{i} \wedge \mathrm{Adm}(\alpha) \to .\ \exists\beta\, [\mathrm{lh}(\beta) = \bar{i} \wedge \alpha \subseteq \beta \wedge \mathrm{Adm}(\beta)]].$$

(d) *For each* $i \in \omega$

$$\mathrm{I}\Sigma_1 \vdash \exists\alpha\, [\mathrm{lh}(\alpha) = \bar{i} \wedge \mathrm{Adm}(\alpha)].$$

Proof. (a) By Lemma 7.4(b) the formula $D[\mathrm{g}(x)]$ is steady for each x and the claim follows by Lemma 5.20(a).

(b) follows from Lemma 5.20(b).

(c) The proof is analogous to that of Lemma 6.9(c). In the present situation, however, we have to do with external induction because the formula we induct on is too arithmetically complex. To carry out the induction step we have to show that if for some i a string α of length i is admitted by $D[\mathrm{g}(x)]$ for cofinally many $x \in \omega$ then there exists a string β of length $i+1$ extending α which is also admitted by $D[\mathrm{g}(x)]$ for infinitely many x. Now if this were true of neither of the two candidates $\beta_1 = \alpha * \langle\bot\rangle$ and $\beta_2 = \alpha * \langle\top\rangle$ ($*$ denotes concatenation) for the role of β then there would exist an $x \in \omega$ such that $D[\mathrm{g}(y)]$ admits neither β_1 nor β_2 for any $y \ge x$. Hence by Lemma 5.20(c), α could not be admitted by $D[\mathrm{g}(x)]$ for unboundedly many x. This proves the claim.

(d) follows from (a) and (c). ∎

In Lemma 7.9(c) and (d) we cannot retain the majestic uniformity of Lemma 6.9. In fact, by using results of Adamowicz [1] combined with methods of §9 it can be shown that there exists a positive numeration μ of a diagonalizable algebra with the strong disjunction property such that the statement

$$\forall i\,\exists\alpha\,[\mathrm{lh}(\alpha)=i\wedge\mathrm{Adm}(\alpha)]$$

for the formula $\mathrm{Adm}(\cdot)$ built up starting from μ is not provable in $\mathrm{I}\Sigma_1$ (nor even in the theory of all the Π_3 truths). Hence the loss of uniformity in many of the succeeding lemmas, most notably in Lemma 7.12(c). The same misfortune befalls $\mathrm{Adm}'(\cdot)$. I do not know whether a cleverer definition of $\mathrm{Adm}(\cdot)$ (or of $\mathrm{T}(\cdot)$) might help.

$$\mathrm{Adm}^+(\alpha)\equiv\mathrm{Adm}(\alpha)\wedge\forall\beta\prec\alpha\,[\mathrm{lh}(\alpha)=\mathrm{lh}(\beta)\to\neg\,\mathrm{Adm}(\beta)]\,. \tag{15}$$

7.10. Lemma. *For each* $i\in\omega$, $\mathrm{I}\Sigma_1$ *proves*

(a) $\exists!\alpha\,[\mathrm{lh}(\alpha)=\bar{i}\wedge\mathrm{Adm}^+(\alpha)]$.

(b) $\forall\alpha\,[\mathrm{lh}(\alpha)\le\bar{i}\wedge\mathrm{Adm}^+(\alpha)\to.\ \exists\beta\,[\mathrm{lh}(\beta)=\bar{i}\wedge\alpha\subseteq\beta\wedge\mathrm{Adm}^+(\beta)]]$.

(c) $\forall\alpha\,\forall\beta\,[\mathrm{Adm}^+(\alpha)\wedge\mathrm{Adm}^+(\beta)\wedge\mathrm{lh}(\alpha)\le\mathrm{lh}(\beta)\le\bar{i}\to.\ \alpha\subseteq\beta]$. ∎

$$\begin{aligned}\mathrm{T}(A)\equiv\exists i\,\exists k\,\exists\lambda\in\mathbf{F}^0(i+k)\,\exists B_1,\dots,B_k\in\mathbf{F}(i)\exists\alpha&\\ [\mathrm{lh}(\alpha)=i+k\wedge\lambda[(\alpha)_1,\dots,(\alpha)_{i+k}]=\top&\\ \wedge A=\lambda(p_1,\dots,p_i,\Box B_1,\dots,\Box B_k)&\\ \wedge\forall j\le i\,[j\ne 0\to.\ (\alpha)_j=\top\leftrightarrow\exists\beta\,[\mathrm{lh}(\beta)=i\wedge\mathrm{Adm}^+(\beta)\wedge(\beta)_j=\top]]&\\ \wedge\forall j\le k\,[j\ne0\to.\ (\alpha)_{i+j}=\top\leftrightarrow\exists x\,D[\mathrm{g}(x)]\vdash_{\mathbf{L}}B_j]]&\,.\end{aligned} \tag{16}$$

7.11. Lemma ($\mathrm{I}\Sigma_1$).

(a) $\forall i\,[\mathrm{T}(p_i)\leftrightarrow\exists\beta\,[\mathrm{lh}(\beta)=i\wedge\mathrm{Adm}^+(\beta)\wedge(\beta)_i=\top]]$.

(b) $\forall A\in\mathbf{F}\,[\mathrm{T}(\Box A)\leftrightarrow\exists x\,D[\mathrm{g}(x)]\vdash_{\mathbf{L}}A]$.

(c) $\forall i\,\forall\lambda\in\mathbf{F}^0(i)\,\forall B_1,\dots,B_i\in\mathbf{F}\,[\mathrm{T}[\lambda(B_1,\dots,B_i)]$
$$\leftrightarrow\exists\alpha\,[\mathrm{lh}(\alpha)=i\wedge\lambda[(\alpha)_1,\dots,(\alpha)_i]=\top\wedge\forall j\le i\,[j\ne0\to.\ (\alpha)_j=\top\leftrightarrow\mathrm{T}(B_j)]]]\,.$$ ∎

7.12. Lemma. *For each* $i\in\omega$, $\mathrm{I}\Sigma_1$ *proves*

(a) $\forall\alpha\Big[\mathrm{lh}(\alpha)\le\bar{i}\to.\ \mathrm{T}\Big[\bigwedge_{1\le j\le\mathrm{lh}(\alpha)}[p_j\leftrightarrow(\alpha)_j]\Big]\leftrightarrow\mathrm{Adm}^+(\alpha)\Big]$.

(b) $\forall A\in\mathbf{F}\,\forall B\in\mathbf{F}\,[\mathrm{T}(\Diamond A\wedge\Box B)\to\mathrm{T}[\Diamond(A\wedge\Box B)]]$.

(c) $\forall A\in\mathbf{F}(\bar{i})\,[\mathrm{T}(\Box A)\to\mathrm{T}(A)]$.

Proof is essentially the same as that of Lemma 6.12. The only new detail is that in (c) we have to use the fact that

$$\mathrm{I}\Sigma_1\vdash\forall A\in\mathbf{F}\,[\exists x\,D[\mathrm{g}(x)]\vdash_{\mathbf{L}}A\leftrightarrow.\ \exists y\,\forall x>y\,D[\mathrm{g}(x)]\vdash_{\mathbf{L}}A]\,.$$ ∎

We define the mapping $^\circ\colon \{p_i\}_{i\in\omega-\{0\}} \to \mathfrak{D}_{\mathrm{T}}$:

(17) $$p_i^\circ \equiv \ell \neq 0 \wedge \ell(\bar{i}) \Vdash \overline{p_i} \vee\!.\ \ell = 0 \wedge \mathrm{T}(\overline{p_i}),$$

and prolong it to all modal formulae.

7.13. Lemma. (a) *For each $i \in \omega$ and for each modal formula $A(p_1,\ldots,p_i)$*

$$\mathrm{I}\Sigma_1 \vdash \ell \neq 0 \to\!.\ [A(p_1,\ldots,p_i)]^\circ \leftrightarrow \ell(\bar{i}) \Vdash \overline{A(p_1,\ldots,p_i)}\,.$$

(b) $\mathrm{I}\Sigma_1 \vdash \forall i\,\forall A \in \mathbf{F}(i)\ \mathrm{Pr}[\overline{\ell \neq 0 \to\!.\ A^\circ \leftrightarrow \ell(\bar{i}) \Vdash \overline{A}}]$.

(c) $\mathrm{I}\Sigma_1 \vdash \ell = 0 \to \forall A \in \mathbf{F}\,[\mathrm{Pr}(\overline{A^\circ}) \to \exists x\, D[\mathrm{g}(x)] \vdash_{\mathbf{L}} A]$.

(d) *For each $i \in \omega$ and for each modal formula $A(p_1,\ldots,p_i)$*

$$\mathrm{I}\Sigma_1 \vdash [A(p_1,\ldots,p_i)]^\circ \leftrightarrow [\ell \neq 0 \wedge \ell(\bar{i}) \Vdash \overline{A(p_1,\ldots,p_i)} \vee\!.\ \ell = 0 \wedge \mathrm{T}[\overline{A(p_1,\ldots,p_i)}]]\,. \ \blacksquare$$

7.14. Lemma. $\mathbb{N} \vDash$ "g is the identity function". ■

7.15. Lemma. *If $A(\nu 1,\nu 2,\ldots) = \top$ for $A(x_1,x_2,\ldots)$ a diagonalizable polynomial then*

$$\mathrm{I}\Sigma_1 \vdash [A(p_1,p_2,\ldots)]^\circ\,. \ \blacksquare$$

Now having defined

(18) $$(\nu i)^* \equiv p_i^\circ$$

we can finish the proof of Theorem 7.1 just as we did in 6.16 with Theorem 6.1. ■

8. An application

In this section we shall apply Theorem 7.1 to give an alternative proof of a proposition which was used in Simmons [43] to obtain some interesting information on the structure of the E-tree.

8.1. Proposition (Simmons [43]). *Let* T *be a Σ_1-sound theory and τ a false Σ_1 sentence. Then there exists a family $\{\sigma_\alpha\}_{\alpha\in\mathbb{Q}}$ of Σ_1 sentences such that*

$$\mathrm{T} \vdash \tau \to \sigma_\alpha$$

and

$$\mathrm{T} \vdash \mathrm{Pr}_{\mathrm{T}}(\overline{\sigma_\alpha}) \to \sigma_\beta$$

whenever $\alpha,\beta \in \mathbb{Q}$ and $\alpha < \beta$ (here $\mathbb{Q}$ is the set of rationals under the natural ordering).

Proof. Consider the following set of modal formulae in propositional letters $\{p_\alpha\}_{\alpha\in\mathbb{Q}}$:

$$\mathcal{S} = \{\Box\Box p_\alpha \to \Box p_\beta \mid \alpha,\beta \in \mathbb{Q} \text{ and } \alpha < \beta\}\,.$$

We shall show that the quotient algebra $\mathbf{F}/\mathcal{S}$ of the free diagonalizable algebra $\mathbf{F}$ on the generators $\{p_\alpha\}_{\alpha\in\mathbb{Q}}$ modulo the τ-filter generated by $\mathcal{S}$ enjoys the strong disjunction property. We fix an effective repetition-free enumeration $\{\alpha_i\}_{i\in\omega}$ of

$\mathbb{Q}$. Then if some diagonalizable polynomial $A(p_{\alpha_0}, p_{\alpha_1}, \ldots)$ hits $\top$ in $\mathbf{F}/\mathcal{S}$ there exists a finite subset $\mathcal{F}$ of $\mathcal{S}$ such that

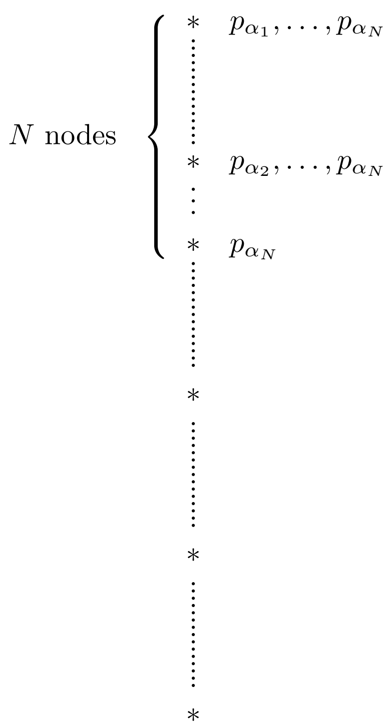
$$\bigwedge\!\!\!\bigwedge \mathcal{F} \vdash_{\mathbf{L}} A(p_{\alpha_0}, p_{\alpha_1}, \ldots)\,.$$

Without loss of generality we may assume that for some $N \in \omega$ the formula $\bigwedge\!\!\!\bigwedge \mathcal{F}$ looks like this:

$$S_N = \bigwedge\!\!\!\bigwedge \{\Box\Box p_{\alpha_i} \to \Box p_{\alpha_j} \mid i, j \leq N \text{ and } \alpha_i < \alpha_j\}\,.$$

Claim. *For each $N \in \omega$ the formula S_N is steady.*

Assume for simplicity that for $i, j \leq N$ we have $\alpha_i < \alpha_j$ if and only if $i < j$. Consider the following Kripke model:



(Only the letters forced are shown at each node.)

