ACTA ARITHMETICA
XLVII (1986)

On linear recurrence relations satisfied by Pisot sequences

by

DAVID W. BOYD* (Vancouver, B.C.)

1. Introduction. The Pisot sequence $E(a_0, a_1)$ is the sequence of positive integers $\{a_n\}$ defined by the non-linear recurrence relation

$$a_{n+2} = N(a_{n+1}^2/a_n), \tag{1.1}$$

where $N(x)$ is the "nearest" integer to x defined by $N(x) = [x+1/2]$.

If $a_1 > a_0+(a_0/2)^{1/2}$ then $a_{n+1}/a_n \to \theta > 1$ and $a_n^{n+1}/a_{n+1}^n \to \lambda > 0$ [8]. We call θ the *ratio* of the Pisot sequence $\{a_n\}$. If ε_n is defined by

$$a_n = \lambda\theta^n + \varepsilon_n \tag{1.2}$$

then ε_n is bounded and satisfies rather stringent inequalities (see Lemma 1).

Pisot [10] showed that if $a_0 = 2$ or 3 then $E(a_0, a_1)$ satisfies a linear recurrence relation. Galyean [9] made a computer study of Pisot sequences for $a_0 \leqslant 10$ and found many that satisfy linear recurrence relations but also many that seemed not to satisfy such relations. Cantor [4] developed a theory of "families" of Pisot sequences which explains many of the recurrences found by Galyean. In [1] we showed that indeed there are non-recurrent Pisot sequences and in [2] gave proofs of non-recurrence for some of the sequences found by Galyean.

If the recurrence relation of minimal length satisfied by $\{a_n\}$ is of the form

$$a_n + q_1 a_{n-1} + \ldots + q_s a_{n-s} = 0 \quad \text{for } n \geqslant n_0, \tag{1.3}$$

then we call $Q(x) = x^s + q_1 x^{s-1} + \ldots + q_s$ the *minimal polynomial* of $\{a_n\}$. From a result of Flor [8], the minimal polynomial of a recurrent Pisot sequence is of the form

$$Q(x) = P(x)K(x), \tag{1.4}$$

* This research was supported in part by NSERC.

where $P(x)$ is the minimal polynomial of a Pisot number or a Salem number and where either $K(x) = 1$ or $K(x)$ is a cyclotomic polynomial with simple roots ([1], p. 90).

Here we study the question of which $Q(x)$ are actually realizable as minimal polynomials of Pisot sequences. For convenience, let $\mathscr{P}$ and $\mathscr{K}$ refer to the sets of polynomials P and K described above. We say that PK is *realizable* if there is a Pisot sequence with minimal polynomial PK.

Flor showed that $Q = P$ is realizable for every $P \in \mathscr{P}$ and gave a necessary and sufficient condition for $Q = P\cdot(x-\varepsilon)$, $\varepsilon = \pm 1$ to be realizable, namely that

$$(\theta-\varepsilon)^2 < \tfrac{1}{2}|P(\varepsilon)|. \tag{1.5}$$

He produced examples of P satisfying (1.5) for both $\varepsilon = +1$ and $\varepsilon = -1$.

The contributions of this paper are most easily summarized by stating a number of increasingly more general questions and our (usually partial) answers to them:

(A) *Given $P \in \mathscr{P}$ and $K \in \mathscr{K}$, is PK realizable?*

Theorem 3 gives a complete answer which reduces to (1.5) in case $K(x) = x-\varepsilon$.

(B) *Given $P \in \mathscr{P}$, determine all $K \in \mathscr{K}$ for which PK is realizable.*

In these circumstances we shall say that K *is admissible* (*for P, or for θ*) and that P (*or θ*) *admits K.*

Our answers here are incomplete and depend on the size of θ relative to its degree d. For example if $\theta > 2^{d-2}+1$ then P admits only $K = 1$. If $d = 1$ or 2 then θ admits only $K = 1$ unless $\theta = (\sqrt{5}+1)/2$ in which case $K = x-1$ is also admissible.

The methods used for (B) usually answer the following question as well:

(C) *Given a Pisot or Salem number θ, must every Pisot sequence $\{a_n\}$ with ratio θ be recurrent?*

In this case we say that θ is *recurrent*. We conjectured in [1] that every Pisot or Salem number is recurrent. Our partial answers support this. For example, all θ of degrees 1 and 2 are recurrent and if $d \geqslant 3$ and $\theta > 8d \log d$ then θ is recurrent (Theorem 4).

(D) *Given K, is there some P for which PK is realizable?*

In this case we say that K is *realizable*. Our answers here consist of producing pairs (P, K) to which Theorem 3 can be applied. Thus, in addition to Flor's 1, $x-1$ and $x+1$ we show that x^2-1, x^2-x+1 and x^4-x^2+1 are realizable. It seems unlikely that all $K \in \mathscr{K}$ can be realized but we have no evidence to support or reject this supposition.

The paper is organized so that the results will appear in their logical order. Thus some results on (B) and (C) appear first in Section 3 while others

appear in Section 4 and 5. The answer to (A) appears in Section 4 together with some Corollaries. The examples appear together in Section 6. The next section contains some preliminary discussion.

2. Preliminary results and notation. Suppose that $\{a_n\}$ is a sequence of integers satisfying (1.3). If $n_0 = s$, we call the recurrence relation *pure*. Since, in this paper, we are only concerned with the minimal polynomial $Q(x)$, we may drop $n_0 - s$ terms of $\{a_n\}$ if necessary and hence assume all recurrence relations are pure. In this case, the generating function of $\{a_n\}$ satisfies

$$\sum_{n=0}^{\infty} \frac{a_n}{x^{n+1}} = \frac{A(x)}{Q(x)}, \tag{2.1}$$

where $A(x)$ is of degree $s-1$ with leading coefficient a_0. Both A and Q have integer coefficients and are relatively prime.

