ACTA ARITHMETICA
VI (1961)

Remarks on the number of factors of an odd perfect number

by

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I. Introduction. It is still unknown whether or not odd perfect numbers exist, but many necessary conditions for their existence have been established. Among the most interesting of these are a theorem of Kühnel ([1]) that an odd perfect number N must have at least six different prime factors, and a result published by Kanold ([2]) stating that $N > 10^{20}$.

In 1888, C. Servais ([3]) showed that if $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ is an odd perfect number with $p_1 < p_2 < \dots < p_k$, then $k \geqslant p_1$. This result was refined by Grün ([4]), who gave the inequality $k > \frac{2}{3}p_1 - 3$. The object of this paper is to make a considerable improvement in Grün's inequality for large p_1 and to give some general theorems concerning the number of prime factors of N as a function of its smallest factor. Other theorems of interest (including two on the size of N) will be given, and a table of numerical results will be included.

We shall let P_n denote the nth prime number, where $P_1 = 2$. The symbol N always denotes an odd perfect number. The function $\operatorname{li}(x)$ is the familiar logarithmic integral:

$$\operatorname{li}(x) = \int_0^x \frac{dt}{\log t} = \lim_{\varepsilon \to 0+0} \left\{ \int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \right\} \frac{dt}{\log t}.$$

We shall commonly write $\log^n f(x)$ in place of $[\log f(x)]^n$. We let $\Theta(x) = \sum_{p \leqslant x} \log p$, where p runs through the primes not exceeding x, and $\pi(x)$ will denote the number of primes not exceeding x.

Now suppose that $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ is perfect, where $3 \leqslant p_1 < p_2 < \dots < p_k$, so

$$(1) \qquad 2 = \prod_{r=1}^{k} \frac{p_r^{a_r+1} - 1}{p_r^{a_r}(p_r - 1)} < \prod_{r=1}^{k} \frac{p_r}{p_r - 1}.$$

If $p_1 = P_n$, then it is plain that (1) implies

$$2 < \prod_{r=n}^{n+k-1} \frac{P_r}{P_r - 1}. \tag{2}$$

Let the function $\alpha(n)$ be defined for $n \geqslant 2$ by the following double inequality:

$$\prod_{r=n}^{n+\alpha(n)-2} \frac{P_r}{P_r - 1} < 2 < \prod_{r=n}^{n+\alpha(n)-1} \frac{P_r}{P_r - 1}. \tag{3}$$

From (1), (2), and (3), it follows that if P_n is the smallest prime factor of N, then N has at least $\alpha(n)$ different prime factors. Also, N must have a prime factor at least as large as $P_{n+\alpha(n)-1} = P_s$. (Throughout the remainder of this paper, the letter $s = s(n)$ will represent $n + \alpha(n) - 1$). The calculation of the exact values of $\alpha(n)$ and P_s for $2 \leqslant n \leqslant 100$ was performed by ILLIAC (the University of Illinois Automatic Digital Computer). These values are found in the table at the end of this paper. The table has four columns, giving the numbers n, P_n, $\alpha(n)$, and P_s, respectively.

I am deeply indebted to my father, Professor H. W. Norton, who spent many hours of careful work preparing the program for ILLIAC. I am also especially grateful to Professor Paul T. Bateman, who read and criticized the manuscript, and whose suggestions (particularly for the proof of Theorem 4) were most helpful.

