The set $z+\Gamma_1$ is a closed convex curve, and it intersects with Δ only several times. We consider one of such intersections of $z+\Gamma_1$ and Δ. This intersection is an arc of the curve $z+\Gamma_1$. We denote the end points of this arc $\tilde{\Gamma}$ by $z+z_1(\theta_0)$ and $z+z_1(\theta_0')$: $\tilde{\Gamma}=\{z+z_1(\theta_1)|\ \theta_0\leqslant\theta_1\leqslant\theta_0'\}$. We can easily show that $\operatorname{dist}(z+z_1(\theta_0), z+z_1(\theta_0'))\ll\delta_N^{1/2}$ in a similar (in fact, simpler) way as in the proof of Lemma 8. Hence, by Lemma 9, we have $|\theta_0'-\theta_0|\ll\delta_N^{1/2}$, and therefore, $\Theta(z)\ll\delta_N^{1/2}$. Hence, by (9.1), we obtain

$$|W_N(R_i)-W_N(R)|\ll\delta_N^{1/2}\ll N^{(1-\sigma_0)/2}(\log N)^{-\sigma_0/2}.$$

This implies Proposition 2, since the same estimate holds for $|W_N(R_y)-W_N(R)|$. The proof of our theorem is thus completed.

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Pisot sequences which satisfy no linear recurrence II

by

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Introduction. In this paper we continue our study of Pisot sequences begun in [1]. Recall that the Pisot sequence $E(a_0, a_1)$ is the sequence of integers defined for $0<a_0<a_1$ by

$$-1/2<a_{n+1}-a_n^2/a_{n-1}\leqslant 1/2. \tag{1}$$

In [1] we proved that there are Pisot sequences satisfying no linear recurrence relation. Our proof made use of an inequality from Pisot's thesis [5]. We recently discovered that the constant in this inequality is incorrect. Since it is used at three separate points in [1], it would appear that many of the details in [1] need to be modified.

Our first purpose here is to show that all the theorems of [1] are correct as stated and to indicate the necessary changes in the proofs. To do this, we prove a number of new inequalities for Pisot sequences. Since these should be useful in other investigations we give more general versions than needed to simply repair the proofs of [1].

Our second purpose is to sketch a simplified proof of the main Theorem 4 of [1], avoiding the use of the Kronecker–Weyl theorem. Combining this proof with results from [2] shows, for example, that none of $E(1089m, 1782m)$ satisfy a linear recurrence for any odd m.

1. The new inequalities. The notation will be as in [1].

If $\{a_n\}=E(a_0, a_1)$, write $\theta_n=a_{n+1}/a_n$ and $\varphi_n=\inf\{\theta_m: m\geqslant n\}$ (misprinted in [1] as "sup"). We write $\theta(a_0, a_1)=\theta=\lim\theta_n$ which always exists. We are interested only in $\theta>1$ for which it is necessary and sufficient that $a_1\geqslant a_0+\sqrt{a_0/2}$, according to results of Pisot [5] and Flor [4]. Let $\lambda=\lim a_n/\theta^n>0$, and define $\varepsilon_n=\lambda\theta^n-a_n$.

LEMMA 1. *For all* $n\geqslant 0$,

$$|\theta-\theta_n|\leqslant 1/(2a_n(\varphi_n-1)), \tag{2}$$

$$|\varepsilon_n|\leqslant 1/(2(\theta-1)(\varphi_n-1)). \tag{3}$$

Proof. By (1), $|\theta_{m+1}-\theta_m| \leqslant 1/(2a_{m+1})$ and clearly

$$a_{m+1} = \theta_m \ldots \theta_n a_n \geqslant \varphi_n^{m-n+1} a_n.$$

Thus

$$|\theta-\theta_n| \leqslant \sum_{m=n}^{\infty} 1/(2a_{m+1}) \leqslant 1/(2a_n(\varphi_n-1)),$$

giving (2).

Now write (2) as

$$|a_n\theta^{-n}-a_{n+1}\theta^{-(n+1)}| \leqslant \theta^{-(n+1)}/(2(\varphi_n-1))$$

and sum to obtain

$$|a_n\theta^{-n}-\lambda| \leqslant \theta^{-n}/(2(\theta-1)(\varphi_n-1)),$$

which gives (3).

Remark. In [5], Pisot claims that if $a_1 \geqslant a_0+2\sqrt{a_0}$ then

(4) $$|\theta-\theta_n| \leqslant 1/(2(a_{n+1}-a_n)).$$

This is a very plausible result in view of (2) since we can write

$$a_{n+1}-a_n = a_n(\theta_n-1).$$

Notice that (2) gives $\theta_n = \theta+O(\theta^{-n})$ and hence

$$\varphi_n = \theta+O(\theta^{-n}).$$

However, Pisot's proof gives only

(5) $$|\theta-\theta_n| \leqslant 1/(a_{n+1}-a_n).$$

Although the proof given for (4) is thus faulty, it is quite conceivable that the result itself is correct. Our results (8) and (9) come close to (4). We note that (9) is true without any assumption on a_0, a_1 beyond $a_1 > a_0$.