Now suppose $Q = PK$ with P and K relatively prime. Let $\deg P = d$ and $\deg K = k$. Then we may write

$$A(x) = E(x)K(x) + D(x)P(x), \tag{2.2}$$

where E and D have rational coefficients and $\deg E < \deg P$, $\deg D < \deg K$. By considering (2.2) as a system of s linear equations for the $d+k$ coefficients of E and D, we recognize the matrix of the system as Sylvester's matrix whose determinant is the resultant $\operatorname{Res}(P, K)$ ([12], pp. 279–283). Hence there is an integer q dividing $\operatorname{Res}(P, K)$ such that

$$A(x) = (E_1(x)K(x) + D_1(x)P(x))/q \tag{2.3}$$

where now $E_1(x)$ and $D_1(x)$ have integer coefficients.

Substituting (2.2) into (2.1) we have

$$\frac{A(x)}{Q(x)} = \frac{E(x)}{P(x)} + \frac{D(x)}{K(x)}, \tag{2.4}$$

so that

$$a_n = b_n + c_n, \tag{2.5}$$

where b_n and c_n are sequences of rationals with common denominator q and generating functions E/P and D/K respectively. Since A is prime to Q we must have E prime to P and D prime to K.

In terms of the zeros $\{\alpha\}$ of P and $\{\omega\}$ of K, we may write

$$b_n = \sum_{\alpha} \frac{A(\alpha)}{P'(\alpha)K(\alpha)}\alpha^n = \sum_{\alpha} \frac{E(\alpha)}{P'(\alpha)}\alpha^n \tag{2.6}$$

and

$$c_n = \sum_{\omega} \frac{A(\omega)}{P(\omega)K'(\omega)}\omega^n = \sum_{\omega} \frac{D(\omega)}{K'(\omega)}\omega^n. \tag{2.7}$$

3. Criteria for admissibility of K. It will be convenient in this section to use the notation of the finite difference calculus. Let E denote the shift operator on sequences $\{x_n\}_{n\geqslant 0}$ defined by $Ex_n = x_{n+1}$. If $B(x) = b_0 x^m + \ldots + b_m$ is a polynomial with complex coefficients then

$$B(E)\,x_n = b_0\,x_{m+n} + \ldots + b_m\,x_n. \tag{3.1}$$

Let $L(B) = |b_0| + \ldots + |b_m|$. If $\{x_n\}$ is a bounded sequence then

$$\limsup |B(E)\,x_n| \leqslant L(B)\,\limsup |x_n|. \tag{3.2}$$

Clearly this is a sharp inequality so in fact $L(B)$ is the norm of $B(E)$ as a linear operator on the space l^∞/c_0, with norm $\limsup |x_n|$. This observation makes a few of the subsequent calculations more transparent.

Except in the examples in Section 6, a Pisot sequence is any sequence of positive integers satisfying

$$|a_{n+2}\,a_n - a_{n+1}^2| \leqslant a_n/2, \qquad n = 0, 1, \ldots \tag{3.3}$$

That is, we need not insist on a particular choice for $N(x)$ except that $|N(x) - x| \leqslant 1/2$.

Lemma 1. *Let $a_n = \lambda\theta^n + \varepsilon_n$ be a Pisot sequence with $\theta > 1$. Then*

$$\limsup |(E-\theta)^2\,\varepsilon_n| \leqslant 1/2, \tag{3.4}$$

$$\limsup |(E-\theta)\,\varepsilon_n| \leqslant 1/(2(\theta-1)), \tag{3.5}$$

$$\limsup |\varepsilon_n| \leqslant 1/(2(\theta-1)^2). \tag{3.6}$$

Conversely, if a_n is a sequence of integers of the form $\lambda\theta^n + \varepsilon_n$ with $\theta > 1$ and ε_n bounded which satisfies (3.4) *with strict inequality then $\{a_n\}_{n\geqslant n_0}$ is a Pisot sequence.*

Proof (see [3], Theorem 1). We sketch the essential points. First check the identity

$$a_{n+2}\,a_n - a_{n+1}^2 = \big((E-\theta)^2\,\varepsilon_n\big)\,a_n - \big((E-\theta)\,\varepsilon_n\big)^2. \tag{3.7}$$

Since $\{\varepsilon_n\}$ is bounded [10] and $a_n \to \infty$, (3.3) and (3.7) imply (3.4). To obtain (3.5) and (3.6), observe that $E-\theta$ is invertible on bounded sequences and its inverse has norm $1/(\theta-1)$.

Finally (3.4) with strict inequality, and (3.7) imply (3.3) for $n \geqslant n_0$.

The following result is a refinement of Lemma 2 of [1].

Theorem 1. *Let $\theta > 1$ be an algebraic number and let $R(x)$ be a polynomial with integer coefficients such that $R(\theta) = 0$. Let $R_\theta(x) = R(x)/(x-\theta)$. Suppose that $\{a_n\}$ is a Pisot sequence with ratio θ.*

(a) (Pisot) *If $L(R) < 2(\theta-1)^2$ then $\{a_n\}$ is recurrent with minimal polynomial dividing $R(x)$.*

(b) *If $L(R_\theta) < 2(\theta-1)$ then the same conclusion holds.*

(c) *If $L(R_\theta) < 2\theta$ then $\{a_n\}$ is recurrent with minimal polynomial dividing $(x-1)\,R(x)$.*

Proof. We begin with (b) since (a) follows from (b).

Consider the sequence of integers $d_n = R(E)\,a_n$. Write $R(E) = R_\theta(E)(E-\theta)$. Since $(E-\theta)\,a_n = (E-\theta)\,\varepsilon_n$ by (1.2), we have

$$\limsup |d_n| = \limsup |R_\theta(E)(E-\theta)\,\varepsilon_n| \leqslant L(R_\theta)\,\limsup |(E-\theta)\,\varepsilon_n|, \tag{3.8}$$

where we use (3.2). By the assumption of (b), together with (3.5) and (3.8), it follows that $|d_n| < 1$ eventually, and hence $d_n = 0$ eventually. So $R(E)\,a_n = 0$ for $n \geqslant n_0$ proving (b).