Imagine two models in which S_N holds grafted above the root of the model depicted. We want to get convinced that the resulting model models $\Box^+ S_N$. Clearly, S_N is forced at every node which is not the root of the new model (for the "old" nodes this is verified by inspection of the picture). As for the root itself, note that it forces $\Box\Box p_{\alpha_i}$ for no $i \leq N$ and hence forces S_N. By Lemma 5.13 conclude that S_N is steady and that the Claim is proved.

From the Claim it follows by Lemma 5.15 that $\mathbf{F}/\mathcal{S}$ possesses the strong disjunction property. Note also that $\mathbf{F}/\mathcal{S}$ is clearly positive and that the theory $\mathrm{T}{+}\neg\tau$ is Σ_1-sound. Hence by Theorem 7.1 there exists an embedding ${}^*\colon \mathbf{F}/\mathcal{S} \to \mathfrak{D}_{\mathrm{T}+\neg\tau}$.

Now let

$$\sigma_\alpha \equiv (\Box p_\alpha)^* .$$

We trivially have $\mathrm{T} \vdash \tau \to \sigma_\alpha$ for each $\alpha \in \mathbb{Q}$ because

$$\begin{aligned} \mathrm{T} \vdash \tau &\to \mathrm{Pr}_\mathrm{T}(\overline{\tau}) \\ &\to \mathrm{Pr}_{\mathrm{T}+\neg\tau}(\overline{\bot}) \\ &\to \mathrm{Pr}_{\mathrm{T}+\neg\tau}(\overline{p_\alpha^*}) \\ &\to (\Box p_\alpha)^* \\ &\to \sigma_\alpha . \end{aligned}$$

Let $\alpha, \beta \in \mathbb{Q}$ and $\alpha < \beta$. Then since $\Box\Box p_\alpha \to \Box p_\beta$ is in $\mathcal{S}$ and * is an embedding we have

$$\begin{aligned} \mathrm{T} + \neg\tau \vdash \mathrm{Pr}_\mathrm{T}(\overline{\sigma_\alpha}) &\to \mathrm{Pr}_{\mathrm{T}+\neg\tau}(\overline{\sigma_\alpha}) \\ &\to (\Box\Box p_\alpha)^* \\ &\to (\Box p_\beta)^* \\ &\to \sigma_\beta . \end{aligned}$$

Combining this with $\mathrm{T} + \tau \vdash \sigma_\beta$ we get

$$\mathrm{T} \vdash \mathrm{Pr}_\mathrm{T}(\overline{\sigma_\alpha}) \to \sigma_\beta . \ \blacksquare$$

Mark that a proof of Proposition 8.1 similar to ours is obtainable by the methods of Jumelet [27].

9. A question of arithmetical complexity

In §§4, 6 and 7 we constructed embeddings $^* : \mathfrak{D} \to \mathfrak{D}_\mathrm{T}$ of positive diagonalizable algebras into diagonalizable algebras of various theories T. This section attempts a close scrutiny of the arithmetical complexity of sentences in rng *. We register the arithmetical complexity of sentences in rng * of the *'s constructed and give a lower bound on this complexity for the case of Σ_2-sound theories under reasonable assumptions on * to the effect that our constructions were fairly optimal in this respect.

9.A. Finite credibility extent. Recall that in §4 we had

$$(\nu i)^* \equiv \ell(\bar{i}) \Vdash \overline{p_i}$$

where $\ell(i)$ denoted the limit of a primitive recursive function $\mathrm{h}(i, \cdot)$ climbing up the Kripke model $\mathcal{M}_i$. Moreover, the part of $\mathcal{M}_i$ a priori accessible to $\mathrm{h}(i, \cdot)$ was finite since it was specified by the condition $a \Vdash \Box^n \bot$, n being the credibility extent of T. Therefore

$$\mathrm{I}\Sigma_1 \vdash \ell(\bar{i}) \Vdash \overline{p_i} \leftrightarrow \bigvee\!\!\!\bigvee \{\ell(\bar{i}) = \overline{a} \mid a \in M_i \text{ and } a \Vdash \Box^n \bot \wedge p_i\} .$$

Now $\ell(\bar{i}) = \overline{a}$ is equivalent to the statement

$$\forall x\, [\mathrm{h}(i,x) R \overline{a} \vee \mathrm{h}(i,x) = \overline{a}] \wedge \exists x\, \mathrm{h}(i,x) = \overline{a}$$

which is a Boolean combination of Σ_1 sentences and therefore each sentence in rng^* is a Boolean combination of Σ_1 sentences over $\mathrm{I}\Sigma_1$. (These considerations come from Solovay [50].)

9.B. Σ_1-ill theories of infinite credibility extent. From §6 we have

$$(\nu i)^* \;\equiv\; \ell \neq 0 \wedge \ell(\bar{i}) \Vdash \overline{p_i} \vee\!.\; \ell = 0 \wedge \mathrm{T}(\overline{p_i})\,.$$

Note that $\ell \neq 0$ is Σ_1, $\ell = 0$ is Π_1 and $\ell(\bar{i}) \Vdash \overline{p_i}$ is Δ_2 over $\mathrm{I}\Sigma_1$. The sentence $\mathrm{T}(\overline{p_i})$ is built from formulas of the form "$D(\omega+\mathsf{s})$ admits α" and some Δ_0 formulas with the help of Boolean connectives and primitive recursively bounded quantification. Now formulas of this sort involving $D(\omega + \mathsf{s})$ can be either written in Σ_1 or in Π_1 form for

$$\begin{aligned} \mathrm{T} \vdash \ldots D(\omega+\mathsf{s})\ldots &\leftrightarrow \exists x\, [x = \mathsf{s} \wedge \ldots D(\omega + x)\ldots] \\ &\leftrightarrow \forall x\, [x = \mathsf{s} \rightarrow \ldots D(\omega + x)\ldots] \end{aligned}$$

and therefore $\mathrm{T}(\overline{p_i})$ is a $\Delta_0(\Sigma_1)$ (and hence Δ_2) sentence over $\mathrm{I}\Sigma_1$.

All this amounts to $(\nu i)^*$ being Δ_2 over $\mathrm{I}\Sigma_1$ and it was shown by Gaifman and Smoryński (cf. Remark 3.5.iii in Chapter 3 of Smoryński [49]) that one cannot generally do with Boolean combinations of Σ_1 sentences.

9.C. Σ_1-sound theories. In §7, $(\nu i)^*$ was defined just as in §6:

$$(\nu i)^* \equiv \ell \neq 0 \wedge \ell(\bar{i}) \Vdash \overline{p_i} \vee\!.\; \ell = 0 \wedge \mathrm{T}(\overline{p_i})\,.$$

The $\mathrm{T}(\cdot)$ of §7 was, however, considerably different from that of §6. The most complex parts of it are subformulas of the form

$$\forall x\, \exists y \geq x \text{ "}D[\mathrm{g}(y)] \text{ admits } \alpha\text{"}$$

where the function g is Δ_0. These formulas are clearly Π_2 over $\mathrm{I}\Sigma_1$ and so $(\nu i)^*$ is a $\Delta_0(\Sigma_2)$ sentence and hence a Boolean combination of Σ_2 sentences.

An important particular case when one can do better than $\Delta_0(\Sigma_2)$ is the case of finitely generated algebras. These algebras only need sentences Δ_2 over $\mathrm{I}\Sigma_1$. That this is so can be seen as follows.

Let $\{x_1, \ldots, x_n\}$ be the generators of such an algebra. When enumerating the relations holding in this algebra we can restrict our attention to formulas in $\mathbf{F}(n)$. Let $\alpha \in \mathbf{A}^0(n)$ be such that $\alpha(x_1, \ldots, x_n)$ is admissible. When designing the sequence $\{D(m)\}_{m\in\omega}$ we can then impose the additional requirement that α be admitted by $D(m)$ for all m (that this requirement is harmless can be seen through Lemma 5.24) and define $\mathrm{Adm}^+(\beta)$ to be the Δ_0 formula $\beta \subseteq \overline{\alpha}$. The formula $\mathrm{T}(\cdot)$ and * are then built from $\mathrm{Adm}^+(\cdot)$ and $\{D(m)\}_{m\in\omega}$ in the same manner as in §7 so that $(\nu i)^*$ is clearly Δ_2 over $\mathrm{I}\Sigma_1$.

Another particular case admitting an improvement is that of a Σ_2-ill theory T. It suffices then to use sentences that are Δ_2 over T. The Σ_2-ill theory T proves that some actually total recursive function f is not total. The following changes are to be introduced in §7:

We inhibit slightly the process of gradually strengthening the formulas $\{D(m)\}_{m\in\omega}$ by the condition that the nth change of the value of D be only allowed to take place after $f(n)$ has converged. Then the formula

$$\mathrm{Adm}(\alpha) \equiv \forall x\, \exists y \geq x \text{ ``}D[\mathrm{g}(y)] \text{ admits } \alpha\text{''}$$

is Δ_2 over T for T proves that the sequence $\{D[\mathrm{g}(x)]\}_{x\in\omega}$ freezes after some $\Delta_0(\Sigma_1)$-definable moment.

In current literature we find particular examples of embeddings of positive diagonalizable algebras into $\mathfrak{D}_{\mathrm{T}}$ for T a Σ_1-sound theory whose range consists of Δ_2 sentences. These sentences suffice to embed into $\mathfrak{D}_{\mathrm{T}}$ the free diagonalizable algebra on countably many generators (cf. Artemov [2], Montagna [32], Boolos [12], Visser [51]). Sentences employed by Jumelet [27] were also Δ_2.

The rest of this section will be devoted to an explanation why the use of highly complex sentences is generally hardly avoidable when embedding positive algebras in diagonalizable algebras of Σ_2-sound theories.

Recall that the embeddings $^*: \mathfrak{D} \to \mathfrak{D}_{\mathrm{T}}$ constructed in preceding sections employed an arbitrary positive numeration ν of $\mathfrak{D}$ and the * we obtained was in each case *recursive* with respect to ν, that is, there always existed a recursive function $^\circ$ such that the following diagram commuted:

$$\begin{array}{ccc} \omega & \xrightarrow{\ \circ\ } & \omega \\ {\scriptstyle\nu}\big\downarrow & & \big\downarrow{\scriptstyle \text{gödel}} \\ \mathfrak{D} & \xrightarrow[\ *\ ]{} & \mathfrak{D}_{\mathrm{T}} \end{array}$$

We are going to show that if T is a Σ_2-sound theory then there exists a positive numeration of a diagonalizable algebra with the strong disjunction property such that the above diagram commutes for no pair $({}^*, {}^\circ)$ with $\operatorname{rng}{}^* \subseteq \Delta_2$ and recursive $^\circ$.

To this end we need some definitions and lemmas.

9.1. Definition. Consider the set $2^{<\omega}$ (which we will think of as consisting of $\bot$-$\top$-strings) ordered by the usual "initial segment" relation $\subseteq$. A *tree* is a downwards closed subset of $2^{<\omega}$. ($2^{<\omega}$ grows upwards.) A tree T is *efflorescent* if it is not empty and for each $\alpha \in T$ there exists a $\beta \in T$ such that α is a proper initial segment of β.