LEMMA 2. *Suppose m is an integer and that $m < \theta_0 < m+1$. Then $m < \theta_n < m+1$ for all n. Also $m < \theta < m+1$ except when $m = 1$ in which case $\theta = 1$ is possible.*

Proof. (a) If $\theta_0 = a_1/a_0 > m$ then $a_1 \geqslant ma_0+1$. By induction, using (1), we obtain

(6) $$a_{n+1} \geqslant ma_n+m^n.$$

Hence $\theta_n > m$ and $\theta \geqslant m$.

If $m \geqslant 2$ we wish to show $\theta > m$. Suppose then that $\theta = m$ and hence $a_n/m^n \to \lambda > 0$. Then (6) gives

$$\theta_n \geqslant m+m^n/a_n \to m+1/\lambda > 0,$$

a contradiction:

(b) If $m < \theta_0 < m+1$ then $a_1 \leqslant (m+1)a_0-1$. Again, by induction,

(7) $$a_{n+1} \leqslant (m+1)a_n-m^n,$$

so $\theta_{n+1} < m+1$ and $\theta \leqslant m+1$.

If $\theta = m+1$ then (7) gives $a_n(\theta-\theta_n) \geqslant m^n \geqslant 1$ which contradicts (2) since $2(\varphi_n-1) \to 2m$ as $n \to \infty$.

COROLLARY 3. *If $\theta \geqslant 2$ then $|\varepsilon_n| \leqslant 1/2$ for all n.*

Proof. By Lemma 2, if $\theta_m < 2$ for any m, then $\theta_n < 2$ for all $n \geqslant m$ and $\theta < 2$. Thus $\theta \geqslant 2$ implies $\theta_n \geqslant 2$ for all n and hence $\varphi_n \geqslant 2$ for all n. Now use (3).

Remark. Lemma 1 and Corollary 3 comprise Lemma 1 of [1]. In order for (2) to be useful when $\theta_0 < 2$, we need a lower bound for φ_n. One such result is the following:

LEMMA 4. *Let $a_{n+1} \geqslant a_n+\sqrt{2a_n}$. Then $\varphi_n \geqslant \psi_n > 1$ where ψ_n is the larger root of*

$$(\psi_n-1)(\theta_n-\psi_n) = 1/(2a_n).$$

Proof. Clearly $\theta_n = \psi_n+1/(2a_n(\psi_n-1)) > \psi_n$. We claim that $\theta_m > \psi_n$ for $m \geqslant n$. The proof is by induction on m. Assume then that $\theta_k > \psi_n$ for $n \leqslant k \leqslant m$ and then, as in the proof of (2),

$$|\theta_{k+1}-\theta_k| \leqslant 1/(2a_{k+1}) < 1/(2a_n\psi_n^{k-n+1}), \qquad n \leqslant k \leqslant m.$$

Summing over k, we have

$$|\theta_{m+1}-\theta_n| < 1/(2a_n(\psi_n-1)) = \theta_n-\psi_n$$

which proves $\theta_{m+1} > \psi_n$, completing the induction. Thus $\varphi_n = \inf\{\theta_m: m \geqslant n\} \geqslant \psi_n$.

COROLLARY 5. *If $\theta_n \geqslant 3/2+1/a_n$ then*

(8) $$|\theta-\theta_n| \leqslant 1/(2(a_{n+1}-a_n-1)).$$

Proof. Let $P(x) = (x-\theta_n)(x-1)+1/(2a_n)$, so $P(\psi_n) = 0$ and $P(\theta_n-1/a_n) < 0$ hence $\psi_n > \theta_n-1/a_n$. Thus $a_n(\varphi_n-1) \geqslant a_n(\psi_n-1) > a_n(\theta_n-1-1/a_n) = a_{n+1}-a_n-1$. Now apply (2).

LEMMA 6. *Suppose $\theta_n \leqslant \alpha$ and $\beta = \alpha+\alpha/(2a_{n+1}(\alpha-1))$ then $\theta < \beta$.*

Proof. If $\theta \leqslant \alpha$ then certainly $\theta < \beta$.

On the other hand, if $\theta > \alpha$ then there is a largest $m \geqslant n$ so that $\theta_m \leqslant \alpha$ but $\theta_k > \alpha$ if $k > m$.

As in the proof of Lemma 1,

$$|\theta-\theta_m| < \alpha/(2a_{m+1}(\alpha-1)) \leqslant \alpha/(2a_{n+1}(\alpha-1))$$

which proves the lemma since $\theta_m \leqslant \alpha$.