Since $R_\theta(E) = R(E)(E-\theta)^{-1}$, taking norms we have $L(R_\theta) \leqslant L(R)(\theta-1)^{-1}$, so (a) follows from (b).

To prove (c), we observe that

$$\limsup |d_{n+1} - \theta d_n| = \limsup |R_\theta(E)(E-\theta)^2\,\varepsilon_n| \leqslant L(R_\theta)/2 < \theta, \tag{3.9}$$

using (3.2), (3.4) and the assumption of $L(R_\theta)$. Thus

$$|d_{n+1} - \theta d_n| < \theta \quad \text{for} \quad n \geqslant n_0, \quad \text{say}. \tag{3.10}$$

We claim that (3.10) implies that $\{d_n\}$ is eventually constant. There are two cases: (i) $d_n = 0$ eventually, or (ii) $d_n \neq 0$ for arbitrarily large n. In case (i) our claim is obvious. In case (ii), suppose $d_n \neq 0$ for some $n = n_1 \geqslant n_0$. Then (3.10) implies that

$$|d_{n+1}| \geqslant |\theta d_n| - |\theta d_n - d_{n+1}| > \theta\,|d_n| - \theta \geqslant |d_n| - 1. \tag{3.11}$$

Since d_n and d_{n+1} are integers, (3.11) implies that $|d_{n+1}| \geqslant |d_n|$. But then $\{|d_n|\}$ is increasing for $n \geqslant n_1$ and since it is bounded (by $L(R_\theta)/(2(\theta-1))$) it is eventually constant. But (3.10) implies that d_n and d_{n+1} have the same sign so $\{d_n\}$ is eventually constant, proving our claim.

Since $(E-1)\,d_n = (E-1)\,R(E)\,a_n = 0$ eventually, (c) then follows.

Theorem 2. *Suppose that θ is a Pisot or Salem number of degree d.*

(a) *If $\theta > 2^{d-2}+1$ then θ is recurrent and admits only $K(x) = 1$.*

(b) *If $\theta > 2^{d-2}$ then θ is recurrent and admits only $K(x) = 1$ and possibly $x-1$.*

Proof. The zeros of $P_\theta(x) = P(x)/(x-\theta)$ all lie in the unit circle so $P_\theta(x)$ is majorized (term-by-term) by $(x+1)^{d-1}$. Thus $L(P_\theta) \leqslant 2^{d-1}$ so (a) and (b) follow from (b) and (c) of Theorem 1.

Corollary 1. *If θ is a Pisot number of degree 1 or 2 then θ is recurrent. Only $K(x) = 1$ is admissible except for $\theta = (\sqrt{5}+1)/2$ when $K(x) = x-1$ is also admissible.*

Proof. If $\deg\theta = 1$ then $\theta \geqslant 2 > 2^{-1}+1$ so the result follows from Theorem 2(a).

If $\deg\theta = 2$ and $\theta \neq (\sqrt{5}+1)/2$ then $\theta > 2 = 2^{2-2}+1$ so (a) applies. If $\theta = (\sqrt{5}+1)/2$ then (b) applies. From Flor's result (1.5), $x-1$ is admissible for θ.

Remark 3.1. Theorem 1 can be used as was Lemma 2 of [1] to show that certain algebraic numbers cannot occur as ratios of Pisot sequences. For if, say, θ satisfies $L(R_\theta) < 2\theta$ and θ is the ratio of a Pisot sequence, then by (c), θ is recurrent and hence a Pisot or Salem number. Thus, for example any quadratic integer θ which dominates its other conjugate is not the ratio of a Pisot sequence unless θ is a Pisot number. As another example, if $\theta = p/q$ is a rational which is not an integer and if $q^2 < 2p$, then θ is not the ratio of a Pisot sequence.

Remark 3.2. A more direct proof of Corollary 1 is possible in the case $\deg\theta = 1$. It can easily be shown that if $\{a_n\} = E(a_0, a_1)$ with $m < a_1/a_0 < m+1$ for integer m, then $m < \theta < m+1$ except in the case $\theta = 1$. Thus an integer $\theta \geqslant 2$ can only occur as the ratio of the Pisot sequences of the form $a_n = a_0\theta^n$.

Remark 3.3. A more careful analysis enables us to deal with the question of whether a recurrence relation must be pure ($n_0 = s$ in (1.3)). We showed in [1] that T-recurrences must be pure if $\theta \geqslant 2$. The methods of Theorem 1 will show, for example, that if $\deg\theta = 2$ then the recurrence must be pure. Since DeLeon [6] has a characterization of all two term pure recurrences for Pisot sequences, this gives a complete characterization of the Pisot sequences whose ratio is Pisot number of degree 2. We do not pursue this further here.

4. The criterion for realizability. The following theorem generalizes Flor's criterion (1.5) for the case $K(x) = x-\varepsilon$ ($\varepsilon = \pm 1$). The proof is very similar except that we must deal with polynomial congruences rather than integer congruences and our method of handling the possibility of equality in (4.1) is necessarily different.

THEOREM 3. *Let $P \in \mathscr{P}$ and $K \in \mathscr{K}$ with $k = \deg K \geqslant 1$.*

Let $\{\omega\}$ denote the set of zeros of K and let N be the smallest integer for which $\omega^N = 1$ for all ω in this set. Then PK is realizable if and only if there is a $C(x)$ with integer coefficients and $\deg C < k$ relatively prime to K such that

$$\max_{0 \leqslant n \leqslant N-1} \left| \sum_{\omega} \frac{C(\omega)}{K'(\omega)P(\omega)} (\omega-\theta)^2 \omega^n \right| < \frac{1}{2}. \tag{4.1}$$

The set of C which can satisfy (4.1) is finite and can be effectively enumerated.

