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Riddles of Representations of Integers

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Part I. Representations of positive integers involving primes

Prime numbers

An integer $p > 1$ is called a **prime** if it cannot be written as ab with $1 < a \leqslant b < p$.

Primes below 100:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,$$
$$41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97.$$

Fundamental Theorem of Arithmetic. Any integer $n > 1$ can be written as a product of finitely many primes. If we ignore the order of the factors, such a decomposition is unique.

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ we denote the nth prime by p_n. Thus

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ p_5 = 11,\ p_6 = 13,\ \dots.$$

Prime Number Theorem. For $x \geqslant 1$ let $\pi(x)$ be the number of primes not exceeding x. Then

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to +\infty.$$

Goldbach's conjecture

Goldbach's Conjecture (1742): Every even number $n > 2$ can be written in the form $p + q$ with p and q both prime.

Goldbach's weak Conjecture [proved by I. M. Vinorgradov (1937) and H. Helfgott (2013)]. Each odd number $n > 6$ can be written as a sum of three primes.

Goldbach's conjecture implies that for any $n > 2$ there is a prime $p \in [n, 2n]$ since $2n \neq p + q$ if p and q are smaller than n.

Jing-run Chen's Theorem (1973). If an even number is large enough, then it can be written as $p + P_2$, where p is a prime, and P_2 is either a prime or a product of two primes.

Lemoine's Conjecture (1894). Any odd integer $n > 6$ can be written as $p + 2q$, where p and q are primes.

Conjectures for twin primes, cousin primes and sexy primes

Conjecture (Sun, 2012-12-22) Any integer $n \geqslant 12$ can be written as $p + q$ with $p, p + 6, 6q \pm 1$ all prime.

Remark. I have verified this for n up to 10^9.

Conjecture (Sun, 2013-01-03) Let

$$A = \{x \in \mathbb{Z}^+ : \ 6x - 1 \text{ and } 6x + 1 \text{ are both prime}\},$$
$$B = \{x \in \mathbb{Z}^+ : \ 6x + 1 \text{ and } 6x + 5 \text{ are both prime}\},$$
$$C = \{x \in \mathbb{Z}^+ : \ 2x - 3 \text{ and } 2x + 3 \text{ are both prime}\}.$$

Then

$$A+B = \{2, 3, \ldots\}, \ B+C = \{5, 6, \ldots\}, \ \ A+C = \{5, 6, \ldots\} \setminus \{161\}.$$

Also, if we set $2X := X + X$ then

$$2A \supseteq \{702, 703, \ldots\}, \ 2B \supseteq \{492, 493, \ldots\}, \ 2C \supseteq \{4006, 4007, \ldots\}.$$

A conjecture refining Bertrand's Postulate

Bertrand's Postulate (proved by Chebyshev in 1850). For any positive integer n, the interval $[n, 2n]$ contains at least a prime.

Conjecture (Sun, 2012-12-18) For each positive integer n, there is an integer $k \in \{0, \ldots, n\}$ such that $n + k$ and $n + k^2$ are both prime.

Conjecture (Sun, 2013-04-15) For any positive integer n there is a positive integer $k \leqslant 4\sqrt{n+1}$ such that $n^2 + k^2$ is prime.

My conjecture on Heath-Brown primes

Heath-Brown's Theorem (2001). There are infinitely many primes of the form $x^3 + 2y^3$ where x and y are positive integers.

Conjecture (Sun, 2012-12-14). Any positive integer n can be written as $x + y$ ($x, y \in \mathbb{N} = \{0, 1, \ldots\}$) with $x^3 + 2y^3$ prime. In general, for each positive *odd* integer m, any sufficiently large integer can be written as $x + y$ ($x, y \in \mathbb{N}$) with $x^m + 2y^m$ prime.

Conjecture (Sun, 2013-04-15) For any integer $n > 4$ there is a positive integer $k < n$ such that $2n + k$ and $2n^3 + k^3$ are both prime.

A general hypothesis on representations

Let's recall

Schinzel's Hypothesis H. If $f_1(x), \ldots, f_k(x)$ are irreducible polynomials with integer coefficients and positive leading coefficients such that there is no prime dividing the product $f_1(q)f_2(q)...f_k(q)$ for all $q \in \mathbb{Z}$, then there are infinitely many $n \in \mathbb{Z}^+$ such that $f_1(n), f_2(n), \ldots, f_k(n)$ are all primes.

The following general hypothesis is somewhat similar to Schinzel's Hypothesis.

General Conjecture on Representations (Sun, 2012-12-28) Let $f_1(x, y), \ldots, f_m(x, y)$ be non-constant polynomials with integer coefficients. Suppose that for large $n \in \mathbb{Z}^+$, those $f_1(x, n-x), \ldots, f_m(x, n-x)$ are irreducible, and there is no prime dividing all the products $\prod_{k=1}^m f_k(x, n-x)$ with $x \in \mathbb{Z}$. If $n \in \mathbb{Z}^+$ is large enough, then we can write $n = x + y$ $(x, y \in \mathbb{Z}^+)$ such that $|f_1(x, y)|, \ldots, |f_m(x, y)|$ are all prime.

Two curious conjectures on primes

Conjecture (Sun, 2013, Prize \$200). Any integer $n > 1$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ such that $x + ny$ and $x^2 + ny^2$ are both prime.

For example, $20 = 11 + 9$ with
$11 + 20 \times 9 = 191$ and $11^2 + 20 \times 9^2 = 1741$ both prime.

