ACTA ARITHMETICA
XXI (1972)

On reciprocally weighted partitions

by

D. H. Lehmer (Berkeley, Calif.)

Dedicated to the memory of Wacław Sierpiński

Introduction. Let n be a positive integer and let

$$P\colon\ a_1+a_2+a_3+\ldots=n \qquad (0<a_1\leqslant a_2\leqslant a_3\leqslant\ldots) \tag{1}$$

be any partition of n into positive integer parts. We assign to this partition the weight

$$w(P)=(a_1a_2a_3\ldots)^{-1}.$$

If the parts a_i are restricted to a set S of positive integers and if we add the weights of all the partitions of n into parts taken from S we obtain

$$W_n(S)=\sum w(P), \tag{2}$$

a rational number we call informally the weighted number of partitions of n into parts belonging to S. If we require that the parts in (1) be distinct we obtain, in lieu of (2) a smaller sum, which we denote by $W_n^*(S)$, called the weighted number of partitions of n into distinct parts belonging to S.

For example if S is unrestricted and if $n=6$ we have eleven partitions.

P	$w(P)$	P	$w(P)$
6	1/6	1+1+1+3	1/3
1+5	1/5	2+2+2	1/8
2+4	1/8	1+1+2+2	1/4
1+1+4	1/4	1+1+1+1+2	1/2
3+3	1/9	1+1+1+1+1+1	1
1+2+3	1/6		

Hence

$$W_6(S)=\frac{581}{180}\quad\text{and}\quad W_6^*(S)=\frac{1}{6}+\frac{1}{5}+\frac{1}{8}+\frac{1}{6}=\frac{79}{120}.$$

If we multiply $W_n(S)$ and $W_n^*(S)$, by $n!$ we obtain two non-negative integers

$$A_n(S) = n!\, W_n(S), \quad A_n^*(S) = n!\, W_n^*(S).$$

That these are indeed integers follows from the fact that every term $w(P)$ becomes an integer when multiplied by $n!$. In fact even

$$\frac{n!}{a_1!\, a_2! \ldots} = \frac{n!\, w(P)}{(a_1-1)!\, (a_2-1)! \ldots}$$

is an integer since it is a multinomial coefficient.

We adopt the convention

$$A_0(S) = A_0^*(S) = 1.$$

Generating functions. If we expand the following products into power series we see that A and A^* are generated by

$$F(x) = F(x, S) = \prod_{m \in S} (1 - x^m/m)^{-1} = \sum_{n=0}^{\infty} A_n(S) x^n/n!, \tag{3}$$

$$F^*(x) = F^*(x, S) = \prod_{m \in S} (1 + x^m/m) = \sum_{n=0}^{\infty} A_n^*(S) x^n/n!. \tag{4}$$

The integers A_n and A_n^* may be computed recursively thus avoiding the generation of all the corresponding partitions of n in terms of which they are defined. To this effect we have

THEOREM 1. *Define* $\Gamma_n(S) = \Gamma_n$ *and* $\Gamma_n^*(S) = \Gamma_n^*$ *by*

$$\Gamma_0 = \Gamma_0^* = 0, \quad \Gamma_n = n! \sum_{\substack{\delta | n \\ \delta \in S}} \delta^{1-n/\delta}, \quad \Gamma_n^* = n! \sum_{\substack{\delta | n \\ \delta \in S}} (-\delta)^{1-n/\delta},$$

where the sums extend over all those divisors of n *which belong to* S.