Proof. (a) Suppose that $Q = PK$ is realized by $\{a_n\}$ and that $A(x)$ is as in (2.1). Let $B(x) \equiv A(x) \pmod{P(x)}$ with $\deg B < d$ and $C(x) \equiv A(x) \pmod{K(x)}$ with $\deg C < k$. Let $\{\alpha\}$ be the set of zeros of P, and let

$$\lambda(\alpha) = A(\alpha)/Q'(\alpha) = B(\alpha)/(P'(\alpha)K(\alpha)) \tag{4.2}$$

and

$$\mu(\omega) = A(\omega)/Q'(\omega) = C(\omega)/(P(\omega)K'(\omega)). \tag{4.3}$$

Then, from (2.5)–(2.7),

$$(E-\theta)^2\varepsilon_n = (E-\theta)^2 a_n = (E-\theta)^2 b_n + (E-\theta)^2 c_n, \tag{4.4}$$

where

$$(E-\theta)^2 b_n = \sum_{\alpha} \lambda(\alpha)\alpha^n(\alpha-\theta)^2 = \xi_n, \quad \text{say} \tag{4.5}$$

and

$$(E-\theta)^2 c_n = \sum_{\omega} \mu(\omega)\omega^n(\omega-\theta)^2 = \delta_n, \quad \text{say}. \tag{4.6}$$

The sequence δ_n is a periodic sequence of real numbers with period dividing N. Hence $\max|\delta_n| = \delta$, say, occurs for n in an arithmetic progression $n \equiv n_1 \pmod{N}$.

The sequence $\xi_n = \xi'_n + \xi''_n$ where ξ'_n is the sum over $|\alpha| < 1$ and ξ''_n the sum over $|\alpha| = 1$ (only present if θ is a Salem number). Thus $\xi'_n \to 0$ and ξ''_n is almost periodic. The α with $|\alpha| = 1$ and $\operatorname{Im}\alpha > 0$ are multiplicatively independent ([11], p. 32) and hence, by Kronecker's approximation theorem ([5], p. 53),

$$\limsup|\xi_n| = \limsup|\xi''_n| = \sum_{|\alpha|=1} |\lambda(\alpha)(\alpha-\theta)^2| = \xi, \quad \text{say}. \tag{4.7}$$

For the same reason, given $\varepsilon > 0$, there are infinitely many $n \equiv n_1 \pmod{N}$ which satisfy $\xi''_n \geqslant \xi - \varepsilon$ and infinitely many which satisfy $\xi''_n \leqslant -\xi + \varepsilon$. Thus

$$\limsup|(E-\theta)^2\varepsilon_n| = \limsup|\xi_n + \delta_n| = \xi + \delta. \tag{4.8}$$

By (3.4), we have $\xi + \delta \leqslant 1/2$ and hence $\delta \leqslant 1/2$.

We wish to show that $\delta < 1/2$ (which is (4.1)). If $\delta = 1/2$ then, for $n = n_1$,

$$c_{n+2} - 2\theta c_{n+1} + \theta^2 c_n = \pm 1/2. \tag{4.9}$$

But $\{c_n\}$ is a sequence of rationals so (4.9) implies that $\deg\theta \leqslant 2$. By Corollary 1 of Theorem 2, this means that $\theta = (\sqrt{5}+1)/2$ and $K(x) = x-1$. Thus c_n is a constant and hence by (4.9) $(\theta-1)^2 = (3-\sqrt{5})/2$ is rational, a contradiction.

(b) For the converse, assume that $C(x)$ exists satisfying (4.1). If θ is a Pisot number, we may choose $A(x) = C(x) + F(x)K(x)$ where $F(x)$ is chosen so that $A(\alpha) \neq 0$ for all roots of P. Then $\{a_n\}$ has generating function $A/(PK)$

and

$$\limsup |(E-\theta)^2 a_n| = \limsup |\xi_n+\delta_n| = \max|\delta_n| < 1/2. \tag{4.10}$$

Hence by Lemma 1, $\{a_n\}$ is eventually a Pisot sequence with minimal polynomial PK.

If θ is a Salem number, $A(x)$ must be chosen more carefully since we must insure that

$$\xi = \sum_{\alpha} |\lambda(\alpha)(\alpha-\theta)^2| < \tfrac{1}{2}-\delta. \tag{4.11}$$

Let $D(x)$ be a polynomial with rational coefficients and $\deg D < k$ satisfying

$$D(x)P(x) \equiv C(x) \pmod{K(x)}. \tag{4.12}$$

Let $D(x) = D_1(x)/q$ where $D_1(x)$ has integer coefficients and q is an integer dividing $\operatorname{Res}(P, K)$; (cf. (2.3)). Let

$$E_0(x) = (qC(x)-D_1(x)P(x))/K(x).$$

Then E_0 has integer coefficients, $\deg E_0 < d$, and

$$E_0(x)K(x)+D_1(x)P(x) \equiv 0 \pmod{q}. \tag{4.13}$$

If $E_1(x)$ is any polynomial of degree $d-1$ satisfying $E_1 \equiv E_0 \pmod{q}$, then the polynomial $A(x)$ defined by (2.3) has integer coefficients and $A(x) \equiv C(x) \pmod{K(x)}$.