II. Auxiliary theorems. Some lemmas due to Rosser ([5], [9]) are indispensable in obtaining our results.

LEMMA 1. *$P_n > n\log n$ for all $n \geqslant 1$.*

LEMMA 2. (a). *If $17 \leqslant x \leqslant e^{100}$, or if $x \geqslant e^{2000}$, then $x/\log x < \pi(x)$.*

(b). *If $6 \leqslant n \leqslant e^{95}$, then $P_n < n\log n + n\log\log n$.*

(c). *For $x \geqslant 2$, we have*

$$\left(1 - \frac{2.85}{\log x}\right)x < \Theta(x) < \left(1 + \frac{2.85}{\log x}\right)x.$$

For $1 < x \leqslant e^{100}$, $\Theta(x) < \left(1 + \frac{1}{\log x}\right)x$, and if $41 \leqslant x \leqslant e^{100}$, then $\left(1 - \frac{1}{\log x}\right)x < \Theta(x)$. If $x \geqslant e^{2000}$, then

$$\left(1 - \frac{1}{\log x}\right)x < \Theta(x) < \left(1 + \frac{1}{\log x}\right)x.$$

III. Main results. We now wish to study the function $\alpha(n)$ in some detail.

THEOREM 1. $\alpha(n) > n^2 - 2n - \frac{n+1}{\log n} - \frac{5}{4} - \frac{1}{2n} - \frac{1}{4n\log n}$.

Proof. Using (3) and Lemma 1, we have

$$\log 2 < \log\left(\frac{P_n}{P_n - 1}\right) + \sum_{r=n+1}^{s}\left(\frac{1}{P_r} + \frac{1}{2P_r^2} + \frac{1}{3P_r^3} + \ldots\right)$$

$$< \log\left(\frac{P_n}{P_n - 1}\right) + \sum_{r=n+1}^{s}\frac{1}{P_r} + \frac{1}{2}\sum_{r=n+1}^{s}\left(\frac{1}{P_r^2} + \frac{1}{P_r^3}\right)$$

$$< \log\left(\frac{P_n}{P_n - 1}\right) + \int_{n+1/2}^{s+1/2}\frac{dx}{x\log x} + \frac{1}{2\log^2(n+1)}\cdot\int_n^s\frac{dx}{x^2} + \frac{1}{2\log^3(n+1)}\cdot\int_n^s\frac{dx}{x^3},$$

so

$$\log(s+\tfrac{1}{2}) > 2\log(n+\tfrac{1}{2})\left(1 - \frac{1}{n\log n}\right)\left\{1 - \frac{1}{2n\log n\log(n+\frac{1}{2})} - \frac{1}{4n^2\log^2 n\log(n+\frac{1}{2})}\right\}$$

$$> 2\log(n+\tfrac{1}{2}) - \frac{2}{n} - \frac{1}{n\log n} - \frac{1}{n^2\log n}.$$

It follows that

$$s + \tfrac{1}{2} > (n+\tfrac{1}{2})^2\left(1 - \frac{2}{n} - \frac{1}{n\log n}\right) = n^2 - n - \frac{n+1}{\log n} - \frac{7}{4} - \frac{1}{2n} - \frac{1}{4n\log n}.$$

This proves the theorem.

THEOREM 2. *Suppose that $t\left(1 - \frac{a}{\log t}\right) < \Theta(t) < t\left(1 + \frac{b}{\log t}\right)$ for $P_n \leqslant t \leqslant P_s$, where a and b are constants and $0 < a \leqslant 3$, $0 < b \leqslant 3$. Then*

$$P_s > e^{-b}P_n^2\left\{1 - \frac{4a + 3b + b^2}{2\log P_n} - \frac{4\log P_n}{P_n}\right\}.$$

In particular,

$$P_s > e^{-b}P_n^2\left\{1 - \frac{4a + 3b + b^2}{2\log 547} - \frac{4\log 547}{547}\right\}, \quad \text{for} \quad n \geqslant 4.$$

Proof. The second inequality follows from the first, since $P_s > P_n^2$ for $4 \leqslant n \leqslant 100$ (see the table of numerical results), and $P_{101} = 547$.