Then, symbolically,

$$nA_n = (A + \Gamma)^n \quad \textit{and} \quad nA_n^* = (A^* + \Gamma^*)^n.$$

In other words,

$$nA_n = \sum_{k=1}^{n} A_{n-k} \binom{n}{k} \Gamma_k, \quad nA_n^* = \sum_{k=1}^{n} A_{n-k}^* \binom{n}{k} \Gamma_k^*.$$

Proof. To prove the first of these conclusions we take the logarithmic derivative of $F(x)$, thus

$$\frac{xF'(x)}{F(x)} = \sum_{m \in S} \frac{x^m}{1 - x^m/m} = \sum_{m \in S} x^m \sum_{n=0}^{\infty} x^{mn} m^{-n} = \sum_{m \in S} \sum_{r=1}^{\infty} x^{mr} m^{1-r}$$
$$= \sum_{k=1}^{\infty} x^k \sum_{\substack{mr=k \\ m \in S}} m^{1-r} = \sum_{k=1}^{\infty} x^k \Gamma_k / k!.$$

Multiplying by $F(x)$ and identifying coefficients of x^n on both sides gives

$$nA_n/n! = \sum_{k=0}^{n} \Gamma_k A_{n-k} / [(k!)(n-k)!]$$

or

$$nA_n = \sum_{k=0}^{n} \Gamma_k \binom{n}{k} A_{n-k} = (A + \Gamma)^n.$$

The formula $nA_n^* = (A^* + \Gamma^*)^n$ is proved in the same way using $F^*(x)$.

COROLLARY. Γ_k *and* Γ_k^* *are integers.*

Proof. By Theorem 1

$$\Gamma_n = nA_n - \sum_{k=1}^{n-1} \Gamma_k \binom{n}{k} A_{n-k}.$$

Hence, by complete induction, Γ_n is an integer since its predecessors are. Similar reasoning applies to Γ_k^*.

Simple limit theorems. Table I gives the first ten values of the six functions A, A^*, Γ, Γ^*, W, W^* in the unrestricted case, i.e., when S is the set of all positive integers.

Table I

n	Γ_n	A_n	W_n	W_n/n	Γ_n^*	A_n^*	W_n^*
1	1	1	1.00000	1.00000	1	1	1.00000
2	4	3	1.50000	.75000	0	1	.50000
3	12	11	1.83333	.61111	12	5	.83333
4	60	56	2.33333	.58333	−12	14	.58333
5	240	324	2.70000	.54000	240	74	.61667
6	1860	2324	3.22778	.53796	−60	474	.65833
7	10080	18332	3.63730	.51961	10080	3114	.61786
8	95760	167544	4.15536	.51942	−15120	24240	.60119
9	766080	1674264	4.61382	.51265	766080	219456	.60476
10	8210160	18615432	5.12991	.51299	−498960	2231280	.61488

Inspection of this small table of W_n^* leads one to guess that W_n^* tends to some limit. This is confirmed by

THEOREM 2. *The weighted number of partitions of* n *into distinct parts tends to* $e^{-\gamma}$ *as* $n \to \infty$. *That is*

$$\lim_{n \to \infty} W_n^* = e^{-\gamma} = .56145948 \ldots,$$

where γ *is Euler's constant.*

Proof. With S the set of all positive integers let

$$G^*(x) = (1-x)F^*(x) = \sum_{n=0}^{\infty} b_n x^n = 1-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\tfrac{1}{30}x^5 \ldots$$

so that

$$W_n^* = A_n/n! = \sum_{k=0}^{n} b_k.$$

Then, by (4),

$$\log G^*(x) = \log(1-x)+\sum_{n=1}^{\infty}\log(1+x^n/n)$$
$$= -\sum_{n=1}^{\infty}\frac{x^n}{n}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{mn^m}x^{mn} = -\sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{(-1)^m}{mn^m}x^{mn}.$$

This series converges for $x=1$ and

$$-\log G^*(1) = \sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{(-1)^m}{mn^m} = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\left(\log\left(1+\frac{1}{n}\right)-\frac{1}{n}\right)\right\}$$
$$= \lim_{N\to\infty}\left\{\log(1+N)-\sum_{n=1}^{N}1/n\right\} = \gamma.$$

Hence

$$e^{-\gamma} = G^*(1) = \lim_{N\to\infty}\sum_{n=0}^{N} b_n = \lim_{N\to\infty} W_N^*$$

which proves the theorem.