Thus c_n is independent of the choice of $E_1(x)$ and b_n is given by (2.6) with $E(x) = E_1(x)/q$. We must thus choose $E_1 \equiv E_0 \pmod{q}$ so that

$$\sum_{|\alpha|=1} \left|\frac{E_1(\alpha)}{qP'(\alpha)}\alpha^n(\alpha-\theta)^2\right| < \frac{1}{2}-\delta. \tag{4.14}$$

There are d coefficients in $E_1(x)$ and only $d-2$ roots α with $|\alpha| = 1$ so by Kronecker's approximation theorem we may simultaneously make all $|E_1(\alpha)|$ sufficiently small so that (4.14) holds. Because of the side condition $E_1 \equiv E_0 \pmod{q}$, this is generally an inhomogeneous problem.

(c) Finally, we must show that the set of $C(x)$ satisfying (4.1) is finite. Since $\omega^N = 1$, the sequence δ_n defined by (4.6) is a finite Fourier transform. Hence

$$\mu(\omega)(\omega-\theta)^2 = \frac{1}{N}\sum_{n=0}^{N-1}\delta_n\omega^{-n}. \tag{4.15}$$

Thus $|\mu(\omega)(\omega-\theta)^2| < 1/2$ and so

$$0 < |C(\omega)| < \tfrac{1}{2}|P(\omega)K'(\omega)|/|\omega-\theta|^2 \quad \text{for all } \omega. \tag{4.16}$$

Since $\deg C < k$ we can recover $C(x)$ from the values of $C(\omega)$ by interpolation, and hence (4.16) defines a finite and effectively determinable set of polynomials.

COROLLARY 2. (a) *If there is a polynomial $C(x)$ with integer coefficients, $\deg C < k$, and prime to $K(x)$, for which*

$$\sum_{\omega}\left|\frac{C(\omega)}{P(\omega)K'(\omega)}(\omega-\theta)^2\right| < \frac{1}{2}, \tag{4.17}$$

then PK is realizable.

(b) *If PK is realizable then there is a $C(x)$ with integer coefficients, $\deg C < k$ and prime to $K(x)$, for which*

$$\sum_{\omega}\left|\frac{C(\omega)}{P(\omega)K'(\omega)}(\omega-\theta)^2\right|^2 < \frac{1}{4}. \tag{4.18}$$

Proof. (a) Clearly (4.17) implies (4.1).

(b) This is a result of Parseval's formula applied to the sequence δ_n.

COROLLARY 3. *The polynomial $K(x) = x^2-1$ is admissible for θ if and only if*

$$\frac{(\theta-1)^2}{|P(1)|}+\frac{(\theta+1)^2}{|P(-1)|} < 1. \tag{4.19}$$

Proof. Clearly, if δ_n is given by (4.3) and (4.6) then

$$\max|\delta_n| = \frac{|C(1)|(\theta-1)^2}{2|P(1)|}+\frac{|C(-1)|(\theta+1)^2}{2|P(-1)|}. \tag{4.20}$$

If (4.1) is to hold with $C(1)$ and $C(-1)$ non-zero integers then (4.20) implies (4.19). On the other hand, by choosing $C(x) \equiv 1$, (4.19) implies (4.1).

COROLLARY 4. *Let θ be a Pisot or Salem number and let $R(x)$, $R_\theta(x)$ be as in Theorem 1.*

(b) *If $L(R_\theta) \leqslant 2(\theta-1)$ then every $K(x)$ admissible for θ must divide $R(x)$.*

(c) *If $L(R_\theta) \leqslant 2\theta$ then every $K(x)$ admissible for θ must divide $(x-1)R(x)$.*

Proof. As in the proof of Theorem 1, let $d_n = R(E)a_n$. Since $R(E)b_n = 0$, we have $d_n = R(E)c_n$. By Theorem 3,

$$\limsup |(E-\theta)^2 c_n| = \delta < 1/2.$$

Hence we obtain the strict inequalities $\limsup |d_n| < 1$ and (3.9) required to complete the proof of Theorem 1.

Remark 4.1. The difference between Theorem 1 and the above Corollary 4 is that we allow equality in the conditions on $L(R_\theta)$ but assume θ is recurrent.

Remark 4.2. By Corollary 3 and Flor's criterion (1.5) we see that if $x-1$ and $x+1$ are admissible for θ then so is x^2-1. Example 6.1 will show that the converse is false.

Remark 4.3. It is possible to apply Theorem 3 without extensive computations involving complex numbers. We observe that the sequence $\{c_n\}$ has generating function $C(x)F(x)/K(x)$ where $F(x)$ has rational coefficients and satisfies $F(x)P(x) \equiv 1 \pmod{K(x)}$. If we generate the periodic sequence of rationals f_n with generating function $F(x)/K(x)$ and the corresponding sequence $\eta_n = (E-\theta)^2 f_n$, then we can produce the sequence $\delta_n = (E-\theta)^2 c_n = C(E)\eta_n$ by cyclic shifts and additions. Example 6.6 illustrates this.

5. Another criterion for the admissibility of K. We will use the box principle to produce suitable $R(x)$ to use in Theorem 1 provided θ is sufficiently large relative to its degree.

Lemma 2. *Let θ be a Pisot or Salem number of degree d. Suppose $t > d$ and $L > 0$ are integers satisfying*

$$L^t > t!(3L)^d \theta^t. \tag{5.1}$$

Then there is a non-trivial polynomial of degree at most t with integer coefficients and $L(R) \leqslant 2L$ such that $R(\theta) = 0$.

Proof. Let V be the set of vectors $(v_0, \ldots, v_t)$ of non-negative integers satisfying $v_0 + \ldots + v_t \leqslant L$. Let $\theta_1 = \theta, \theta_2, \ldots, \theta_d$ be the conjugates of θ and consider the mapping $F(v) = (f_1, \ldots, f_d)$ defined by $f_i = v_0\theta_i^t + \ldots + v_t$. We have $0 \leqslant f_1 \leqslant L\theta^t$ and $|f_i| \leqslant L$ if $i \geqslant 2$. If we can find two different $v^{(1)}$ and $v^{(2)}$ in V so that $|f_i^{(1)} - f_i^{(2)}| < 1$, then $R(x) = (v_0^{(1)} - v_0^{(2)})x^t + \ldots + (v_t^{(1)} - v_t^{(2)})$ will have $|R(\theta_i)| < 1$ for all i. But $R(\theta_1), \ldots, R(\theta_d)$ are all the roots of a monic polynomial with integer coefficients, hence this implies $R(\theta_i) = 0$ for all i. In particular $R(\theta) = 0$.