Now, for $r \geqslant 2$,

$$\log\left(\frac{P_r}{P_r - 1}\right) < \frac{1}{P_r} + \frac{1}{P_r^2},$$

so

$$\log 2 < \log \prod_{r=n}^{s} \frac{P_r}{P_r - 1} < \sum_{r=n}^{s} \frac{1}{P_r} + \sum_{r=n}^{\infty} \frac{1}{P_r^2} < \int_{P_n}^{P_s} \frac{d\Theta(t)}{t \log t} + \frac{2}{P_n}$$

$$= \frac{2}{P_n} + \frac{\Theta(P_s)}{P_s \log P_s} - \frac{\Theta(P_n)}{P_n \log P_n} + K(n),$$

where

$$K(n) = \int_{P_n}^{P_s} \frac{\Theta(t)(\log t + 1)\,dt}{t^2 \log^2 t} < \log\log P_s - \log\log P_n +$$

$$+ \frac{1+b}{\log P_n} - \frac{1+b}{\log P_s} + \frac{b}{2\log^2 P_n} - \frac{b}{2\log^2 P_s}.$$

Hence,

$$\log\log P_s - \frac{b}{\log P_s} + \frac{b}{2\log^2 P_s} > \log\log P_n + \log 2 - \frac{b}{\log P_n} - \frac{2a+b}{2\log^2 P_n} - \frac{2}{P_n}.$$

Now, $e^{-x} < 1 - x + x^2/2$ for $0 < x$. Since $0 < b \leqslant 3$ and $P_s \geqslant 7$, it follows that

$$0 < \frac{b}{\log P_s} - \frac{b}{2\log^2 P_s},$$

so we have

$$\log P_s - b + \frac{b+b^2}{2\log P_s} > (\log P_s) \exp\left(-\frac{b}{\log P_s} + \frac{b}{2\log^2 P_s}\right)$$

$$> (2\log P_n)\left\{1 - \frac{b}{\log P_n} - \frac{2a+b}{2\log^2 P_n} - \frac{2}{P_n}\right\},$$

and since

$$\frac{b+b^2}{2\log P_s} < \frac{b+b^2}{2\log P_n},$$

it follows that

$$P_s > e^{-b} P_n^2 \exp\left(-\frac{4a+3b+b^2}{2\log P_n} - \frac{4\log P_n}{P_n}\right)$$

$$> e^{-b} P_n^2 \left\{1 - \frac{4a+3b+b^2}{2\log P_n} - \frac{4\log P_n}{P_n}\right\}.$$

COROLLARY. *For $2 \leqslant n < e^{42}$, and for $n > e^{1993}$, we have*

$$P_s > \frac{P_n^2}{e}\left\{1 - \frac{4}{\log P_n} - \frac{4\log P_n}{P_n}\right\}.$$

Proof. Using Lemma 2(b), it is not hard to show that $n < e^{42}$ implies $s < e^{95}$ and $P_s < e^{100}$. Also, for $n > e^{1993}$, $P_n > n \log n > e^{2000}$. We now apply Lemma 2(c) and Theorem 2, noting that the result is obvious for $2 \leqslant n \leqslant 12$.

Recent unpublished theorems of Rosser and Schoenfeld (with better inequalities for $\Theta(x)$) indicate that the inequality of the Corollary can be improved in various ways. For example,

$$P_s > \frac{P_n^2}{\sqrt{e}}\left\{1 - \frac{15}{8\log P_n} - \frac{4\log P_n}{P_n}\right\}$$

for all $n \geqslant 2$. Using these results and Lemma 2(a), various inequalities for $a(n)$ can easily be deduced.

We wish to make some remarks on the size of N. It was proved by Euler that any odd perfect number N can be written in the form

$$N = p^a q_1^{2b_1} q_2^{2b_2} \ldots q_m^{2b_m}, \tag{4}$$

where $p \equiv a \equiv 1 \pmod 4$. By two theorems of Kanold ([7], [8]), N is not perfect if $2b_1 < 10$ and $b_2 = b_3 = \ldots = b_m = 1$, or if $b_1 = b_2 = 2$ and $b_3 = b_4 = \ldots = b_m = 1$. From this, we obtain without difficulty the following lemma:

LEMMA 3. *If P_n is the smallest prime factor of N, then*

$$N \geqslant P_n^6 P_{n+1}^4 (P_{n+2} P_{n+3} \ldots P_{s-1})^2 P_s.$$

Taking logarithms, we get $\log N \geqslant 2\Theta(P_s) - 2\Theta(P_n) + 6\log P_n + 2\log P_{n+1} - \log P_s$. Using Lemma 2(c), the proof of the Corollary, the table of numerical results, and Kanold's theorem that $N > 10^{20}$, we have

THEOREM 3. *If P_n is the smallest factor of N, where $2 \leqslant n < e^{42}$ or $n > e^{1993}$, then*

$$\log N > 2P_s\left(1 - \frac{1}{\log P_s}\right) - 2P_n\left(1 + \frac{1}{\log P_n}\right) + 6\log P_n + 2\log P_{n+1} - \log P_s.$$

Using the recent results of Rosser and Schoenfeld, this inequality can be improved to read

$$\log N > 2P_s\left(1 - \frac{1}{2\log P_s}\right) - 2P_n\left(1 + \frac{1}{2\log P_n}\right) + 6\log P_n + 2\log P_{n+1} - \log P_s$$

for all $n \geqslant 9$.

We now obtain a result giving the true order of magnitude of $\alpha(n)$ and P_s. From this we can easily get a function which is a lower bound for $\log N$. The statement and proof of the following theorem are due to Professor Paul T. Bateman, who communicated to the author his improvements of the author's proof of a similar theorem.

THEOREM 4. *Let N be an odd perfect number with smallest prime factor P_n, and let b be any number less than $\frac{4}{7}$. Then:*

(1) *N has at least $\alpha(n)$ different prime factors, where*

$$\alpha(n) = \mathrm{li}(P_n^2)+O(n^2e^{-\log^b n}).$$

(2) *N has a prime factor at least as large as*

$$P_s = P_n^2+O(n^2e^{-\log^b n}).$$

(3) $\log N > 2P_n^2+O(n^2e^{-\log^b n})$.

Proof. It is known that $\Theta(x) = x+\Delta(x)$, where $\Delta(x) = O(xe^{-\log^a x})$ and a is any number less than $\frac{4}{7}$ (see Prachar [10]). Hence,

$$\sum_{p\leqslant x}\frac{1}{p} = \frac{1}{2}+\int_e^x\frac{d\Theta(t)}{t\log t} = \frac{1}{2}+\int_e^x\frac{dt+d\Delta(t)}{t\log t}$$

$$= \log\log x+\frac{\Delta(x)}{x\log x}+B-\int_x^\infty \Delta(t)\left\{\frac{1}{t^2\log t}+\frac{1}{t^2\log^2 t}\right\}dt$$

$$= \log\log x+B+O\left(\frac{e^{-\log^a x}}{\log x}\right)+O\left(\int_x^\infty\frac{e^{-\log^a t}}{t\log t}\,dt\right),$$

where B is a constant. Let $b < a < \frac{4}{7}$. Then

$$\int_x^\infty\frac{e^{-\log^a t}}{t\log t}\,dt = O(e^{-\log^b x}).$$

Therefore,

$$\sum_{p\leqslant x}\frac{1}{p} = \log\log x+B+O(e^{-\log^a x})\quad\text{for all}\quad a<\tfrac{4}{7}.$$

It follows that

$$\sum_{p\leqslant x}-\log\left(1-\frac{1}{p}\right) = \sum_{p\leqslant x}\frac{1}{p}+\sum_p\left(-\log\left(1-\frac{1}{p}\right)-\frac{1}{p}\right)+O\left(\frac{1}{x}\right)$$

$$= \log\log x+c+O(e^{-\log^a x}),$$

where c is a constant. Thus, for $h = o(x^2)$,

$$\sum_{x\leqslant p\leqslant x^2+h}-\log\left(1-\frac{1}{p}\right) = \log\log(x^2+h)-\log\log x+O(e^{-\log^a x}).$$