Further inspection of Table I suggests that W_n/n also tends to a limit. This is confirmed by

THEOREM 3. *The weighted number of unrestricted partitions of n is asymptotic to $e^{-\gamma}n$. That is*

$$\lim_{n\to\infty} W_n/n = e^{-\gamma}.$$

Before proving this theorem it is convenient to prove

THEOREM 4. *The weighted number of partitions of n into parts > 1 tends to $e^{-\gamma}$ as $n\to\infty$.*

Proof. Let S_1 now be the set of all integers $\geqslant 2$ and let $F(x)$ mean $F(x, S_1)$, then

$$G(x) = (1-x)F(x) = \sum c_n x^n = \frac{x^2}{2}-\frac{x^3}{6}+\ldots$$

so that

$$W_n(S_1) = A_n(S_1)/n! = \sum_{k=0}^{n} c_k.$$

Then by (3)

$$\log G(x) = \log(1-x)-\sum_{n=2}^{\infty}\log(1-x^n/n)$$
$$= \sum_{n=2}^{\infty}\sum_{m=1}^{\infty}\frac{x^{mn}}{mn^m}-\sum_{n=1}^{\infty}\frac{x^n}{n} = -x+\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{x^{mn}}{mn^m}$$

a series that converges at $x=1$.

Hence

$$\log G(1) = -1+\lim_{N\to\infty}\sum_{n=2}^{N}\sum_{m=2}^{\infty}\frac{1}{mn^m} = -1+\lim_{N\to\infty}\sum_{n=2}^{N}\left\{\sum_{m=1}^{\infty}\frac{(n^{-1})^m}{m}-\frac{1}{n}\right\}$$
$$= \lim_{N\to\infty}\left\{\sum_{n=2}^{N}\log\left(\frac{n}{n-1}\right)-\sum_{n=1}^{N}\frac{1}{n}\right\} = \lim_{N\to\infty}\left\{\log N-\sum_{n=1}^{N}\frac{1}{n}\right\} = -\gamma.$$

Hence

$$e^{-\gamma} = G(1) = \lim_{N\to\infty}\sum_{n=0}^{N} c_n = \lim_{N\to\infty} W_N(S_1),$$

which proves Theorem 4.

To prove Theorem 3 we observe that unrestricted partitions of n are of two types:

(a) Those involving the part 1.

(b) Those with all parts >1.

Those of type (a) correspond uniquely to an unrestricted partition of $n-1$ simply by supressing one unit part. These two partitions have the same weight. Hence

$$W_n - W_{n-1} = W_n(S_1).$$

Therefore

$$\lim_{N\to\infty}\frac{W_N}{N} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(W_n-W_{n-1}) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}W_n(S_1).$$

By Theorem 4 this average must tend to $e^{-\gamma}$. This proves Theorem 3.

Arithmetical progressions. In order to treat partitions whose parts lie in an arithmetical progression we need two lemmas.

LEMMA 1. *Let $c_n \geqslant 0$ and let $f(x) = \sum_{n=0} c_n x^n$ be such that for some $\lambda > 0$*

$$\lim_{x\to 1-0}(1-x)^{\lambda} f(x) = C.$$

Then

$$\lim_{n\to\infty} n^{-\lambda}(c_0+c_1+\ldots+c_n) = C/\Gamma(\lambda).$$

Proof. For $\lambda=1$ this is a Tauberian theorem of Hardy ([1], Theorem 96, p. 155). For a general λ we need to observe that as $x\to 1-0$

$$f(x)\sim C(1-x)^{-\lambda} = C\sum_{n=0}^{\infty}\binom{n+\lambda-1}{n}x^{n}.$$

Furthermore

$$\binom{n+\lambda-1}{n} = \frac{\Gamma(n+\lambda)}{\Gamma(n+1)\Gamma(\lambda)} = \frac{1}{\Gamma(\lambda)}(n+\lambda-1)(n+\lambda-2)\dots(n+1)\sim n^{\lambda-1}/\Gamma(\lambda).$$

With these modifications the proof goes through.