By induction, the number of vectors in V exceeds $L^t/t!$.

On the other hand we claim that the image $F(V)$ can be covered by at most $3^d L^d \theta^t$ "rectangles" of the form $\prod\{|x_i - b_i| < 1/2\}$, where $b_j = \overline{b_i}$ if $\theta_j = \overline{\theta_i}$. For, the interval $0 \leqslant f_i \leqslant L\theta^t$ can be covered by $2[L\theta^t]+2 < 3L\theta^t$ intervals of length 1. If $i \geqslant 2$ and θ_i is real then the interval $|x_i| \leqslant L$ can be covered by $2L+1 < 3L$ intervals of length 1. If θ_i is complex, then the disk $|f_i| \leqslant L$ can be covered by at most $(2\sqrt{2}L)^2$ squares of diameter 1 and hence by $< (3L)^2$ disks of diameter 1. The conjugate of this disk covers f_j if $f_j = \overline{f_i}$. This proves the claim.

Now the lemma follows by the box principle since the assumption (5.1) insures that two distinct points in V have images in the same small rectangle.

Theorem 4. *Let θ be a Pisot or Salem number of degree d. If $\theta > 8d \log d$ then θ is recurrent. Furthermore the set of K admissible for θ is finite and effectively determinable.*

Proof. The case $d \leqslant 2$ is handled by Corollary 1 of Theorem 2 hence assume $d \geqslant 3$. Let $L = 1 + [\theta^2/2]$. Thus, using $t! < t^t e^{1-t}$, (5.1) will hold if

$$\theta^t 2^{-t} > t^t e^{1-t}(3\theta^2/2)^d. \tag{5.2}$$

Now take $t = [(8.8)d \log d] + 1$ and assume $\theta > 8d \log d > 9t$ and that $d \geqslant 3$. Then it can be verified that (5.2) holds.

Thus by Lemma 2, there is a polynomial R of degree at most t with $L(R) \leqslant 2L < \theta^2 + 2 < 2(\theta-1)^2$. By Theorem 1(a), θ is recurrent and if K is admissible then K divides R. There are only a finite number of possible K. Since R is effectively constructible, the set of admissible K is effectively determinable.

6. Examples illustrating the above theory.

Example 6.1. Let $R(x) = x^m(x^2-x-1)-1$ for $m \geqslant 1$. Then R has a unique positive root θ known to be a Pisot number [7]. The minimal polynomial of θ is

$$P(x) = \begin{cases} R(x), & \text{if } m \text{ is odd,} \\ R(x)/(x+1), & \text{if } m \text{ is even.} \end{cases} \tag{6.1}$$

We compute $R_\theta(x) = x^{m+1} + r_1 x^m + \ldots \equiv 1\ r_1 \ldots r_{m+1}$ to be

$$R_\theta = 1,\ \theta-1,\ \theta^2-\theta-1,\ \theta^3-\theta^2-\theta,\ \ldots,\ \theta^{m+1}-\theta^m-\theta^{m-1}. \tag{6.2}$$

Since $\theta^2-\theta-1 > 0$, R_θ has positive coefficients so

$$L(R_\theta) = R_\theta(1) = R(1)/(1-\theta) = 2/(\theta-1) < 2\theta.$$

By Theorem 1(c), θ is recurrent and admissible $K(x)$ must divide $(x-1)R(x)$. If $m = 2$, $L(R_\theta) < 2(\theta-1)$ so $K(x)$ must divide $R(x)$.

By Flor's criterion (1.5), if $\theta < 1 + (\sqrt{2}/2)$ then $(\theta-1)^2 < 1/2 \leqslant |P(1)|/2$ so $x-1$ is admissible for such θ. This applies here unless $m = 2$, and indeed $x-1$ is not admissible for $m = 2$. By (6.1) and the above, only 1 and $x-1$ are admissible for m odd.

If m is even, then $\theta \leqslant 1.7548\ldots$ and $|P(-1)| = |R'(-1)| = m+3$ so by (1.5), $x+1$ is admissible for all even m. Since $P(1) = -1$, x^2-1 will be admissible only if (4.19) holds, i.e.

$$(\theta-1)^2 + (\theta+1)^2/(m+3) < 1. \tag{6.3}$$

If $m \geqslant 10$ then $\theta \leqslant 1.62158\ldots$ so (6.3) holds while if $m \leqslant 8$ then $\theta \geqslant 1.62710\ldots$ so (6.3) is false. Hence, x^2-1 is admissible if and only if $m \geqslant 10$.

Example 6.2. Let $R(x) = x^m(x^2-x-1)+1$ for $m \geqslant 2$ which again has a root $\theta > 1$ which is a Pisot number whose minimal polynomial is

$$P(x) = \begin{cases} R(x)/(x-1) & \text{if } m \text{ is even,} \\ R(x)/(x^2-1) & \text{if } m \text{ is odd.} \end{cases} \tag{6.4}$$

Again R_θ is given by (6.2) but now $\theta^2-\theta-1 < 0$ so only the first two coefficients are positive. Hence

$$L(R_\theta) = -R_\theta(1) + 2 + 2(\theta-1) = -R(1)/(1-\theta) + 2\theta = 2\theta.$$

Thus Theorem 1(c) just fails to apply but we can use Corollary 4 of Theorem 3 to determine the admissible $K(x)$. We find that $K(x)$ must divide $x-1$ if m is even and x^2-1 if m is odd.