9.2. Definition. Let ν be a numeration of a diagonalizable algebra $\mathfrak{D}$ with the strong disjunction property. The *admissibility tree* of ν is then the set

$$\Big\{\alpha \in 2^{<\omega} \,\Big|\, \bigwedge_{1\leq i\leq \mathrm{lh}(\alpha)} [\nu i \leftrightarrow (\alpha)_i] \text{ is an admissible element of } \mathfrak{D}\Big\}.$$

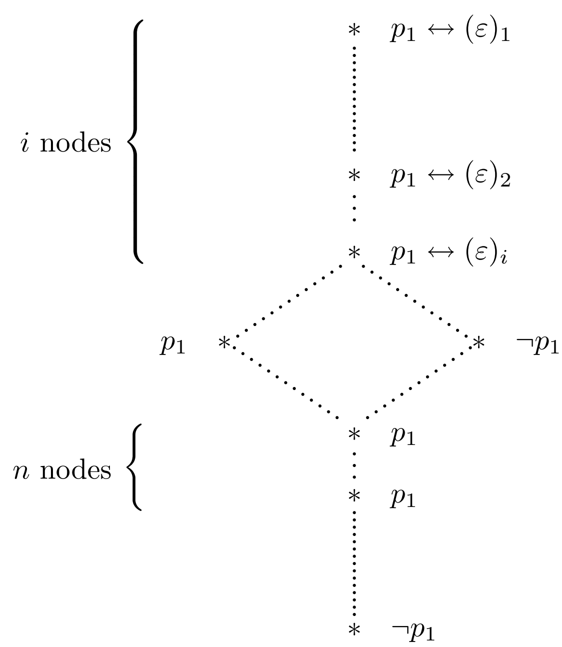
9.3. Lemma. *The admissibility tree of a* (*positive*) *numeration* ν *of a diagonalizable algebra* $\mathfrak{D}$ *with the strong disjunction property is a* (Π^0_2) *efflorescent tree.*

Proof. That the admissibility tree is a tree follows from Lemma 5.17(b). Efflorescence is an immediate consequence of Lemma 5.17(a) and (c). If ν is positive then the equalities of diagonalizable polynomials with elements of rng* as variables that hold in $\mathfrak{D}$ are recursively enumerable and hence admissibility is Π^0_2 by inspection of Definition 5.16. ■

9.4. Lemma. *Let* T *be an arbitrary* Π^0_2 *efflorescent tree. There exists a positive numeration* μ *of a diagonalizable algebra with the strong disjunction property whose admissibility tree is* T.

Proof. Let $\mathbf{F}$ be the free diagonalizable algebra on the generators $\{p_i\}_{i\in\omega-\{0\}}$. We shall compile an r.e. set $\mathcal{C}$ of formulas in $\{p_i\}_{i\in\omega-\{0\}}$ and let μ be the numeration of $\mathbf{F}/\mathcal{C}$ induced by the natural numeration of $\mathbf{F}$.

Define a family of Kripke models $\{\mathcal{I}^\varepsilon_n\}^{\varepsilon\in 2^{<\omega}}_{n\in\omega}$ by letting $\mathcal{I}^\varepsilon_n$ be the p_1-model shown in the picture (we assume $i = \mathrm{lh}(\varepsilon)$):



For typographical reasons the name of the model $\mathcal{I}^\varepsilon_n$ will also stand for the formula $\Psi_{\mathcal{I}^\varepsilon_n}(p_1)$.

Claim 1. $\vdash_{\mathbf{L}} \mathcal{I}^\varepsilon_n \to \neg\mathcal{I}^\delta_m$ *unless* $\varepsilon = \delta$ *and* $n = m$.

This is an easy consequence of Lemma 3.3(c) and (d) since the models $\mathcal{I}^\varepsilon_n$ are clearly differentiated and $\mathcal{I}^\varepsilon_n$ is not isomorphic to $\mathcal{I}^\delta_m$ unless $\varepsilon = \delta$ and $n = m$.

CLAIM 2. $\vdash_{\mathbf{L}} \mathcal{I}_n^\varepsilon \to \Box\neg\mathcal{I}_m^\delta$ *for all* $\varepsilon, \delta \in 2^{<\omega}$ *and all* $n, m \in \omega$.

This follows from Lemma 3.5(b) because $\mathcal{I}_m^\delta$ is not isomorphic to any proper cone of $\mathcal{I}_n^\varepsilon$.

Now let

$$T = \{\alpha \in 2^{<\omega} \mid \forall x\, \exists y\, R(\alpha, x, y)\}$$

with $R(\alpha, x, y)$ decidable. Consider a particular element ε of $2^{<\omega}$. Let ε be of length i. We describe the construction of a set $\mathcal{C}^\varepsilon$ by recursive stages. One begins with Stage 0 at the beginning of which $\mathcal{C}^\varepsilon$ is empty. In what follows $*$ denotes concatenation of strings.

S t a g e n. Look for j_1 and j_2 in ω such that $R[\varepsilon*\langle\bot\rangle, n, j_1]$ and $R[\varepsilon*\langle\top\rangle, n, j_2]$. On finding j_1 add the formula

$$\Big[\bigwedge_{1\le j\le i} [p_j \leftrightarrow (\varepsilon)_j]\Big] \wedge p_{i+1} \to. \Box\neg\mathcal{I}_n^\varepsilon$$

to $\mathcal{C}^\varepsilon$. Once j_2 is found one adds to $\mathcal{C}^\varepsilon$ the formula

$$\Big[\bigwedge_{1\le j\le i} [p_j \leftrightarrow (\varepsilon)_j]\Big] \wedge \neg p_{i+1} \to. \Box\neg\mathcal{I}_n^\varepsilon\,.$$

If one has found both j_1 and j_2 then $\neg\mathcal{I}_n^\varepsilon$ is appended to $\mathcal{C}^\varepsilon$ and we go to Stage $n+1$.

Note that it very well may happen that the number of stages is finite. In this case the last stage takes an infinite amount of "time".

Let $\mathcal{C}_n^\varepsilon$ be the part of $\mathcal{C}^\varepsilon$ compiled during the first n stages. Put

$$\mathcal{C}_N = \bigcup\{\mathcal{C}_n^\varepsilon \mid \varepsilon \in 2^{\le N},\ n \le N\}\,, \quad C_N = \bigwedge \mathcal{C}_N\,,$$
$$\mathcal{C} = \bigcup\{\mathcal{C}^\varepsilon \mid \varepsilon \in 2^{<\omega}\} = \bigcup\{\mathcal{C}_N \mid N \in \omega\}\,.$$

Clearly $A(p_1, p_2, \ldots) = \top$ in $\mathbf{F}/\mathcal{C}$ if and only if $C_N \vdash_{\mathbf{L}} A$ for some $N \in \omega$.

Let N and M be arbitrary natural numbers and let δ be a non-empty string of length $N+1$. Consider the following $(p_1, \ldots, p_{N+1})$-model $\mathcal{Q}$:

$$2M+2 \text{ nodes} \left\{ \begin{array}{ll} * & p_1, \ldots, p_{N+1} \\ \vdots & \\ * & p_1, \ldots, p_{N+1} \end{array} \right.$$
$$\begin{array}{ll} \vdots & \\ * & p_1 \leftrightarrow (\delta)_1, \ldots, p_{N+1} \leftrightarrow (\delta)_{N+1} \end{array}$$

Suppose $\delta \in T$ and picture two models in which C_M holds grafted above the root of $\mathcal{Q}$. Call the resulting model $\mathcal{N}$. If new propositional letters come into play then extend the forcing relation at the "old" nodes of $\mathcal{N}$ so that the root of $\mathcal{N}$ force

$$p_1 \leftrightarrow (\varepsilon)_1,\ \ldots,\ p_{M+1} \leftrightarrow (\varepsilon)_{M+1}$$

with $\delta \subseteq \varepsilon \in T$ (this is possible because T is efflorescent). We are going to show that C_M also holds in $\mathcal{N}$.

Clearly C_M cannot fail at any of the "new" nodes of $\mathcal{N}$. As for the "old" non-bottom nodes, note that these force $\Box^+\neg\mathcal{I}_n^\varepsilon$ for all $\varepsilon \in 2^{<\omega}$, $n \in \omega$ because the root of $\mathcal{I}_n^\varepsilon$ forces $\neg p_1$. Therefore we only have to consider the root of $\mathcal{N}$.

Let α^0 denote the string α with the last "digit" deleted and let α^- stand for α with the last digit replaced by its negation. Neither ^{0} nor $^-$ is meant to be applied to the empty string Λ.

Let us take a closer look at C_M. The conjuncts of this formula are formulas of either of the two forms

$$\neg\mathcal{I}_m^\varepsilon \quad \text{and} \quad \bigwedge_{1\le j\le \mathrm{lh}(\alpha)} [p_j \leftrightarrow (\alpha)_j] \rightarrow. \Box\neg\mathcal{I}_m^{\alpha^0}$$

with $\varepsilon \in 2^{\le M}$, $\alpha \in 2^{\le M+1}$ and $m \le M$.

All the formulas of the first form are forced at the root of $\mathcal{N}$ because $\mathcal{Q}$ (and hence $\mathcal{N}$) is very high. Suppose that the antecedent of the formula

$$\bigwedge_{1\le j\le \mathrm{lh}(\alpha)} [p_j \leftrightarrow (\alpha)_j] \rightarrow. \Box\neg\mathcal{I}_m^{\alpha^0}$$

is forced at the root of $\mathcal{N}$ (we then have $\alpha \in T$) and that this formula is a conjunct of C_M. This indicates that there exists a $j \in \omega$ such that $R(\alpha^-, m, j)$. Since $\alpha \in T$ there also exists an $i \in \omega$ such that $R(\alpha, m, i)$ and so the formula $\neg\mathcal{I}_m^{\alpha^0}$ should likewise be a conjunct of C_M. As was shown above $\neg\mathcal{I}_m^{\alpha^0}$ holds in $\mathcal{N}$ and therefore

$$\bigwedge_{1\le j\le \mathrm{lh}(\alpha)} [p_j \leftrightarrow (\alpha)_j] \rightarrow. \Box\neg\mathcal{I}_m^{\alpha^0}$$

is forced at the root of $\mathcal{N}$. Thus $\mathcal{N} \Vdash C_M$.

In view of Lemmas 5.13 and 5.21 these arguments prove that each of the formulas C_M is steady and if $\alpha \in T$ then C_M admits α. By Lemmas 5.15 and 5.24 this implies the strong disjunction property of $\mathbf{F}/\mathcal{C}$ and that each element of T is in the admissibility tree of μ.

Next pick a string ε outside T. Let us see that the formula $\bigwedge\{p_j \leftrightarrow (\varepsilon)_j \mid 1 \le j \le \mathrm{lh}(\varepsilon)\}$ is not an admissible element of $\mathbf{F}/\mathcal{C}$. Since T is efflorescent there exists a string $\delta \subseteq \varepsilon$ such that $\delta^- \in T$ and $\delta \notin T$ and by Lemma 5.17(b) we shall be done once we show $\bigwedge\{p_j \leftrightarrow (\delta)_j \mid 1 \le j \le \mathrm{lh}(\delta)\}$ to be inadmissible.

By the choice of $R(\cdot,\cdot,\cdot)$ there exists an $N \in \omega$ such that $\exists j\, R(\delta^-, N, j)$ and $\forall i\, \neg R(\delta, N, i)$. By the construction of $\mathcal{C}^{\delta^0}$ we then know that the formula $\neg\mathcal{I}_N^{\delta^0}$ will never get in $\mathcal{C}$ while in $\mathbf{F}/\mathcal{C}$ one has

$$\bigwedge_{1\le j\le \mathrm{lh}(\delta)} [p_j \leftrightarrow (\delta)_j] \rightarrow. \Box\neg\mathcal{I}_N^{\delta^0} = \top\,.$$

Fix an arbitrary $M \in \omega$. Take the model $\mathcal{I}_N^{\delta^0}$ and extend the forcing relation at the nodes of this model to the tuple $(p_1, \ldots, p_{M+1})$ in an arbitrary way. C_M

holds in this model because by Claim 1 it forces $\neg\mathcal{I}_n^\alpha$ unless $\alpha = \delta^0$ and $n = N$, the formula $\neg\mathcal{I}_N^{\delta^0}$ is not a conjunct of C_M and by Claim 2 the model forces $\Box\neg\mathcal{I}_n^\alpha$ for all $\alpha \in 2^{<\omega}$, $n \in \omega$.