LEMMA 2. *If* $a>0$,

$$\lim_{x\to 1-0}(1-x)^{1/a}\prod_{n=1}^{\infty}\left(1-\frac{x^{an+b}}{an+b}\right)^{-1} = a^{-1/a}e^{-\gamma/a}\left\{\Gamma\left(1+\frac{b-1}{a}\right)\Big/\Gamma\left(1+\frac{b}{a}\right)\right\}.$$

Proof. Let

$$G(x) = (1-x^{a})^{-1/a}\prod_{n=1}^{\infty}\left(1-\frac{x^{an+b}}{an+b}\right).$$

For typographical simplicity set

$$y=x^{a},\quad \alpha=1/a,\quad b\alpha=c,$$

then

$$(1-x^{a})^{1/a} = (1-y)^{\alpha} = \exp\{\alpha\log(1-y)\} = \exp\left(\sum_{n=1}^{\infty}\frac{-\alpha y^{n}}{n}\right) = \prod_{n=1}^{\infty}e^{-\alpha y^{n}/n}.$$

Hence

$$G(x) = \prod_{n=1}^{\infty}\left\{\left(1-\frac{\alpha y^{n+c}}{n+c}\right)e^{\alpha y^{n}/n}\right\}$$
$$= e^{\alpha y}\left(1-\frac{\alpha y^{c+1}}{c+1}\right)\prod_{n=1}^{\infty}\left\{\left(1-\frac{\alpha y^{n+c+1}}{n+c+1}\right)e^{\alpha y^{n+1}/(n+1)}\right\}$$
$$= e^{\alpha y}\frac{c+1-\alpha y^{c+1}}{c+1}\prod_{n=1}^{\infty}\left\{\left(1+\frac{c+1-\alpha y^{n+c+1}}{n}\right)e^{-(c+1-\alpha y^{n+1+c})/n}\right\}\times$$
$$\times\left\{\prod_{n=1}^{\infty}\left\{\left(1+\frac{c+1}{n}\right)e^{-(c+1)/n}\right\}\right\}^{-1}\left\{\prod_{n=1}^{\infty}\exp\left\{\alpha\left[\frac{y^{c}}{n}-\frac{1}{n+1}\right]y^{n+1}\right\}\right\}^{-1}.$$

Since

$$\Gamma(z) = z^{-1}e^{-\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^{-1}e^{z/n}\right\}$$

we have

$$G(x) = \Gamma(1+c)e^{\gamma(1+c)}(c+1-\alpha y^{c+1})\times$$
$$\times\exp\left\{\alpha\left[y+\sum_{n=1}^{\infty}\frac{y^{n+1}}{n+1}-y^{c+1}\sum_{n=1}^{\infty}\frac{y^{n}}{n}\right]\right\}P(y)$$

where

$$P(y) = \prod_{n=1}^{\infty}\left\{\left(1+\frac{c+1-\alpha y^{n+c+1}}{n}\right)e^{-(c+1-\alpha y^{n+c+1})/n}\right\}.$$

The expression inside the square brackets is simply

$$-(1-y^{c+1})\log(1-y)$$

and this vanishes as y tends to 1.

Next we consider

$$\lim_{y\to 1-0}P(y).$$

The logarithm of the nth factor is

$$T_n = -\frac{c+1-\alpha y^{n+c+1}}{n}+\log\left(1+\frac{c+1-\alpha y^{n+c+1}}{n}\right)$$
$$= \sum_{m=2}^{\infty}\frac{(-1)^{m+1}(c+1-\alpha y^{c+1+n})^{m}}{mn^{m}}.$$

Let

$$\nu = c+1-\alpha$$

and let

$$N>2\nu.$$

Then for $n>N$

$$|T_n| < \frac{\nu^2}{n^2}\left(1+\frac{|\nu|}{n}+\frac{|\nu|^2}{n^2}+\dots\right) < \frac{N^2}{4n^2}\sum_{\lambda=0}^{\infty}2^{-\lambda} = \frac{1}{2}\left(\frac{N}{n}\right)^{2}.$$