As above, $\theta < (\sqrt{5}+1)/2 < 1+1/\sqrt{2}$ so $x-1$ is admissible in all cases.

If m is odd then $P(1) = -(m-1)/2$ and $P(-1) = -(m+3)/2$. By (1.5), $(x+1)$ is admissible only if $m \geqslant 25$. However, (4.19) reduces to

$$(\theta-1)^2/(m-1)+(\theta+1)^2/(m+3) < 1/2 \tag{6.5}$$

which is true if $m \geqslant 13$ but false if $m \leqslant 11$. Thus x^2-1 is admissible only if $m \geqslant 13$.

Note that if $13 \leqslant m \leqslant 23$ then x^2-1 is admissible but $x+1$ is not.

Note that the smallest Pisot number $\theta_0 = 1.3247\ldots$ with minimal polynomial x^3-x-1 is the case $m=2$ of this example. Although Theorem 1(c) does not show that θ_0 is recurrent, it is possible to do so by considering $d_n = P(E)\,a_n$ which, in addition to $|(E-\theta)\,d_n| \leqslant L(P_\theta)/2$, satisfies $|(E-\theta)^2\,d_n| \leqslant L(P)/2$. Using both inequalities one can deduce that d_n is eventually constant.

Example 6.3. The other small Pisot numbers near $(\sqrt{5}+1)/2$ have minimal polynomials

$$P(x) = x^m(x^2-x-1)+\varepsilon(x^2-1)$$

with $m \geqslant 1$ and $\varepsilon = \pm 1$ ([7]).

If $\varepsilon = 1$ and $m = 1$, we again have x^3-x-1.

If $\varepsilon = 1$ and $m = 2$, then $P(x) = x^4-x^3-1$ with $\theta = 1.38027\ldots$ the second smallest Pisot number. Here P_θ has no-negative coefficients and $L(P_\theta) = 1/(\theta-1) < 2\theta$ so by Theorem 3(c), θ is recurrent and only 1 and $x-1$ are admissible. (As usual (1.5) shows $x-1$ is admissible.)

For other choices of m and ε, we have been unable to answer (B).

With $m = 3$, $\varepsilon = 1$, for example, we have $\theta = 1.44326\ldots$ and $L(P_\theta) > 2\theta$. If we examine $|P(e^{i\varphi})|$ we find that it is relatively large for $70° \leqslant \varphi \leqslant 140°$ suggesting that $K(x) = x^2+1$ might be admissible. However, if we compute

$$\delta_n = \frac{1}{13}\operatorname{Re}\{C(i)(-2+3i)\,i^n(i-\theta)^2\}, \tag{6.6}$$

the conditions (4.16) give

$$0 < |C(i)| < \sqrt{13}/(\theta^2+1) \equiv 1.694\ldots.$$

So, without loss, we may take $C(x) \equiv 1$. But then $\delta_1 = (3\theta^2+1)/13 = .55762\ldots > 1/2$. Hence x^2+1 is not admissible for θ.

Example 6.4. Consider the polynomial of degree 16

$$P = 1\ \ -2\ \ 2\ \ -3\ \ 2\ \ -2\ \ 1\ \ 0\ \ 0\ \ 1\ \ -1\ \ 2\ \ -2\ \ 2\ \ -2\ \ 1\ \ -1 \tag{6.7}$$

which is found in [2] and has $\theta = 1.62165\ldots$. If $\omega_1 = \exp(i30°)$ and $\omega_2 = \exp(i\,150°)$ are 12th roots of unity then $|P(\omega_1)| = 3.3007\ldots$ and $|P(\omega_2)| = 9.3330\ldots$ are relatively large suggesting that $K(x) = x^4-x^2+1$ may be admissible for θ. We find that the choice $C(x) = x^3-x-1$ satisfies (4.17) so indeed K is admissible.

In some sense, K is a natural partner for P since

$$P(x)\,K(x)(x+1) = x^{19}(x^2-x-1)-x^{11}+x^{10}-1, \tag{6.8}$$

which is a more appealing expression than (6.7).

To determine an explicit Pisot sequence with minimal polynomial PK we need only examine the sequence $\{a_n\}$ generated by C/PK. We can verify $|a_{n+2}\,a_n - a_{n+1}^2| < a_n/2$ by use of (3.7) since $(E-\theta)^2\,\varepsilon_n$ and $(E-\theta)\,\varepsilon_n$ can be expressed as sums of the form (4.4). We find that $E(a_m, a_{m+1})$ is a Pisot sequence if and only if $m \geqslant 106$. Explicitly, PK is the minimal polynomial for

$$E(7995086938825650416629,\ 12965300600822297402505).$$

We should emphasize that the proof that PK is realizable requires only a hand calculator. It is only in determining a specific sequence $E(a, b)$ realizing PK that a small computer is helpful.

Example 6.5. We give an example with θ a Salem number to illustrate the proof of Theorem 3 and to show that x^2-x+1 can be realized.

Let $P(x) = 1\ 0\ -1\ -1\ -1\ 0\ 1$ with $\theta = 1.40126\ldots$ and let $K(x) = x^2-x+1$ with roots $\omega = \exp(i60°)$ and $\bar{\omega}$. Then $|P(\omega)| = 4$, $K'(\omega) = \sqrt{3}i$ and $|\theta-\omega|^2 = 1.56223\ldots$. Thus the choice $C(x) = 1$ gives

$$|\delta_n| \leqslant 2|\omega-\theta|^2/|P(\omega)\,K'(\omega)| = .45099\ldots$$

so PK is realizable by Theorem 3.

To find a specific sequence realizing PK we begin by computing $D(x)\,P(x) \equiv 1 \pmod{K(x)}$. We find $D(x) = 1/4$ hence $D_1(x) = 1$ and $q = 4$. (Note that $\operatorname{Res}(P, K) = 16$. One can show that q^2 divides $\operatorname{Res}(P, K)$ if P and K are reciprocal, i.e. θ is a Salem number and $K(1) \neq 0$.)