Thus we have shown that $C_M \vdash_{\mathbf{L}} \neg\mathcal{I}_N^{\delta^0}$ for no $M \in \omega$ and therefore

$$\neg\mathcal{I}_N^{\delta^0} \neq \top$$

in $\mathbf{F}/\mathcal{C}$. Therefore $\bigwedge\!\!\!\bigwedge_{1\le j\le \mathrm{lh}(\delta)}[p_j \leftrightarrow (\delta)_j]$ is not admissible and hence δ is not in the admissibility tree of μ. ∎

9.5. Lemma. *Let* T *be a* Σ_2*-sound theory. Then the admissibility tree of the gödelnumbering of* $\mathfrak{D}_{\mathrm{T}}^{\Delta_2}$ *enjoys an infinite* Δ_2^0 *branch.*

Proof. Each sentence which is Δ_2 over T may be thought of as a pair (σ, π) of sentences σ in Σ_2 and π in Π_2 such that $\mathrm{T} \vdash \sigma \leftrightarrow \pi$. By our assumption on T, σ is true if and only if π is true. Also note that if a Δ_2 sentence is true in the above sense then it corresponds to an admissible element of $\mathfrak{D}_{\mathrm{T}}^{\Delta_2}$ because if $\mathrm{T} \vdash \sigma \to \mathrm{Pr}(\overline{\varphi})$ then $\mathrm{Pr}(\overline{\varphi})$ is true and hence $\mathrm{T} \vdash \varphi$. Now the truth of Σ_2 sentences as well as falsity of Π_2 sentences is verifiable by a Turing machine with an oracle for $\mathbf{0}'$ and hence the set of true sentences Δ_2 over T is recursive in $\mathbf{0}'$ and therefore is Δ_2^0. Thus the true Δ_2 sentences give rise to an infinite Δ_2^0 branch in the admissibility tree of (the gödelnumbering of) $\mathfrak{D}_{\mathrm{T}}^{\Delta_2}$. ∎

9.6. Lemma. *There exists an efflorescent* Π_2^0 *tree with no infinite* Σ_2^0 *branch.*

Proof. Left as an exercise for the reader. Alternatively, the reader may check that if $\exists x\,\forall y\, D(\alpha, x, y, z)$ is a Σ_2^0 predicate on $\bot$-$\top$-strings α universal in z with $D(\alpha, x, y, z)$ decidable then the set of strings δ satisfying

$$\begin{aligned}\forall z\,\forall\alpha\,[\mathrm{lh}(\alpha) = z + 1 \wedge \alpha \subseteq \delta \to.\ \forall x[\exists y\, \neg D(\alpha, x, y, z) \vee \exists\beta\,[\mathrm{lh}(\beta) = \mathrm{lh}(\alpha)\\ \wedge.\ \alpha \prec \beta \to \exists w \le x\,\forall v\, D(\beta, w, v, z) \wedge.\ \beta \prec \alpha \to \exists w < x\,\forall v\, D(\beta, w, v, z)]]]\end{aligned}$$

is such a tree. ∎

To give an example of a positive numeration μ of a diagonalizable algebra with the strong disjunction property which is not embeddable into $\mathfrak{D}_{\mathrm{T}}^{\Delta_2}$ for a Σ_2-sound T recursively with respect to μ it suffices to take a positive numeration $\mu\colon \omega \to \mathfrak{D}$ whose admissibility tree is the tree T constructed in Lemma 9.6. For if there existed an embedding of $\mathfrak{D}$ into $\mathfrak{D}_{\mathrm{T}}^{\Delta_2}$ recursive with respect to μ then by Lemma 9.5 we could single out an infinite Δ_2^0 branch in T which does not exist.

9.7. Corollary. *Let* T *be a* Σ_2*-sound theory. There exists no recursive* (*nor even recursive in* $\mathbf{0}'$) *embedding of* $\mathfrak{D}_{\mathrm{T}}^{\Delta_0(\Sigma_2)}$ *into* $\mathfrak{D}_{\mathrm{T}}^{\Delta_2}$. ∎

In connection with Corollary 9.7 we should mention a result of Pour-El & Kripke [38] which shows that if the diagonalizable structures of $\mathfrak{D}_{\mathrm{T}}^{\Delta_2}$ and $\mathfrak{D}_{\mathrm{T}}^{\Delta_0(\Sigma_2)}$ are forgotten then the underlying Boolean algebras are recursively isomorphic.

10. Arbitrary subalgebras. Σ_1-ill theories

10.1. THEOREM. *Let* T *be a* Σ_1*-ill theory. A denumerable diagonalizable algebra* $\mathfrak{D}$ *is embeddable in* $\mathfrak{D}_{\mathrm{T}}$ *iff*

(i) $\mathfrak{D}$ *is locally positive and*
(ii) *the height of* $\mathfrak{D}$ *equals the credibility extent of* T.

We only prove Theorem 10.1 for the case of theories T of infinite credibility extent. The case of finite credibility extent is much simpler.

Our proof will exploit the exposition of §6 whenever possible. Thus we take the function h_0, the terms ℓ_0 and s and the sentence $\ell = 0$ to be just the same as what they were in §6.

Let $\nu\colon \omega - \{0\} \to \mathfrak{D}$ be a locally positive numeration of $\mathfrak{D}$. Thus for each $i \in \omega - \{0\}$ there exists a Δ_0 enumeration $\{A_i(m)\}_{m\in\omega}$ of diagonalizable polynomials in propositional letters $p_1, \ldots, p_i$ sent to $\top$ of $\mathfrak{D}$ by substitution of νj for p_j. We shall work with a rearranged family of sequences $\{D_i(m)\}_{m\in\omega\cdot 2}$ (here i ranges over ω) defined as follows (the auxiliary functions $\mathrm{k}_i(\cdot)$ are only defined for $0 < i < \omega$):

(1) $$D_0(x) = \top\,,$$

(2) $$D_{i+1}(0) = \top\,,$$

(3) $$\mathrm{k}_{i+1}(0) = 0\,,$$

(4) $$D_{i+1}(x+1) = \begin{cases} A_{i+1}(x) & \text{if (i) } A_{i+1}(x) \vdash_{\mathbf{L}} D_{i+1}(x), \\ & \text{(ii) } A_{i+1}(x) \vdash_{\mathbf{L}} A_{i+1}[\mathrm{k}_{i+1}(x)] \text{ and} \\ & \text{(iii) } \boxdot^+ A_{i+1}(x) \text{ is } i\text{-conservative over } \boxdot^+ D_i(x)\,, \\ D_{i+1}(x) & \text{otherwise,} \end{cases}$$

(5) $$\mathrm{k}_{i+1}(x+1) = \begin{cases} \mathrm{k}_{i+1}(x)+1 & \text{if } D_{i+1}(x+1) \vdash_{\mathbf{L}} A_{i+1}[\mathrm{k}_{i+1}(x)], \\ \mathrm{k}_{i+1}(x) & \text{otherwise,} \end{cases}$$

(6) $$D_0(\omega + x) = \top\,.$$

(7) Suppose $D_i(\omega + x)$ is steady and $\boxdot^+ D_{i+1}(x)$ (and hence $\boxdot^+[D_{i+1}(x) \wedge D_i(\omega + x)]$) is i-conservative over $\boxdot^+ D_i(\omega + x)$. Then let $D_{i+1}(\omega + x)$ be the formula provided by Lemma 5.25 such that

(i) $D_{i+1}(\omega + x) \vdash_{\mathbf{L}} D_{i+1}(x)$,
(ii) $D_{i+1}(\omega + x) \vdash_{\mathbf{L}} D_i(\omega + x)$,
(iii) $D_{i+1}(\omega + x)$ is steady,
(iv) $\boxdot^+ D_{i+1}(\omega + x)$ is i-conservative over $\boxdot^+ D_i(\omega + x)$ and
(v) $D_{i+1}(\omega + x)$ admits every formula in $\mathbf{F}^0(i)$ that $D_i(\omega + x)$ does.

(Note that, unlike x, the i in $D_i(x)$ is an index rather than a free variable.)

10.2. LEMMA. *For each* $i \in \omega$ *such that* $i > 0$,

(a) $\mathrm{I}\Sigma_1 \vdash \forall x \in \omega\, \forall y \in \omega\, [x \le y \to D_i(y) \vdash_{\mathbf{L}} D_i(x)]$.
(b) $\mathrm{I}\Sigma_1 \vdash \forall x \in \omega\, D_i(\omega + x) \vdash_{\mathbf{L}} D_i(x)$.

(c) $\mathrm{I}\Sigma_1 \vdash \forall x \in \omega\, D_{i+1}(\omega + x) \vdash_{\mathbf{L}} D_i(\omega + x)$.

(d) $\mathrm{I}\Sigma_1 \vdash \forall x \in \omega$ "$D_i(\omega + x)$ is steady".

(e) $\mathrm{I}\Sigma_1 \vdash \forall x \in \omega \cdot 2$ "$\boxdot^+ D_{i+1}(x)$ is $\bar{i}$-conservative over $\boxdot^+ D_i(x)$".

(f) $\mathrm{I}\Sigma_1 \vdash \forall x \in \omega\, \forall \alpha\, [\mathrm{lh}(\alpha) \le i \rightarrow$. "$D_i(\omega + x)$ admits α" $\leftrightarrow$ "$D_{i+1}(\omega + x)$ admits α"$]$.

(g) *For each* $y \in \omega$ *there exists an* $x \in \omega$ *such that* $D_i(x) \vdash_{\mathbf{L}} A_i(y)$.

(h) *For each* $x \in \omega$ *there exists a* $y \in \omega$ *such that* $D_i(x) = A_i(y)$.

Proof. (a) and (h) are easy.

To prove clause (e) for the case $x \in \omega$ it suffices to notice that if a formula A is i-conservative over a formula B then it also is i-conservative over $B \wedge C$ for any formula C.

Claim. *For all* $i \in \omega$, $\mathrm{I}\Sigma_1$ *proves that for each* $x \in \omega$

(i) $D_i(\omega + x) \vdash_{\mathbf{L}} D_i(x)$.

(ii) "$D_i(\omega + x)$ is steady".

(iii) "$\boxdot^+ D_{i+1}(x)$ is i-conservative over $\boxdot^+ D_i(\omega + x)$".

(iv) "The premises of (7) are satisfied for i".

The Claim is proved by induction on i. For $i = 0$ this is clear: (1) implies (i) and (ii) of the Claim and since $\boxdot^+ D_1(x)$ has to be conservative by (iii) of (4), (iii) and (iv) of the Claim for $i = 0$ follow.

Suppose that $i > 0$ and that the Claim holds for $i - 1$. Then by (iv) of the induction hypothesis we see on inspection of (7) that $D_i(\omega + x) \vdash_{\mathbf{L}} D_i(x)$, $D_i(\omega + x)$ is steady, and since by the ω-part of (e), $\boxdot^+ D_{i+1}(x)$ is i-conservative over $\boxdot^+ D_i(x)$, this implies that $\boxdot^+ D_{i+1}(x)$ is also i-conservative over $\boxdot^+ D_i(\omega + x)$. Thus the Claim is proved.

(b) and (d) are direct consequences of the Claim.

(c) follows from (iv) of the Claim combined with (ii) of (7).

Clause (e) for $x > \omega$ is implied by (iv) of the Claim and (iv) of (7).

(f) $(\rightarrow)$ is inferred from (v) of (7) plus (iv) of the Claim.

$(\leftarrow)$ follows from (c)–(e) by Corollary 5.23.