Let $h = x^2e^{-\log^b x}$, where $b<a<\frac{4}{7}$. Then

$$\sum_{x\leqslant p\leqslant x^2+h}-\log\left(1-\frac{1}{p}\right) = \log 2+\frac{e^{-\log^b x}}{2\log x}+O\left(\frac{e^{-2\log^b x}}{\log x}\right)+O(e^{-\log^a x}) > \log 2$$

for all sufficiently large x. Also, if $h = -x^2e^{-\log^b x}$, where $b<a<\frac{4}{7}$, then

$$\sum_{x\leqslant p\leqslant x^2+h}-\log\left(1-\frac{1}{p}\right) = \log 2-\frac{e^{-\log^b x}}{2\log x}+O\left(\frac{e^{-2\log^b x}}{\log x}\right)+O(e^{-\log^a x}) < \log 2$$

for all sufficiently large x. Hence, if $g(x)$ is the smallest number such that

$$\sum_{x\leqslant p\leqslant g(x)}-\log\left(1-\frac{1}{p}\right)\geqslant\log 2,$$

then $g(x) = x^2+O(x^2e^{-\log^a x})$ for all $a<\frac{4}{7}$. Therefore, if $s = n+\alpha(n)-1$ is the smallest integer such that

$$\prod_{r=n}^{s}\frac{P_r}{P_r-1} > 2,$$

then $\displaystyle\sum_{P_n\leqslant p\leqslant P_s}-\log\left(1-\frac{1}{p}\right)>\log 2$, so $P_s = P_n^2+O(P_n^2e^{-\log^a(P_n^2)})$, or $P_s = P_n^2+O(n^2e^{-\log^b n})$ for any $b<a$. This proves statement (2) of the theorem.

Now, it is known that $\pi(x) = \mathrm{li}(x)+O(xe^{-\log^a x})$ for all $a<\frac{4}{7}$ (see Prachar [10]). Hence,

$$\begin{aligned}\alpha(n) &= s-n+1 = \pi[P_n^2+O(n^2e^{-\log^b n})]-n+1\\ &= \pi(P_n^2)+O(n^2e^{-\log^b n})\\ &= \mathrm{li}(P_n^2)+O(P_n^2e^{-\log^a(P_n^2)})+O(n^2e^{-\log^b n})\\ &= \mathrm{li}(P_n^2)+O(n^2e^{-\log^b n}),\end{aligned}$$

where we have assumed as before that $b<a<\frac{4}{7}$. This proves part (1) of the theorem.

Finally, if the odd perfect number N has smallest prime factor P_n, then (cf. Lemma 3)

$$\log N > \log \prod_{r=n}^{s-1} P_r^2 = 2\Theta(P_{s-1}) - 2\Theta(P_{n-1}) = 2\Theta(P_s) + O(P_n)$$

$$= 2P_s + O(P_s e^{-\log^a P_s}) + O(P_n) = 2P_n^2 + O(n^2 e^{-\log^b n}).$$

The proof is complete.

Using known results (see Landau [6]) for P_n as a function of n, we can rewrite parts (1) and (2) in the (weaker) form

$$(5)\qquad a(n) = \frac{1}{2} n^2 \log n + \frac{1}{2} n^2 \log\log n - \frac{3}{4} n^2 + \frac{n^2 \log\log n}{2\log n} + O\left(\frac{n^2}{\log n}\right)$$

$$= \frac{1}{2} n P_n - \frac{1}{4} n^2 + O\left(\frac{n^2}{\log n}\right);$$

$$(6)\qquad P_s = n^2 \log^2 n + 2n^2 \log n \log\log n - 2n^2 \log n + n^2 \log^2 \log n + O(n^2).$$

The inequality $N > \exp[2P_n^2 + O(n^2 e^{-\log^b n})]$, while rather crude, serves to show the rapidity with which N increases as a function of n (or P_n).