Thus $E_0(x) = (4-D_1\,P)/K \equiv -1\ \ -1\ \ 1\ \ -1\ \ -1 \pmod 4$. Now we want $E_1(x) \equiv E_0(x) \pmod 4$ for which $|E_1(\alpha)|$ is small for the conjugates of θ with $|\alpha| = 1$.

Writing $P(x) = x^3\,R(x+x^{-1})$ with $R(t) = t^3-4t-1$, and $\varrho = \theta+\theta^{-1}$, we see that $P(x)/(x-\theta)(x-\theta^{-1}) = x^2\,R_\varrho(x+x^{-1})$ vanishes for all α with $|\alpha| = 1$. Here

$$R_\varrho = 1,\ \varrho,\ \varrho^{-1} \approx 1,\ 2.11490,\ .47283$$

so a natural choice for $E_1(x)$ is an approximate multiple of $x^2 R_\varrho(x+x^{-1})$. Say $E_1(x) = x^2\,S(x+x^{-1})$ where $S(t) = s_0\,t^2+s_1\,t+s_2$, where

$$s_1/s_0 \approx \varrho, \quad s_2/s_0 \approx 1/\varrho \tag{6.9}$$

and

(6.10) $$s_0, s_1, s_2 \equiv -1, -1, -1 \pmod 4),$$

which is equivalent to the requirement $E_1 \equiv E_0 \pmod 4$.

Take $s_0 \equiv -1 \pmod 4$ and define s_1 and s_2 to be the nearest integers to $s_0\varrho$ and $s_0\varrho^{-1}$ respectively. Then, by Weyl's uniform distribution theorem ([5], p. 66), asymptotically $(1/4)^2$ of the choices of s_0 will satisfy (6.10) and, among these, $\varepsilon = \max(|s_1 - s_0\varrho|, |s_2 - s_0\varrho^{-1}|)$ can be made arbitrarily small.

A simple search yields the triple

$$(s_0, s_1, s_2) = (24167, 51111, 11427)$$

with $\varepsilon = .02943$. For the corresponding $E_1(x)$, we have

$$\xi = \sum_{|\alpha|=1} |\lambda(\alpha)(\alpha-\theta)^2| = .022367\ldots$$

and since $\delta = \max|\delta_n| = .450634\ldots$ we do have the required $\xi + \delta < 1/2$.

Next computing $A(x) = 6042, 6736, 8204, 10615, \ldots$ we know that $A/(PK)$ generates the required Pisot sequence once the contribution from θ^{-1} has died out. We thus find that $E(a_{30}, a_{31}) = E(306174824, 429033096)$ realizes PK.

EXAMPLE 6.6. To illustrate Remark 4.3 we take again $K(x) = x^4 - x^2 + 1$ and now $P(x) = x^{10} - x^6 - x^5 - x^4 + 1$ which defines a Salem number $\theta = 1.21639\ldots$. The sequence f_n of period 12 is given by

$$(f_n) = (11)^{-1}(2\ -1\ 4\ -1\ 2\ 0\ -2\ 1\ -4\ 1\ -2\ 0\ldots)$$

so, truncating to 3 decimal places,

$$(\eta_n) = ((E-\theta)^2 f_n) = (.854, -1.110, .941, -.577, .087, .533, -.854, \ldots).$$

Thus, by inspection (using $\eta_{n+6} = -\eta_n$), we have

$$((E+1)\eta_n) = (.320, -.256, -.169, .364, -.489, .620, -.320, \ldots)$$

and

$$((E+1)^2\eta_n) = (-.299, .064, -.425, .195, -.125, .130, .299, \ldots).$$

Since this has all components less than 1/2, $C(x) = (x+1)^2$ satisfies (4.1), so $x^4 - x^2 + 1$ is admissible for θ.

We leave as an exercise the determination of a specific Pisot sequence realizing PK.

References

[1] D. W. Boyd, *Pisot sequences which satisfy no linear recurrence*, Acta Arith. 32 (1977), pp. 89–98.

[2] — *Pisot and Salem numbers in intervals of the real line*, Math. Comp. 32 (1978), pp. 1244–1260.

[3] D. W. Boyd, *Some integer sequences related to Pisot sequences*, Acta Arith. 34 (1979), pp. 295–305.

[4] D. G. Cantor, *On families of Pisot E-sequences*, Ann. Sci. École Norm. Sup., (4), 9 (1976), pp. 283–308.

[5] J. W. S. Cassels, *An introduction to diophantine approximation*, Cambridge University Press, 1957.

[6] M. J. DeLeon, *Pisot sequences*, J. Reine Angew. Math. 249 (1971), pp. 20–30.

[7] J. Dufresnoy et Ch. Pisot, *Etude de certaines fonctions méromorphes bornées sur le circle unité, application à un ensemble fermé d'entiers algébriques*, Ann. Sci. École Norm. Sup. (3) 72 (1955), pp. 69–92.

[8] P. Flor, *Über eine Klasse von Folgen naturlicher Zahlen*, Math. Ann. 140 (1960), pp. 299–307.

[9] P. Galyean, *On linear recurrence relations for E-sequences*, thesis, University of California, Los Angeles 1971.

[10] Ch. Pisot, *La répartition modulo 1 et les nombres algébriques*, Ann. Scuola Norm. Sup. Pisa 7 (1938), pp. 205–248.

[11] R. Salem, *Algebraic numbers and Fourier analysis*, D. C. Heath and Co., 1963.

[12] J. V. Uspensky, *Theory of equations*, McGraw-Hill, 1948.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, B.C., CANADA, V6T 1Y4

Received on 16.8.1984 (1445)