(g) We proceed by induction on i. Suppose there exists an $y \in \omega$ such that $D_{i+1}(x) \vdash_{\mathbf{L}} A_{i+1}(y)$ for no $x \in \omega$. Pick the minimal such y. Then $\lim_{x \to \infty} \mathrm{k}_{i+1}(x) = y$ and $\lim_{x \to \infty} D_{i+1}(x) = A_{i+1}(z)$ for some $z < y$. Consider the formula $A_{i+1}(z) \wedge A_{i+1}(y)$. Let C be the formula in $\mathbf{F}(i)$ such that $A_{i+1}(z) \wedge A_{i+1}(y) \vdash_{\mathbf{L}} C \vdash_{\mathbf{L}} D$ whenever $A_{i+1}(z) \wedge A_{i+1}(y) \vdash_{\mathbf{L}} D$ and $D \in \mathbf{F}(i)$ (here we use the Uniform Craig Interpolation Lemma). Since $\{A_i(m)\}_{m \in \omega}$ and $\{A_{i+1}(m)\}_{m \in \omega}$ enumerate the relations of one and the same diagonalizable algebra we infer that $A_i(w) \vdash_{\mathbf{L}} C$ for some $w \in \omega$. Therefore by the induction hypothesis there exists a $v \in \omega$ such that $D_i(v) \vdash_{\mathbf{L}} C$. By the choice of C the formula $\boxdot^+[A_{i+1}(z) \wedge A_{i+1}(y)]$ is i-conservative over $\boxdot^+ C$ and hence over $\boxdot^+ D_i(w)$ for all $w \ge v$. Now formulas $\mathbf{L}$-equivalent to $A_{i+1}(z) \wedge A_{i+1}(y)$ occur unboundedly often in $\{A_{i+1}(m)\}_{m \in \omega}$

and, as we have shown, infinitely many such occurrences satisfy (i)–(iii) of (4). Thus we have reached a contradiction and therefore proved (g). ∎

We now define a sequence of Δ_0 functions $\{\mathrm{g}_i(\cdot)\}_{i\in\omega-\{0\}}$:

$$\mathrm{g}_{i+1}(x) = \begin{cases} z & \text{if } z \text{ satisfies (i)–(iv) below,} \\ x & \text{if } x < \mathsf{s} \text{ and no } z \text{ satisfying (i)–(iii) exists,} \\ \omega + \mathsf{s} & \text{if } x \geq \mathsf{s} \text{ and no } z \text{ satisfying (i)–(iii) exists,} \end{cases} \tag{8}$$

(i) $z < x$,

(ii) $z < \mathsf{s}$,

(iii) there exists a formula $A \in \mathbf{F}(i+1)$ such that

$$\mathrm{Prf}[z, \overline{\ell_{i+1} \Vdash \overline{\neg A} \to \exists y\, [0R\mathrm{h}_{i+1}(y)R\ell_{i+1}]}] \quad \text{and} \quad \Big[\bigwedge_{0<j\leq i+1} D_j[\mathrm{g}_j(z)]\Big] \nvdash_{\mathbf{L}} A\,,$$

or there exists a j such that $0 < j \leq i$ and $\mathrm{g}_j(x) = z$,

(iv) z is minimal among those satisfying (i)–(iii).

We are going to construct the functions h_i and terms ℓ_i later. Clearly the Fixed Point Lemma gives us a free hand to use $\mathrm{g}_0, \ldots, \mathrm{g}_i$ when doing so.

10.3. Lemma. *For each $i \in \omega$ such that $i > 0$, $\mathrm{I}\Sigma_1$ proves*

(a) $\forall x\, \mathrm{g}_i(x) \geq \mathrm{g}_{i+1}(x)$.

(b) $\forall x$ "$\Box^+ D_{i+1}[\mathrm{g}_{i+1}(x)]$ is $\bar{i}$-conservative over $\Box^+ D_i[\mathrm{g}_i(x)]$".

(c) $\forall x\, \forall y\, [x \leq y \to D_i[\mathrm{g}_i(y)] \vdash_{\mathbf{L}} D_i[\mathrm{g}_i(x)]]$.

(d) $\forall x = \mathsf{s}\, \forall y\, D_i(\omega + x) \vdash_{\mathbf{L}} D_i[\mathrm{g}_i(y)]$.

(e) $$\forall x\, \forall A \in \mathbf{F}(\bar{i}) \Big[\mathrm{Prf}[x, \overline{\ell_i \Vdash \overline{\neg A} \to \exists y\, [0R\mathrm{h}_i(y)R\ell_i]}] \wedge \Big[\bigwedge_{0<j\leq i} D_j[\mathrm{g}_j(x)]\Big] \nvdash_{\mathbf{L}} A \to .\ \forall y \geq x\, \mathrm{g}_i(y) = \mathrm{g}_i(x)\Big].$$

(f) $$\forall x < \mathsf{s} \Big[\mathrm{g}_i(x) \neq x \to .\ \exists A \in \mathbf{F}(\bar{i})\, [\mathrm{Pr}[\overline{\ell_i \Vdash \overline{\neg A} \to \exists y\, [0R\mathrm{h}_i(y)R\ell_i]}] \wedge \forall z\, D_i[\mathrm{g}_i(z)] \nvdash_{\mathbf{L}} A] \vee .\ \bigvee_{0<j<i} \mathrm{g}_j(x) \neq x\Big].$$

(g) $$\forall x = \mathsf{s} \Big[\mathrm{g}_i(x) \neq \omega + x \to .\ \exists A \in \mathbf{F}(\bar{i})\, [\mathrm{Pr}[\overline{\ell_i \Vdash \overline{\neg A} \to \exists y\, [0R\mathrm{h}_i(y)R\ell_i]}] \wedge \forall z\, D_i[\mathrm{g}_i(z)] \nvdash_{\mathbf{L}} A] \vee .\ \bigvee_{0<j<i} \mathrm{g}_j(x) \neq \omega + x\Big].$$

Proof. (a) is obvious.

(b) is proved by induction on x using the fact that if a formula A is i-conservative over a formula B then A is i-conservative over any formula that is stronger than B.

(c)–(g) are proved in the same way as corresponding clauses of Lemma 6.7. ∎

Next we write down a sequence of Solovay functions.

$$\mathrm{h}_{i+1}(0) = 0\,, \tag{9}$$

$$\mathrm{h}_{i+1}(x+1) = \begin{cases} a & \text{if (i)–(viii) below hold,} \\ \mathrm{h}_{i+1}(x) & \text{if no } a \text{ satisfying (i)–(vii) exists;} \end{cases} \tag{10}$$

(i) $a \in M_{i+1}$,
(ii) $\mathrm{h}_i(x) \neq \mathrm{h}_i(x+1)$,
(iii) $\mathrm{h}_{i+1}(x) R a$,
(iv) $a \lhd \mathrm{h}_i(x+1)$,
(v) if $\mathrm{h}_{i+1}(x) = 0$ then $a \vDash D_{i+1}[\mathrm{g}_{i+1}(x)]$,
(vi) if $\mathrm{h}_{i+1}(x) = 0$ then for each b satisfying (i)–(v) in place of a one has

$$\begin{aligned} \forall z \le x \Big[\exists A \in \mathbf{F}(\overline{i+1}) \Big[&\mathrm{Prf}[z, \overline{\ell_{i+1} \Vdash \overline{\neg A} \to \exists y\,[0 R \mathrm{h}_{i+1}(y) R \ell_{i+1}]}] \\ &\wedge \Big[\bigwedge\!\!\!\bigwedge_{0<j\le i+1} D_j[\mathrm{g}_j(z)] \Big] \nvdash_{\mathbf{L}} A \wedge b \nVdash A \Big] \\ \to \exists w \le z\, \exists A \in \mathbf{F}(\overline{i+1}) \Big[&\mathrm{Prf}[w, \overline{\ell_{i+1} \Vdash \overline{\neg A} \to \exists y\,[0 R \mathrm{h}_{i+1}(y) R \ell_{i+1}]}] \\ &\wedge \Big[\bigwedge\!\!\!\bigwedge_{0<j\le i+1} D_j[\mathrm{g}_j(w)] \Big] \nvdash_{\mathbf{L}} A \wedge a \nVdash A \Big]\Big]\,, \end{aligned}$$

(vii) if $\mathrm{h}_{i+1}(x) \neq 0$ then for each b satisfying (i)–(v) in place of a one has

$$\begin{aligned} &\forall z \le x\, [\mathrm{Prf}[z, \overline{\ell_{i+1} = \overline{b} \to \exists y\, [\overline{\mathrm{h}_{i+1}(x)} R \mathrm{h}_{i+1}(y) R \overline{b}]}] \\ &\qquad \to \exists w \le z\ \mathrm{Prf}[w, \overline{\ell_{i+1} = \overline{a} \to \exists y\, [\overline{\mathrm{h}_{i+1}(x)} R \mathrm{h}_{i+1}(y) R \overline{a}]}]]\,, \end{aligned}$$

(viii) a is minimal among those c that satisfy (i)–(vii) in place of a;

$$\ell_{i+1} = \begin{cases} \lim\limits_{x\to\infty} \mathrm{h}_{i+1}(x) & \text{if } \mathrm{h}_{i+1} \text{ reaches a limit,} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

10.4. Lemma. *For each $i, j \in \omega$ such that $i < j$,* $\mathrm{I}\Sigma_1$ *proves*

(a) $\forall x \forall y\, [x \le y \to .\ \mathrm{h}_i(x) = \mathrm{h}_i(y) \vee \mathrm{h}_i(x) R \mathrm{h}_i(y)]$.
(b) $\forall x\, \mathrm{h}_{i+1}(x) \lhd \mathrm{h}_i(x)$.
(c) $\forall x\, \mathrm{h}_j(x) \lhd \mathrm{h}_i(x)$.
(d) $\ell_i = \lim_{x\to\infty} \mathrm{h}_i(x)$.
(e) $\forall x\, [\mathrm{h}_i(x) R \ell_i \vee \mathrm{h}_i(x) = \ell_i]$.
(f) $\ell_j \lhd \ell_i$.

Proof. We only prove (b) and that, as usual, by induction on x.

Assume $\mathrm{h}_{i+1}(x) \lhd \mathrm{h}_i(x) \neq \mathrm{h}_i(x+1)$. The case when $\mathrm{h}_i(x) \neq 0$ is treated as in §6. So let $0 = \mathrm{h}_{i+1}(x) = \mathrm{h}_i(x) \neq \mathrm{h}_i(x+1)$.

Once we show that there exists a node $a \in M_{i+1}$ such that $a \lhd \mathrm{h}_i(x+1)$ and $a \vDash D_{i+1}[\mathrm{g}_{i+1}(x)]$ we are done because then it only takes the (Δ_0) least number principle to satisfy (vi)–(viii). Assuming there is no such a we have by Lemmas 3.3(d) and 3.7(d)

$$\vdash_{\mathbf{L}} \Box^{+} D_{i+1}[\mathrm{g}_{i+1}(x)] \to \neg\Psi_{\mathrm{h}_i(x+1)}\,.$$

Now from Lemma 10.3(b) we know that $\Box^{+} D_{i+1}[\mathrm{g}_{i+1}(x)]$ is i-conservative over $\Box^{+} D_i[\mathrm{g}_i(x)]$ and hence

$$\vdash_{\mathbf{L}} \Box^{+} D_{i}[\mathrm{g}_{i}(x)] \to \neg\Psi_{\mathrm{h}_i(x+1)}\,.$$

But note that since $0 = \mathrm{h}_i(x) \neq \mathrm{h}_i(x+1)$ we see by (v) of (10) that $\mathrm{h}_i(x+1) \vDash D_i[\mathrm{g}_i(x)]$, ergo

$$\vdash_{\mathbf{L}} \Psi_{\mathrm{h}_i(x+1)} \to \Box^{+} D_i[\mathrm{g}_i(x)]$$

by Lemma 3.3(d). The contradiction proves (b). ∎

10.5. Lemma. *For each* $i \in \omega$, $\mathrm{I}\Sigma_1 + \ell \neq 0$ *proves*

(a) $\mathrm{Pr}(\overline{\overline{\ell_i} R \ell_i})$.

(b) $\forall a \in M_{\bar{i}}\, [\ell_i R a \to \neg \mathrm{Pr}[\overline{\ell_i = \overline{a} \to \exists y\, [\overline{\ell_i} R \mathrm{h}_i(y) R \overline{a}]}]]$.

(c) $\forall a \in M_{\bar{i}}\, [\ell_i R a \to \neg \mathrm{Pr}(\overline{\ell_i \neq \overline{a}})]$.

Proof. Cf. Lemma 6.6. ∎

10.6. Lemma. *For each* $i \in \omega$ *such that* $i > 0$, $\mathrm{I}\Sigma_1$ *proves*

(a) $\ell \neq 0 \to \forall x \Big[\mathrm{h}_i(x) = 0 \to \ell_i \vDash \bigwedge\!\!\!\bigwedge_{0<j\le i} D_j[\mathrm{g}_j(x)]\Big]$.

(b) $\ell \neq 0 \to \forall x\, \forall A \in \mathbf{F}(\bar{i}) \Big[\mathrm{h}_i(x) = 0 \wedge \Big[\bigwedge\!\!\!\bigwedge_{0<j\le i} D_j[\mathrm{g}_j(x)]\Big] \vdash_{\mathbf{L}} A \to .\ \ell_i \Vdash A\Big]$.