IV. Numerical results

n	P_n	$a(n)$	P_s	n	P_n	$a(n)$	P_s
2	3	3	7	23	83	1010	8231
3	5	7	23	24	89	1114	9173
4	7	15	61	25	97	1222	10151
5	11	27	127	26	101	1331	11197
6	13	41	199	27	103	1448	12343
7	17	62	337	28	107	1572	13487
8	19	85	479	29	109	1704	14779
9	23	115	677	30	113	1845	16097
10	29	150	937	31	127	1994	17599
11	31	186	1193	32	131	2138	19087
12	37	229	1511	33	137	2289	20563
13	41	274	1871	34	139	2445	22109
14	43	323	2267	35	149	2609	23761
15	47	380	2707	36	151	2774	25469
16	53	443	3251	37	157	2948	27259
17	59	509	3769	38	163	3127	29123
18	61	577	4349	39	167	3311	31081
19	67	653	5009	40	173	3502	33029
20	71	733	5711	41	179	3699	35081
21	73	818	6451	42	181	3900	37199
22	79	912	7321	43	191	4112	39461

n	P_n	$a(n)$	P_s	n	P_n	$a(n)$	P_s
44	193	4324	41761	73	367	13215	143329
45	197	4546	44101	74	373	13611	148147
46	199	4775	46643	75	379	14014	152777
47	211	5016	49253	76	383	14423	157721
48	223	5255	51827	77	389	14841	162823
49	227	5493	54449	78	397	15266	168067
50	229	5738	57073	79	401	15696	173263
51	233	5993	59779	80	409	16134	178609
52	239	6255	62861	81	419	16576	183919
53	241	6524	65837	82	421	17021	189257
54	251	6804	69019	83	431	17477	194827
55	257	7084	72101	84	433	17935	200383
56	263	7371	75367	85	439	18404	206249
57	269	7663	78691	86	443	18880	211873
58	271	7961	82003	87	449	19364	218083
59	277	8270	85577	88	457	19856	224057
60	281	8585	89237	89	461	20353	230059
61	283	8909	92831	90	463	20858	236549
62	293	9243	96643	91	467	21376	242971
63	307	9579	100559	92	479	21902	249383
64	311	9913	104491	93	487	22429	255961
65	313	10254	108497	94	491	22962	262807
66	317	10607	112601	95	499	23503	269387
67	331	10969	116911	96	503	24049	276251
68	337	11328	121039	97	509	24605	283207
69	347	11693	125303	98	521	25168	290489
70	349	12059	129581	99	523	25732	297797
71	353	12437	134129	100	541	26308	304961
72	359	12823	138581				

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Reçu par la Rédaction le 20. 3. 1960

ACTA ARITHMETICA
VI (1961)

Criteria for Kummer's congruences

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1. Kummer ([2]) obtained the well-known congruence for the Euler numbers

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}E_{n+s(p-1)}\equiv 0\,(\mathrm{mod}\,p^r)\qquad (n\geqslant r,\ p>2)$$

by means of the following general result. Let p be a fixed prime $\geqslant 2$. Then if

$$\sum_{n=0}^{\infty}a_n\frac{x^n}{n!}=\sum_{k=0}^{\infty}A_k(e^x-1)^k, \tag{1.1}$$

where the a_n, A_k are integral (mod p), it follows that

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}a_{n+s(p-1)}\equiv 0\,(\mathrm{mod}\,p^r)\qquad (n\geqslant r). \tag{1.2}$$

Indeed since (1.1) is equivalent to

$$a_n=\sum_{k=0}^{n}A_kC_n^{(k)},\qquad \text{where}\qquad C_n^{(k)}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n,$$

we have

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}a_{n+s(p-1)}=\sum_{k=0}^{n+r(p-1)}A_k\sum_{s=0}^{r}(-1)^s\binom{r}{s}C_{n+s(p-1)}^{(k)}$$
$$=\sum_{k=0}^{n+r(p-1)}A_k\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n(1-j^{p-1})^r$$

and (1.2) follows by Fermat's theorem.