Since

$$\sum_{n>N}N^2/n^2$$

converges it follows that $\log P(y)$ is analytic for $|y|\leqslant 1$, and hence $P(y)$ tends to

$$P(1) = \prod_{n=1}^{\infty}\left\{\left(1+\frac{\nu}{n}\right)e^{-\nu/n}\right\} = e^{-\nu\gamma}/\{\nu\Gamma(\nu)\}.$$

Hence

$$\lim_{x\to 1-0} G(x) = \Gamma(c+1)e^{\gamma(c+1)}\nu P(1) = e^{a\gamma}\Gamma(c+1)/\Gamma(c+1-a).$$

It remains to observe that

$$\lim_{x\to 1}\frac{(1-x)^{1/a}}{(1-x^{a})^{1/a}} = a^{-1/a}$$

and to restore the variables a and b in terms of α and c to obtain the theorem.

THEOREM 5. *Let S_w denote the set of all odd numbers. Then for the weighted number $W_n(S_w)$ of partitions of n into odd parts we have*

$$W_n(S_w)\sim\frac{2}{\pi}\sqrt{2e^{-\gamma}n} = .6746124\sqrt{n}.$$

Proof. In Lemma 2 we put $a=2$ and $b=1$, we find

$$\lim_{x\to 1-0}(1-x)^{1/2}\prod_{n=1}^{\infty}\left(1-\frac{x^{2n+1}}{2n+1}\right)^{-1} = 2^{-1/2}e^{-\gamma/2}/\Gamma(3/2) = \sqrt{\frac{2}{e^{\gamma}\pi}}.$$

Since the first factor $(1-x)^{-1}$ of the generator of $W_n(S_w)$ is missing from the above product we have

$$\lim_{x\to 1-0}(1-x)^{3/2}\sum_{n=0}^{\infty}W_n(S_w)x^n = \sqrt{\frac{2}{e^{\gamma}\pi}}.$$

By Lemma 1 therefore

$$\sum_{r=0}^{n}W_r(S_w)\sim n^{3/2}\sqrt{\frac{2}{e^{\gamma}\pi}}\Big/\Gamma(3/2) = n^{3/2}\sqrt{8e^{-\gamma}}/\pi.$$

Since $W_n(S_w)$ is an increasing function of n

$$\lim_{n\to\infty}W_n(S_w)n^{-1/2} = \lim_{n\to\infty}n^{-3/2}\sum_{r=0}^{n}W_r(S_w) = \sqrt{8e^{-\gamma}}/\pi.$$

THEOREM 6. *Let S_e denote the set of all even numbers. Then for the weighted number $W_n(S_e)$ of partitions of n into even parts we have $W_n(S_e)=0$ if n is odd, while*

$$W_n(S_e)\sim\sqrt{2e^{-\gamma}}/\sqrt{n} = 1.0596787/\sqrt{n}$$

if n is even.

Proof. If we put $a=2$ and $b=0$ in Lemma 2 we find

$$\lim_{x\to 1-0}(1-x)^{1/2}\prod_{n=1}^{\infty}\left(1-\frac{x^{2n}}{2n}\right)^{-1} = 2^{-1/2}e^{-\gamma/2}\Gamma(1/2) = \sqrt{\frac{\pi}{2e^{\gamma}}}.$$

Hence by Lemma 1 we have

$$\sum_{r=0}^{n}W_r(S_e)\sim n^{1/2}\sqrt{\frac{\pi}{2e^{\gamma}}}\Big/\Gamma(1/2) = n^{1/2}/\sqrt{2e^{\gamma}}.$$

or

$$\frac{2}{n}\sum_{r=0}^{n}\sqrt{n}W_r(S_e)\sim\sqrt{2e^{-\gamma}}.$$

Only half the terms of this sum are non-zero. So we have an average of the even ordered terms.