(c) $\ell = 0 \to \forall A \in \mathbf{F}(\bar{i}) \Big[\mathrm{Pr}[\overline{\ell_i \Vdash \overline{\neg A} \to \exists y\, [0 R \mathrm{h}_i(y) R \ell_i]}]$

$$\to \exists x \Big[\bigwedge\!\!\!\bigwedge_{0<j\le i} D_j[\mathrm{g}_j(x)]\Big] \vdash_{\mathbf{L}} A\Big].$$

(d) $\ell = 0 \to \forall A \in \mathbf{F}(\bar{i}) \Big[\mathrm{Pr}[\overline{\ell \neq 0 \to \ell_i \Vdash \overline{A}}] \to \exists x \Big[\bigwedge\!\!\!\bigwedge_{0<j\le i} D_j[\mathrm{g}_j(x)]\Big] \vdash_{\mathbf{L}} A\Big]$.

Proof. (a) We prove this by induction on i. For $i = 0$ the claim is trivial. If $\mathrm{h}_{i+1}(x) = 0$ then by (v) of (10) and by Lemma 10.3(c), h_{i+1} can only jump to a node at which $D_{i+1}[\mathrm{g}_{i+1}(x)]$ holds and therefore $\ell_{i+1} \vDash D_{i+1}[\mathrm{g}_{i+1}(x)]$. On the other hand, by Lemma 10.4(f) we have $\ell_{i+1} \lhd \ell_i$ and $\ell_i \vDash \bigwedge\!\!\!\bigwedge_{0<j\le i} D_j[\mathrm{g}_j(x)]$ by the induction hypothesis, ergo $\ell_{i+1} \vDash \bigwedge\!\!\!\bigwedge_{0<j\le i} D_j[\mathrm{g}_j(x)]$. Thus $\ell_{i+1} \vDash \bigwedge\!\!\!\bigwedge_{0<j\le i+1} D_j[\mathrm{g}_j(x)]$ and we have carried out the induction step.

(b) follows from (a).

(c) Once again, induction on x. Let $i = 0$ and let A be a formula in $\mathbf{F}(0)$ such that $\nvdash_{\mathbf{L}} A$. Suppose

$$\mathrm{Pr}[\overline{\ell_0 \Vdash \overline{\neg A} \to \exists y\, [0 R \mathrm{h}_0(y) R \ell_0]}]\,.$$

Then there exists a node $a \in M_0$ such that $a \nVdash A$ and we have

$$\mathrm{Pr}[\overline{\ell_0 = \overline{a} \to \exists y\,[0R\mathrm{h}_0(y)R\overline{a}]}]\,.$$

But this combined with $\ell = 0$ contradicts Lemma 6.2(d).

Now we let $i > 0$, assume that the claim holds for i, deny it for $i+1$ and look for a contradiction. Let $x \in \omega$ be such that

$$\mathrm{Prf}[x, \overline{\ell_{i+1} \Vdash \overline{\neg A} \to \exists y\,[0R\mathrm{h}_{i+1}(y)R\ell_{i+1}]}]$$

and $[\bigwedge\!\!\!\bigwedge_{0<j\le i+1} D_j[\mathrm{g}_j(x)]] \nvdash_{\mathbf{L}} A$ for some $A \in \mathbf{F}(i)$. By the (Δ_0) least number principle we can assume that x is minimal among the proofs of this kind. We then find by (8) that $\forall y \ge x\, \mathrm{g}_{i+1}(y) = \mathrm{g}_{i+1}(x)$ so $D_{i+1}[\mathrm{g}_{i+1}(z)] \vdash_{\mathbf{L}} A$ for no z.

Consider the weakest formula B in $\mathbf{F}(i)$ such that

$$\vdash_{\mathbf{L}} B \to .\, \Box^+ D_{i+1}[\mathrm{g}_{i+1}(x)] \to A$$

which exists by the Uniform Craig Interpolation Lemma. Let a be an arbitrary node of $\mathcal{M}_i$ forcing $\neg B$. We show that there is a node $b \in M_{i+1}$ such that $b \lhd a$ and $b \Vdash \neg A \wedge \Box^+ D_{i+1}[\mathrm{g}_{i+1}(x)]$. For if this were not the case then $e \Vdash \Box^+ D_{i+1}[\mathrm{g}_{i+1}(x)] \to A$ for each $e \in M_{i+1}$ such that $e \lhd a$ and therefore by Lemmas 3.3(d) and 3.7(d)

$$\vdash_{\mathbf{L}} \Psi_a \to .\, \Box^+ D_{i+1}[\mathrm{g}_{i+1}(x)] \to A\,,$$

whence $\vdash_{\mathbf{L}} \Psi_a \to B$ because B is the weakest formula in $\mathbf{F}(i)$ implying

$$\Box^+ D_{i+1}[\mathrm{g}_{i+1}(x)] \to A\,.$$

But this contradicts the assumption $a \Vdash \neg B$.

Reason in $\mathrm{I}\Sigma_1$. If h_i jumps from 0 directly to a node forcing $\neg B$ then h_{i+1} will have to jump directly to a node forcing $\neg A$ (this is because conditions (i)–(v) of (10) will obviously be satisfied and because of the minimality condition imposed on x so that (vi) of (10) will also hold). So outside $\mathrm{I}\Sigma_1$ we have

$$\mathrm{Pr}[\overline{\ell_i \Vdash \overline{\neg B} \to .\, \exists y\,[0R\mathrm{h}_i(y)R\ell_i] \vee \ell_{i+1} \Vdash \overline{\neg A}}]\,,$$

which together with Lemma 10.4(b) and the assumption on A gives

$$\mathrm{Pr}[\overline{\ell_i \Vdash \overline{\neg B} \to .\, \exists y\,[0R\mathrm{h}_i(y)R\ell_i]}]\,.$$

Now note that $[\bigwedge\!\!\!\bigwedge_{0<j\le i} D_j[\mathrm{g}_j(z)]] \vdash_{\mathbf{L}} B$ for no $z \in \omega$ because otherwise

$$\Big[\bigwedge\!\!\!\bigwedge_{0<j\le i} D_j[\mathrm{g}_j(z)]\Big] \vdash_{\mathbf{L}} B \vdash_{\mathbf{L}} \Box^+ D_{i+1}[\mathrm{g}_{i+1}(x)] \to A$$

and hence $[\bigwedge\!\!\!\bigwedge_{0<j\le i+1} D_j[\mathrm{g}_j(z)]] \vdash_{\mathbf{L}} A$ for z large enough contrary to assumptions. The situation in which B has found itself can now easily be seen to contradict the induction hypothesis and so (c) is proved.

(d) follows from (c). ∎

Next we deal with 0 (in $\mathrm{I}\Sigma_1+\sigma$). The formulas $\mathrm{Adm}_i(\cdot)$ are defined for $i\in\omega-\{0\}$:

$$\mathrm{Adm}_i(\alpha)\equiv \mathrm{lh}(\alpha)\le \bar{i}\wedge \text{“}D_i(\omega+\mathsf{s})\text{ admits }\alpha\text{”}. \tag{12}$$

10.7. Lemma. *For each* $i\in\omega-\{0\}$, $\mathrm{I}\Sigma_1+\sigma$ *proves*

(a) $\mathrm{Adm}_i(\overline{\Lambda})$.

(b) $\forall\alpha\,\forall\beta\,[\mathrm{Adm}_i(\beta)\wedge\alpha\subseteq\beta \to.\ \mathrm{Adm}_i(\alpha)]$.

(c) $\forall\alpha\,[\mathrm{Adm}_i(\alpha)\to\exists\beta\,[\mathrm{lh}(\beta)=\bar{i}\wedge\alpha\subseteq\beta\wedge\mathrm{Adm}_i(\beta)]]$.

(d) $\exists\alpha\,[\mathrm{lh}(\alpha)=\bar{i}\wedge\mathrm{Adm}_i(\alpha)]$.

(e) $\forall\alpha\,[\mathrm{lh}(\alpha)\le\bar{i}\to.\ \mathrm{Adm}_i(\alpha)\leftrightarrow\mathrm{Adm}_{i+1}(\alpha)]$.

P r o o f. (a)–(d) are proved as in Lemma 6.9.

(e) is a straightforward consequence of Lemma 10.2(f). ∎

$$\mathrm{Adm}_i^+(\alpha)\equiv\mathrm{Adm}_i(\alpha)\wedge\forall\beta\prec\alpha\,[\mathrm{lh}(\alpha)=\mathrm{lh}(\beta)\to\neg\,\mathrm{Adm}_i(\beta)]. \tag{13}$$

10.8. Lemma. *For each* $i\in\omega$ *such that* $i>0$, $\mathrm{I}\Sigma_1+\sigma$ *proves*

(a) $\exists!\alpha\,[\mathrm{lh}(\alpha)=\bar{i}\wedge\mathrm{Adm}_i^+(\alpha)]$.

(b) $\forall\alpha\,[\mathrm{Adm}_i^+(\alpha)\to\exists\beta\,[\mathrm{lh}(\beta)=i\wedge\alpha\subseteq\beta\wedge\mathrm{Adm}_i^+(\beta)]]$.

(c) $\forall\alpha\,\forall\beta\,[\mathrm{Adm}_i^+(\alpha)\wedge\mathrm{Adm}_i^+(\beta)\wedge\mathrm{lh}(\alpha)\le\mathrm{lh}(\beta)\to.\ \alpha\subseteq\beta]$.

(d) $\forall\alpha\,[\mathrm{lh}(\alpha)\le\bar{i}\to.\ \mathrm{Adm}_i^+(\alpha)\leftrightarrow\mathrm{Adm}_{i+1}^+(\alpha)]$.

(e) $\forall\alpha\,\forall\beta\,[\mathrm{Adm}_i^+(\alpha)\wedge\mathrm{Adm}_{i+1}^+(\beta)\wedge\mathrm{lh}(\alpha)\le\mathrm{lh}(\beta)\to.\ \alpha\subseteq\beta]$.

P r o o f. Clauses (a)–(c) are proved in full analogy with those of Lemma 6.10.

(d) follows from Lemma 10.7(e).

(e) follows from (c) and (d). ∎

$$\begin{aligned}\mathrm{T}_i(A)\equiv\exists k\,\exists\lambda\in\mathbf{F}^0(\bar{i}+k)\,\exists B_1,\ldots,B_k\in\mathbf{F}(\bar{i})\,\exists\alpha&\\ [\mathrm{lh}(\alpha)=\bar{i}+k\wedge\lambda[(\alpha)_1,\ldots,(\alpha)_{\bar{i}+k}]=\top&\\ \wedge A=\lambda(\overline{p_1},\ldots,\overline{p_i},\Box B_1,\ldots,\Box B_k)\wedge\forall j\le\bar{i}\,[j\ne0&\\ \to.\ (\alpha)_j=\top\leftrightarrow\exists\beta\,[\mathrm{lh}(\beta)=\bar{i}\wedge\mathrm{Adm}_i^+(\beta)\wedge(\beta)_j=\top]]&\\ \wedge\forall j\le k\,[j\ne0\to.\ (\alpha)_{\bar{i}+j}=\top\leftrightarrow D_i(\omega+\mathsf{s})\vdash_{\mathbf{L}}B_j]].&\end{aligned} \tag{14}$$

10.9. Lemma. *For each* $i,j\in\omega$ *such that* $0<i\le j$, $\mathrm{I}\Sigma_1+\sigma$ *proves*

(a) $\mathrm{T}_j(\overline{p_i})\leftrightarrow\exists\beta\,[\mathrm{lh}(\beta)=\bar{j}\wedge\mathrm{Adm}_j^+(\beta)\wedge(\beta)_{\bar{i}}=\top]$.

(b) $\forall A\in\mathbf{F}(\bar{i})\,[\mathrm{T}_i(\Box A)\leftrightarrow D_i(\omega+\mathsf{s})\vdash_{\mathbf{L}}A]$.

(c) $$\begin{aligned}\forall k\,\forall\lambda\in\mathbf{F}^0(k)\,\forall B_1,\ldots,B_k\in\mathbf{F}\,[\mathrm{T}_i[\lambda(B_1,\ldots,B_k)]&\\ \leftrightarrow\exists\alpha\,[\mathrm{lh}(\alpha)=k\wedge\lambda[(\alpha)_1,\ldots,(\alpha)_k]=\top&\\ \wedge\forall l\le k\,[l\ne0\to.\ (\alpha)_l=\top\leftrightarrow\mathrm{T}_i(B_l)]]].&\end{aligned}$$

(d) $\forall A\in\mathbf{F}(\bar{i})\,[\mathrm{T}_i(A)\leftrightarrow\mathrm{T}_{i+1}(A)]$.