Corollary 7. *For any* $0 < a_0 < a_1$,

(9) $$\theta - \theta_n < 1/(2(a_{n+1} - a_n)).$$

Proof. Take $\alpha = \theta_n$ in Lemma 6.

Corollary 8. *Suppose* $\theta \geqslant 1.6$ *and* $a_1 \geqslant 15$. *Then* $\theta_n \geqslant \theta - 1/10$ *and* $|\varepsilon_n| \leqslant 5/3$ *for all* n.

Proof. If $\theta_n \leqslant \theta - 1/10 = \alpha$ then Lemma 6 gives $\theta < \theta - 1/10 + 3/(2a_{n+1})$ which implies $a_1 \leqslant a_{n+1} < 15$, contrary to assumption. Now use (3) and $\varphi_n \geqslant \theta - 1/10 \geqslant 3/2$ and $\theta - 1 \geqslant 3/5$ to prove $|\varepsilon_n| \leqslant 5/3$.

2. Corrections to the proofs of [1]. The unproved (4) was used in three places in [1]. Two of these are in the proof of Lemma 1 of [1] and, as already mentioned, are handled by Lemma 1 and Corollary 3 of Section 1.

The third use of (4) is in the final paragraph of the proof of Theorem 1. This is avoided by the use of Corollary 8. Note that $a_1 \geqslant 2 + 13/(\theta^2 - \theta - 1)$ and $\theta \leqslant 2$ imply $a_1 \geqslant 15$, explaining the peculiar choice of constants in Corollary 8.

3. A new proof of the main theorem of [1]. We use the following criterion for T-recurrence which differs slightly from that stated in Theorem 3 of [1]. The same techniques as used there give this result if one uses the improved inequalities of Section 1.

Lemma 9. *If* $E(a_0, a_1)$ *satisfies a pure T-recurrence then*

(10) $$\|a_0(a_0 + a_2)/a_1\| \leqslant (1+\theta)/(2\theta^2) + 1/a_1.$$

Theorem. *The set of limits* $\theta(a_0, a_1)$ *corresponding to non-recurrent E-sequences is dense in* $[\tau, \infty)$, *where* $\tau = (\sqrt{5}+1)/2$.

Proof. Let (α, β) be an interval in the complement of the Pisot numbers, and let p/q be any rational number in this interval with p even and q odd. Let m be an odd multiple of $p/2$. Consider $E(a_0, a_1)$ with $a_0 = mq^2$, $a_1 = mpq$ and hence $a_2 = mp^2$.

Then
$$a_0(a_0 + a_2)/a_1 = mq(q^2+p^2)/p \equiv 1/2 \pmod 1.$$

But $1/2 > (1+\theta)/(2\theta^2) + 1/a_1$ if $\theta > \tau$ and m is sufficiently large.

Theorem 1 of [1] and Lemma 9 thus show that $E(a_0, a_1)$ is not T-recurrent for large m. Since $\theta(a_0, a_1) \to p/q \notin S$ as $m \to \infty$, $E(a_0, a_1)$ is not S-recurrent for large m either, hence is non-recurrent. The set of rationals considered is dense in $[\tau, \infty)$, which completes the proof.

Corollary 10. $E(1089m, 1782m) = E(9 \cdot 11^2 \cdot m, 2 \cdot 9^2 \cdot 11 \cdot m)$ *does not satisfy a linear recurrence relation for any odd* m.

Proof. As in the proof of the theorem, $a_2 = 4 \cdot 9^3 \cdot m$ and
$$a_0(a_0 + a_2)/a_1 \equiv 1/2 \pmod 1.$$

The nearest Pisot numbers to $\theta_0 = 18/11$ are, by [3], $\alpha = 1.6326907332$ with minimal polynomial $z^8(z^2-z-1)-(z^2-1)$ and $\beta = 1.6407279391$ with minimal polynomial $(z^6(z^2-z-1)-1)/(z+1)$ (so $\alpha = \hat{\theta_7}$ and $\beta = \theta_6$ in the terminology of [3]).

By Corollary 5, $|\theta - \theta_0| < 1/(1384m)$ showing that

$$\alpha < \theta < \beta \quad \text{if} \quad m > 5.$$

For $m = 1, 3$ and 5 this can be verified by calculation of $\theta(a_0, a_1)$. Hence $\theta \notin S$ so $E(a_0, a_1)$ is not S-recurrent for any m.

Since $a_1 \geqslant 2 + 13/(\theta^2 - \theta - 1)$ and

$$\|a_0(a_0 + a_2)/a_1\| = 1/2 > (1+\alpha)/(2\alpha^2) + 1/a_1 > (1+\theta)/(2\theta^2) + 1/a_1,$$

$E(a_0, a_1)$ is not T-recurrent either.

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