Hence when n is even

$$\sqrt{n}W_n(S_e)\to\sqrt{2e^{-\gamma}}\quad\text{as}\quad n\to\infty.$$

For the general arithmetic progression we have

THEOREM 7. *Let $a>1$ and denote by $S_{a,b}$ the set of all positive integers congruent to b modulo a with $0\leqslant b<a$. Then as $n\to\infty$*

$$W_n(S_{a,0})\sim\left\{a^{1-1/a}e^{-\gamma/a}\,\Gamma\left(1-\frac{1}{a}\right)\Gamma\left(\frac{1}{a}\right)\right\}n^{1/a-1}\quad\text{if}\quad a|n,$$

$$W_n(S_{a,0}) = 0.\quad\text{if}\quad a\nmid n.$$

$$W_n(S_{a,1})\sim\left\{a^{-1/a}e^{-\gamma/a}\Big/\left[\Gamma\left(1+\frac{1}{a}\right)\right]^2\right\}n^{1/a}.$$

If $b\neq 0$ and $b\neq 1$,

$$W_n(S_{a,b})\sim\left\{a^{1-1/a}e^{-\gamma/a}\,\Gamma\left(\frac{b-1}{a}\right)\Gamma\left(\frac{b}{a}\right)\Big/\Gamma\left(\frac{1}{a}\right)\right\}n^{1/a-1}.$$

The proof follows the lines of the proofs of Theorems 5 and 6.

Numerical values. In conclusion we give in Table II brief numerical evidence that the various limits discussed above as approached at rather leisurely rates. The five functions of the following table refer to partitions into distinct parts, parts >1, unrestricted parts, odd parts and even parts respectively.

Table II

n	W_n^*	$W_n(S_1)$	W_n/n	$W_n(S_w)/\sqrt{n}$	$\sqrt{n}W_n(S_e)$
100	.566786	.555790	.542158	.669193	1.072995
101	.566726	.555423	.542289	.669292	0
102	.566691	.555910	.542423	.669277	1.071161
103	.566634	.555546	.542550	.669369	0
104	.566584	.556011	.542680	.669432	1.072575
Limit	.561459	.561459	.561459	.674612	1.05968

The slight irregularities in these functions are not due to inaccuracy. They reflect the existence of an asymptotic, or possibly convergent, series for each entry.

Reference

[1] G. H. Hardy, *Divergent Series*, Oxford 1949.

Received on 17. 8. 1971 (206)

ACTA ARITHMETICA
XXI (1972)

Some diophantine equations solvable by identities

by

A. MĄKOWSKI (Warszawa)

Dedicated to the memory
of my teacher Wacław Sierpiński

1. W. Sierpiński in many of his papers investigated the triangular numbers $t_n = \frac{1}{2}n(n+1)$ and tetrahedral numbers $T_n = \frac{1}{6}n(n+1)(n+2)$.

From the identity given by A. Gérardin [1] we get immediately the following identity

$$(27n^6)^2-1 = (9n^4-3n)^3+(9n^3-1)^3 = (9n^4+3n)^3-(9n^3+1)^3.$$

With n odd and positive the last identity provides infinitely many integer solutions of the equation

$$(2x+1)^2-1 = (2y)^3+(2z)^3 = (2u)^3-(2v)^3$$

which is equivalent to

$$t_x = y^3+z^3 = u^3-v^3.$$

Thus there exist infinitely many triangular numbers which are simultaneously representable as sums and differences of two positive cubes.

We have the identity $3aT_{a-1} = t_{a^2-1}$. Since there exist infinitely many tetrahedral numbers divisible by 3: $T_m = 3a$ we infer that there exist infinitely many triangular numbers which are products of two tetrahedral numbers >1.

2. The numbers $x = 6^2pr^2n^3+6^6p^4r^5n^9$, $y = 6^2pr^2n^3-6^6p^4r^5n^9$, $z = 6^5p^3r^4n^7$ satisfy the equation

$$p(x^3+y^3-z^3) = r(x-y).$$

This answers a question posed by A. Oppenheim in [3].

3. L. J. Mordell [2] investigated the equation $z^2 = ax^3+by^3+c$. It may be noticed that the equation

$$z^2 = ax^{2k+1}+by^{2k+1}+c$$