(e) $\forall A \in \mathbf{F}(\bar{\imath})\,[\mathrm{T}_i(A) \leftrightarrow \mathrm{T}_j(A)]$.

P r o o f. (a)–(c) are proved as in Lemma 6.11.

(d) For A a propositional letter in $\mathbf{F}(i)$ we have this by Lemma 10.8(d) and (e), and (a) of the present lemma. For A of the form $\Box B$ this follows from (b) combined with Lemma 10.2(c) and (e). Finally, one executes induction on the Boolean structure of A with the help of (c).

(e) follows from (d). ∎

10.10. Lemma. *For each $i \in \omega$ such that $i > 0$, $\mathrm{I}\Sigma_1 + \sigma$ proves*

(a) $\forall\alpha\Big[\mathrm{lh}(\alpha) \le \bar{\imath} \rightarrow.\ \mathrm{T}_i\Big[\bigwedge\!\!\!\bigwedge_{1\le j\le \mathrm{lh}(\alpha)} [p_j \leftrightarrow (\alpha)_j]\Big] \leftrightarrow \mathrm{Adm}_i^+(\alpha)\Big]$.

(b) $\forall A \in \mathbf{F}(\bar{\imath})\,\forall B \in \mathbf{F}(\bar{\imath})\,[\mathrm{T}_i(\Diamond A \wedge \Box B) \rightarrow \mathrm{T}_i[\Diamond(A \wedge \Box B)]]$.

(c) $\forall A \in \mathbf{F}(\bar{\imath})\,[\mathrm{T}_i(\Box A) \rightarrow \mathrm{T}_i(A)]$.

P r o o f. Analogous to Lemma 6.12. ∎

Now we map $\{p_i\}_{i\in\omega-\{0\}}$ (and hence the whole of $\mathbf{F}$) into $\mathfrak{D}_{\mathrm{T}}$:

$$p_i^\circ \equiv \ell \ne 0 \wedge \ell_i \Vdash \overline{p_i} \vee.\ \ell = 0 \wedge \mathrm{T}_i(\overline{p_i})\,. \tag{15}$$

10.11. Lemma. *For each $i \in \omega$ such that $i > 0$ there holds*

(a) *For each modal formula $A(p_1, \ldots, p_i)$ one has*

$$\mathrm{I}\Sigma_1 \vdash \ell \ne 0 \rightarrow.\ [A(p_1, \ldots, p_i)]^\circ \leftrightarrow \ell_i \Vdash \overline{A(p_1, \ldots, p_i)}\,.$$

(b) $\mathrm{I}\Sigma_1 \vdash \forall A \in \mathbf{F}(\bar{\imath})\,\mathrm{Pr}[\overline{\ell \ne 0 \rightarrow.\ A^\circ \leftrightarrow \ell_i \Vdash \overline{A}}]$ (*note that* $^\circ$ *restricted to* $\mathbf{F}(i)$ *is representable by a* Δ_0 *function*).

(c) $\mathrm{I}\Sigma_1 \vdash \ell = 0 \rightarrow \forall A \in \mathbf{F}(\bar{\imath})\Big[\mathrm{Pr}(\overline{A^\circ}) \rightarrow \exists x\Big[\bigwedge\!\!\!\bigwedge_{0<j\le i} D_j[\mathrm{g}_j(x)]\Big] \vdash_{\mathbf{L}} A\Big]$.

(d) $\mathrm{I}\Sigma_1 + \sigma \vdash \ell = 0 \rightarrow \mathrm{g}_i(\mathrm{s}) = \omega + \mathrm{s}$.

(e) *For each modal formula $A(p_1, \ldots, p_i)$ one has*

$$\mathrm{I}\Sigma_1 + \sigma \vdash [A(p_1, \ldots, p_i)]^\circ \leftrightarrow [\ell \ne 0 \wedge \ell_i \Vdash \overline{A(p_1, \ldots, p_i)} \vee.\ \ell = 0 \wedge \mathrm{T}_i[\overline{A(p_1, \ldots, p_i)}]]\,.$$

P r o o f. Fairly similar to the proof of Lemma 6.13. When proving (d) use induction on i and Lemmas 10.3(g) and 10.6(c). When carrying out the induction step in (e) corresponding to a Boolean connective use Lemma 10.9(e). ∎

10.12. Lemma. *For each $i \in \omega$ such that $i > 0$*

$$\mathbb{N} \models \text{“}\mathrm{g}_i \text{ is the identity function”}\,.$$

P r o o f. This is proved by induction on i with the help of Lemmas 10.3(f) and 10.6(c). ∎

10.13. LEMMA. *If $A(\nu 1, \nu 2, \ldots) = \top$ for $A(x_1, x_2, \ldots)$ a diagonalizable polynomial then*

$$\mathrm{I}\Sigma_1 + \sigma \vdash [A(p_1, p_2, \ldots)]^\circ .$$

Proof. Cf. Lemma 6.15. ∎

Finally, set

$$(\nu i)^* \equiv p_i^\circ$$

and use Lemmas 10.6(d), 10.11(e) and 10.13 to finish the proof of Theorem 10.1. ∎

10.14. COROLLARY. *Each locally positive diagonalizable algebra can be embedded into a positive diagonalizable algebra.* ∎

11. Arbitrary subalgebras. Σ_1-sound theories

In this section we show that the straightforward attempt to generalize Theorem 7.1 to arbitrary subalgebras of diagonalizable algebras of Σ_1-sound theories the way Theorem 10.1 generalizes Theorems 4.1 and 6.1 fails. We shall present an example of a locally positive diagonalizable algebra with the strong disjunction property which is not isomorphic to any subalgebra of $\mathfrak{D}_{\mathrm{T}}$ for a Σ_1-sound theory T. To do so we first need to list some definitions and facts from recursion theory.

11.1. DEFINITION. Consider Turing machines with two distinct halting states 0 and 1. Such machines are said to be 0-1-*Turing*. A 0-1-valued partial recursive function f is *computed* by a 0-1-Turing machine M if

$$f(n) = i \iff M \text{ halts in state } i \text{ on input } n .$$

A mapping φ of ω onto the set of 0-1-valued partial recursive functions (we denote the image of i under φ by φ_i) such that $\varphi_i(n)$ is a binary recursive function is called a *numbering* (of 0-1-valued partial recursive functions). A numbering is *acceptable* (cf. Rogers [39]) if there exists a total recursive function τ such that for each $i \in \omega$, $\tau(i)$ is a φ-index for the function computed by the ith 0-1-Turing machine (that is, $\varphi_{\tau(i)}$ is this function).

11.2. DEFINITION. Let φ be an acceptable numbering. A binary partial recursive function $\Phi_i(n)$ is called a *Blum complexity measure* (*for* φ) (cf. Blum [10]) if

(i) $\Phi_i(n)$ converges if and only if $\varphi_i(n)$ converges and
(ii) the ternary relation $\Phi_i(n) \le m$ is decidable.

In 11.3–11.5 we shall assume that Φ is a Blum complexity measure for an acceptable numbering φ.

We quote a theorem due to Blum which will help us to construct the desired diagonalizable algebra. Although in Blum [10] the theorem was claimed for acceptable numberings of all the partial recursive functions, an easy inspection of

Blum's proofs shows it to be also valid for acceptable numberings of 0-1-valued partial recursive functions.

11.3. PROPOSITION (Blum [10]). *To each partial recursive function f there corresponds a* 0-1-*valued partial recursive function g such that*

(i) $\operatorname{dom} g = \operatorname{dom} f$ *and*

(ii) *if i is a φ-index for g then $\Phi_i(n) \geq f(n)$ for all but finitely many $n \in \operatorname{dom} f$.* ∎

From this theorem we infer an easy corollary.

11.4. COROLLARY. *Let X be an r.e. subset of ω. There exists a sequence of* 0-1-*valued partial recursive functions $\{g_i\}_{i\in\omega}$ such that*

(i) *for each $i \in \omega$, $\operatorname{dom} g_i = X$ and*

(ii) *for each partial recursive function f with $\operatorname{dom} f = X$ there exists an $i \in \omega$ such that for each φ-index j for g_i one has $f(n) < \Phi_j(n)$ for all but finitely many $n \in X$.*

(*Clearly $\{g_i\}_{i\in\omega}$ cannot be recursive in i.*) ∎

11.5. LEMMA. *Let X be a nonrecursive r.e. subset of ω and let the domain of φ_i be X. Then for each total recursive function f one has $f(n) \leq \Phi_i(n)$ for infinitely many $n \in X$.*

Proof. If this were not so then X could easily be shown to be decidable. ∎

Define the modal formula $\#_n$ to be $\Box^{n+1}\bot \wedge \Diamond^n\top$. In this section we shall give way to an unmerciful intrusion of modal-logical notation on arithmetic.

11.6. DEFINITION. Let T be a theory of infinite credibility extent. Consider a particular numbering δ^{T} of 0-1-valued partial recursive functions which will take (gödelnumbers of) arithmetic sentences for indices:

$$\delta^{\mathrm{T}}_\alpha(n) = \begin{cases} 0 & \text{if } \mathrm{T} \vdash \#_{\overline{n}} \to \alpha, \\ 1 & \text{if } \mathrm{T} \vdash \#_{\overline{n}} \to \neg\alpha, \\ \text{divergent} & \text{if } \mathrm{T} + \#_{\overline{n}} \text{ does not decide } \alpha. \end{cases}$$

Further define

$$\Delta^{\mathrm{T}}_\alpha(n) = \begin{cases} p & \text{if } p \text{ is the least proof in T of either } \#_{\overline{n}}\to\alpha \text{ or } \#_{\overline{n}}\to\neg\alpha, \\ \text{divergent} & \text{if } \mathrm{T} + \#_{\overline{n}} \text{ does not decide } \alpha. \end{cases}$$

11.7. LEMMA. *If* T *is a theory of infinite credibility extent then*

(a) δ^{T} *is an acceptable numbering*,

(b) Δ^{T} *is a Blum complexity measure for* δ^{T}.

Proof. (a) Note that for each $n\in\omega$ the theory $\mathrm{T}+\#_{\overline{n}}$ is consistent. By Corollary 2 in Smoryński [45] or by the Uniform Dual Semi-Representability Theorem of Smoryński [47] or by a Theorem in Smoryński [48] there exists a formula $\sigma(y,x)$ such that for all $i,m\in\omega$

$$\begin{aligned} \mathrm{T}\vdash\sigma(\bar{i},\overline{m}) \quad &\text{iff for some } n\in\omega,\quad \mathrm{T}+\#_{\overline{n}}\vdash\sigma(\bar{i},\overline{m})\\ &\text{iff for all } n\in\omega,\quad \mathrm{T}+\#_{\overline{n}}\vdash\sigma(\bar{i},\overline{m})\\ &\text{iff the } i\text{th 0-1-Turing machine halts in state 0 on input } m \end{aligned}$$

and

$$\begin{aligned} \mathrm{T}\vdash\neg\sigma(\bar{i},\overline{m}) \quad &\text{iff for some } n\in\omega,\quad \mathrm{T}+\#_{\overline{n}}\vdash\neg\sigma(\bar{i},\overline{m})\\ &\text{iff for all } n\in\omega,\quad \mathrm{T}+\#_{\overline{n}}\vdash\neg\sigma(\bar{i},\overline{m})\\ &\text{iff the } i\text{th 0-1-Turing machine halts in state 1 on input } m\,. \end{aligned}$$

Let $\tau(i)$ be the formula

$$\forall x\,[\#_x\to\sigma(\bar{i},x)]\,.$$

Then $\tau(i)$ is a δ^{T}-index for the function computed by the ith 0-1-Turing machine because

$$\mathrm{T}\vdash\forall x\,\forall y\,(\#_x\wedge\#_y\to.\ x=y)$$

and therefore

$$\mathrm{T}\vdash\#_{\overline{n}}\to.\ \forall x\,[\#_x\to\sigma(\bar{i},x)]\leftrightarrow\sigma(\bar{i},\overline{n})\,.$$

(b) is obvious. ∎

11.8. Proposition. *Let* T *be a* Σ_1*-sound theory. There exists a locally positive diagonalizable algebra* $\mathfrak{D}$ *with the strong disjunction property which is not isomorphic to any subalgebra of* $\mathfrak{D}_{\mathrm{T}}$.

Proof. As usual we take $\mathfrak{D}$ to be the quotient of the free diagonalizable algebra **F** modulo a suitable set $\mathcal{D}$ of its elements. The generators of **F** are $\{q\}\cup\{p_i\}_{i\in\omega}$. We now describe $\mathcal{D}$.

Let X be a nonrecursive r.e. subset of ω.

First, we put in $\mathcal{D}$ all the formulas

$$\#_n\to q\qquad\text{for } n\in X\,.$$

Second, formulas

$$\Box(\#_n\to q)\to.\ \Box(\#_n\to p_i)\vee\Box(\#_n\to\neg p_i)\qquad\text{for all } i,n\in\omega\,.$$

Third, with the help of Lemma 11.7 we fix a sequence $\{g_i\}_{i\in\omega}$ of 0-1-valued partial recursive functions such that $\operatorname{dom} g_i=X$ for each $i\in\omega$ and for which the conditions of Corollary 11.4 hold with φ replaced by δ^{T} and Φ replaced by Δ^{T}. Put in $\mathcal{D}$ the formulas

$$\#_n\to p_i \text{ for all } i\in\omega, n\in X \text{ such that } g_i(n)=0$$

and

$$\#_n\to\neg p_i \text{ for all } i\in\omega, n\in X \text{ such that } g_i(n)=1\,.$$

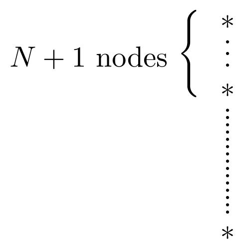
Let D_N be the conjunction of all formulas in $\mathcal{D}$ that neither speak of $\#_m$ for $m > N$ nor of p_i with $i > N$.

Clearly $A(q, p_0, \ldots) = \top$ in $\mathbf{F}/\mathcal{D}$ if and only if for some $N \in \omega$ one has $D_N \vdash_{\mathbf{L}} A$.

It can easily be shown that if $n \notin X$ then neither $\#_n \to p_i = \top$ nor $\#_n \to \neg p_i = \top$ for any $i \in \omega$.

Now we show that for each $N \in \omega$ the formula D_N is steady.

Consider the $(q, p_0, \ldots, p_N)$-model $\mathcal{H}$ (the forcing relation is not indicated in the picture):



We do not care which propositional letters are forced at the bottom node. Each of the rest nodes forces $\#_n$ for a unique $n \leq N$. Call this node a_n.

Let q be forced precisely at those nodes a_n for which $n \in X$.

For each $n \leq N$ such that $n \in X$ and each $i \in \omega$ precisely one of the formulas

$$\#_n \to p_i \quad \text{and} \quad \#_n \to \neg p_i$$

is a conjunct of D_N. In the first case let $a_n \Vdash p_i$ and in the second case let $a_n \Vdash \neg p_i$. The forcing of letters p at the nodes a_n with $n \notin X$ is quite irrelevant.

We leave it to the reader to graft a pair of models in which D_N holds above the root of $\mathcal{H}$ and to check that D_N holds in the resulting model.

This having been done, one can apply Lemmas 5.13 and 5.15 to see that D_N is steady and that $\mathbf{F}/\mathcal{D}$ enjoys the strong disjunction property. Moreover, $\mathbf{F}/\mathcal{D}$ is clearly locally positive.

It remains to prove that $\mathbf{F}/\mathcal{D}$ is not embeddable into $\mathfrak{D}_{\mathrm{T}}$.

We need three total recursive functions n, mp and sd (necessitation, modus ponens and strong disjunction) with the following properties:

— n is such that whenever p is a proof in T of a sentence α and $p \leq x$ there exists a proof q in T of $\Box\alpha$ satisfying $q \leq \mathrm{n}(x)$. Clearly we can stipulate that $\mathrm{n}(x) \geq x$ for each x;

— mp is such that whenever p_1 and p_2 are proofs in T of sentences α and $\alpha \to \beta$ and $p_1, p_2 \leq x$ there is a proof q in T of β satisfying $q \leq \mathrm{mp}(x)$;

— sd is such that whenever p is a proof in T of a sentence of the form $\Box\alpha \vee \Box\beta$ and $p \leq x$ there exists a proof q in T either of α or of β satisfying $q \leq \mathrm{sd}(x)$.

Note that by Parikh's Theorem cited in §1 or, alternatively, by Proposition 1.1 the function sd cannot be provably recursive whereas the functions n and mp can, under any reasonable gödelnumbering, be chosen primitive recursive.

Suppose * is a hypothetical embedding of $\mathbf{F}/\mathcal{D}$ into $\mathfrak{D}_{\mathrm{T}}$. We want a contradiction.

Since p_i^* is a δ^{T}-index for g_i and because the functions g_i were chosen in accordance with Corollary 11.4 there is a number $i \in \omega$ such that

$$\mathrm{sd} \circ \mathrm{mp} \circ \mathrm{n} \circ \Delta^{\mathrm{T}}_{q^*}(n) < \Delta^{\mathrm{T}}_{p_i^*}(n)$$

for all but finitely many $n \in X$ (we know that for each $n \in X$ both the l.h.s. and the r.h.s. converge).

Let D be a total recursive function such that for each $n \in \omega$ $D(n)$ is a proof in T of

$$\Box(\#_{\overline{n}} \to q^*) \to . \Box(\#_{\overline{n}} \to p_i^*) \vee \Box(\#_{\overline{n}} \to \neg p_i^*)\,.$$

By Lemma 11.5 we have

$$D(n) \leq \Delta^{\mathrm{T}}_{q^*}(n) \leq \mathrm{n} \circ \Delta^{\mathrm{T}}_{q^*}(n)$$

for infinitely many $n \in X$. Therefore for infinitely many $n \in X$ there are proofs $\leq \mathrm{n} \circ \Delta^{\mathrm{T}}_{q^*}(n)$ of both formulas

$$\Box(\#_{\overline{n}} \to q^*) \quad \text{and} \quad \Box(\#_{\overline{n}} \to q^*) \to . \Box(\#_{\overline{n}} \to p_i^*) \vee \Box(\#_{\overline{n}} \to \neg p_i^*)\,.$$

Therefore for infinitely many $n \in X$ there is a proof $\leq \mathrm{mp} \circ \mathrm{n} \circ \Delta^{\mathrm{T}}_{q^*}(n)$ of the formula

$$\Box(\#_{\overline{n}} \to p_i^*) \vee \Box(\#_{\overline{n}} \to \neg p_i^*)$$

and hence for infinitely many $n \in X$ there is a proof $\leq \mathrm{sd} \circ \mathrm{mp} \circ \mathrm{n} \circ \Delta^{\mathrm{T}}_{q^*}(n)$ of at least one of the formulas

$$\#_{\overline{n}} \to p_i^* \quad \text{and} \quad \#_{\overline{n}} \to \neg p_i^*\,,$$

that is,

$$\Delta^{\mathrm{T}}_{p_i^*}(n) \leq \mathrm{sd} \circ \mathrm{mp} \circ \mathrm{n} \circ \Delta^{\mathrm{T}}_{q^*}(n)$$

for infinitely many $n \in X$. But this cannot be the case by our assumption on p_i^*. Thus we have obtained the required contradiction and have therefore completed the proof. ∎

Roughly speaking, the reason for Proposition 11.8's holding true is that the strong disjunction property in locally positive diagonalizable algebras can, in a sense, be even less effective than in the diagonalizable algebras of Σ_1-sound theories. A reason for this reason lies in the fact that the conjunction of two steady formulas is not necessarily steady. If one, starting from Theorem 7.1, were to attempt disproving Proposition 11.8 following the lines that led us from Theorems 4.1 and 6.1 to Theorem 10.1 one would meet difficulties in arranging recursive sequences of formulas $\{D_i(m)\}_{m \in \omega}$ in such a way that for each $i, m \in \omega$

(i) $D_i(m+1) \vdash_{\mathbf{L}} D_i(m)$,
(ii) $\Box^+ D_{i+1}(m)$ is i-conservative over $\Box^+ D_i(m)$ and
(iii) $D_i(m)$ is steady.

This happens because we cannot guarantee that $D_{i+1}(m) \wedge D_i(m+1)$ is steady from the condition that both $D_{i+1}(m) \wedge D_i(m)$ and $D_i(m+1)$ are. In other words, the introduction of a stronger steady formula $D_i(m+1)$ may create new disjunction problems for the sequence D_{i+1} and since in locally positive algebras D_i knows nothing about D_{i+1} this situation is generally unavoidable. (These sentiments also explain how the proof of Proposition 11.8 came to the author's mind.)

A weak kind of consolation is presented by Corollary 11.10 and Proposition 11.11. Proposition 11.9 was kindly pointed out to me by Professor Franco Montagna.

11.9. PROPOSITION. *Let* T_1 *and* T_2 *be* Σ_1*-sound theories. Then* $\mathfrak{D}_{\mathrm{T}_1}$ *is embeddable in* $\mathfrak{D}_{\mathrm{T}_2}$.

Proof. A straightforward consequence of Theorem 7.1. ■

11.10. COROLLARY. *Diagonalizable algebras of all* Σ_1*-sound theories possess one and the same collection of subalgebras.* ■

11.11. PROPOSITION. *A locally positive diagonalizable algebra* $\mathfrak{D}$ *can be embedded into the diagonalizable algebra of a* Σ_1*-sound theory* T *if and only if it can be embedded into some positive diagonalizable algebra with the strong disjunction property.*

Proof. Any positive diagonalizable algebra can by Theorem 7.1 be embedded into $\mathfrak{D}_{\mathrm{T}}$ and hence we can also embed $\mathfrak{D}$ into $\mathfrak{D}_{\mathrm{T}}$ by just taking the composition of embeddings. ■

Thus in contrast to Corollary 10.14 we have

11.12. COROLLARY. *There exists a locally positive diagonalizable algebra with the strong disjunction property which is not embeddable into any positive diagonalizable algebra with the strong disjunction property.* ■

Proposition 11.11 does not constitute a satisfactory characterization of subalgebras of diagonalizable algebras of Σ_1-sound theories. One would of course want something more informative.

So the problem of characterizing subalgebras of diagonalizable algebras of Σ_1-sound theories T remains open. It appears that a locally positive strongly disjunctive algebra $\mathfrak{D}$ can be embedded in $\mathfrak{D}_{\mathrm{T}}$ if the generators of $\mathfrak{D}$ abstain from trying to bury one another in increasingly complex disjunction problems. Thus the free product of a countable number of positive diagonalizable algebras with the strong disjunction property is always embeddable in $\mathfrak{D}_{\mathrm{T}}$.

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Added in proof (January 1993). Andrew M. Pitts proves the Uniform Craig Interpolation Lemma for Intuitionistic Propositional Logic by cut-eliminationary methods (A. M. Pitts, *On an interpretation of second order quantification in first order propositional logic,* J. Symbolic Logic 57 (1992), 33–52).

Domenico Zambella has answered the one remaining question of the present paper about arbitrary subalgebras of diagonalizable algebras of Σ_1-sound theories. He has shown that a denumerable diagonalizable algebra $\mathfrak{D}$ is embeddable into $\mathfrak{D}_{\mathrm{T}}$ for T a Σ_1-sound theory if and only if $\mathfrak{D}$ is locally positive and *uniformly strongly disjunctive*, that is, there exists a locally positive numeration ν of $\mathfrak{D}$ and, for each $i \in \omega$, recursive sequences $\{D_i(m)\}_{m\in\omega}$ of diagonalizable polynomials in $\nu 1, \ldots, \nu i$ exhausting the unit of $\mathfrak{D}$ modulo $\vdash_{\mathbf{L}}$, such that for all $i, m \in \omega$, $D_i(m+1) \vdash_{\mathbf{L}} D_i(m)$, $D_{i+1}(m) \vdash_{\mathbf{L}} D_i(m)$ and $D_i(m)$ is steady (yet unpublished). Moreover, he has shown that in all our embeddability results the requirements on the strength of the theory T can be weakened down to $\mathrm{I}\Delta_0 + \exp$ (D. Zambella, *Shavrukov's theorem on the subalgebras of diagonalizable algebras for theories containing* $\mathrm{I}\Delta_0 + \exp$, ILLC Prepublication Series ML–92–05, Institute for Logic, Language and Computation, University of Amsterdam, 1992).