Conjecture (Sun, 2013, Prize \$1000). Any integer $n > 1$ can be written as $k + m$ $(k, m \in \mathbb{Z}^+)$ with $2^k + m$ prime.

Remark. I have verified this for $n \leqslant 10^7$. For example, $8 = 3 + 5$ with $2^3 + 5 = 13$ prime. Also,

$$9302003 = 311468 + 8990535$$

with

$$2^{311468} + 8990535$$

a prime of 93762 decimal digits.

Unification of Goldbach's conjecture and the twin prime conjecture

Unification of Goldbach's Conjecture and the Twin Prime Conjecture (Sun, 2014-01-29). For any integer $n > 2$, there is a prime q with $2n - q$ and $p_{q+2} + 2$ both prime.

We have verified the conjecture for n up to 10^8. Clearly, it is stronger than Goldbach's conjecture. Now we explain why it implies the twin prime conjecture.

In fact, if all primes q with $p_{q+2} + 2$ prime are smaller than an even number $N > 2$, then for any such a prime q the number $N! - q$ is composite since

$$N! - q \equiv 0 \pmod q \ \text{ and } \ N! - q \geqslant q(q+1) - q > q.$$

Example. $20 = 3 + 17$ with 3, 17 and $p_{3+2} + 2 = 11 + 2 = 13$ all prime.

Super Twin Prime Conjecture

If $p, p+2$ and $\pi(p)$ are all prime, then we call $\{p, p+2\}$ a *super twin prime pair*.

Super Twin Prime Conjecture (Sun, 2014-02-05). Any integer $n > 2$ can be written as $k + m$ with k and m positive integers such that $p_k + 2$ and $p_{p_m} + 2$ are both prime.

Example. $22 = 20 + 2$ with $p_{20} + 2 = 71 + 2 = 73$ and $p_{p_2} + 2 = p_3 + 2 = 5 + 2 = 7$ both prime.

Remark. If all those positive integer m with $p_{p_m} + 2$ prime are smaller than an integer $N > 2$, then by the conjecture, for each $j = 1, 2, 3, \ldots$, there are positive integers $k(j)$ and $m(j)$ with $k(j) + m(j) = jN$ such that $p_{k(j)} + 2$ and $p_{p_{m(j)}} + 2$ are both prime, and hence $k(j) \in ((j-1)N, jN)$ since $m(j) < N$; thus

$$\sum_{j=1}^{\infty} \frac{1}{p_{k(j)}} \geqslant \sum_{j=1}^{\infty} \frac{1}{p_{jN}},$$

which is impossible since the series on the right-hand side diverges while the series on the left-hand side converges by Brun's theorem.

Alternating sums of primes

Let p_n be the nth prime and define

$$s_n = p_n - p_{n-1} + \cdots + (-1)^{n-1} p_1.$$

For example,

$$s_5 = p_5 - p_4 + p_3 - p_2 + p_1 = 11 - 7 + 5 - 3 + 2 = 8.$$

Note that

$$s_{2n} = \sum_{k=1}^{n} (p_{2k} - p_{2k-1}) > 0, \quad s_{2n+1} = \sum_{k=1}^{n} (p_{2k+1} - p_{2k}) + p_1 > 0.$$

Let $1 \leqslant k < n$. If $n - k$ is even, then

$$s_n - s_k = (p_n - p_{n-1}) + \cdots + (p_{k+2} - p_{k+1}) > 0.$$

If $n - k$ is odd, then

$$s_n - s_k = \sum_{l=k+1}^{n} (-1)^{n-l} p_l - 2 \sum_{j=1}^{k} (-1)^{k-j} p_j \equiv n - k \equiv 1 \pmod 2.$$

So, $s_1, s_2, s_3, \ldots$ are pairwise distinct.

An amazing recurrence for primes

We may compute the $(n+1)$-th prime p_{n+1} in terms of $p_1, \ldots, p_n$.

Conjecture (Z. W. Sun, J. Number Theory 2013). For any positive integer $n \neq 1, 2, 4, 9$, the $(n+1)$-th prime p_{n+1} is the least positive integer m such that

$$2s_1^2, \ldots, 2s_n^2$$

are pairwise distinct modulo m.

Remark. I have verified the conjecture for $n \leqslant 2 \times 10^5$, and proved that $2s_1^2, \ldots, 2s_n^2$ **are indeed pairwise distinct modulo** p_{n+1}.

Let $1 \leqslant j < k \leqslant n$. Then

$$0 < |s_k - s_j| \leqslant \max\{s_k, s_j\} \leqslant \max\{p_k, p_j\} \leqslant p_n < p_{n+1}.$$

Also, $s_k + s_j \leqslant p_k + p_j < 2p_{n+1}$. If $2 \nmid k - j$, then

$$s_k + s_j = p_k - p_{k-1} + \ldots + p_{j+1} \leqslant p_k < p_{n+1}.$$

If $2 \mid k - j$, then $s_k \equiv s_j \pmod 2$ and hence $s_k + s_j \neq p_{n+1}$. Thus $2s_k^2 - 2s_j^2 = 2(s_k - s_j)(s_k + s_j) \not\equiv 0 \pmod{p_{n+1}}$.

Conjecture on alternating sums of consecutive primes

Conjecture (Z. W. Sun, J. Number Theory, 2013). For any positive integer m, there are consecutive primes $p_k, \ldots, p_n$ $(k < n)$ not exceeding $2m + 2.2\sqrt{m}$ such that

$$m = p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k.$$

(Moreover, we may even require that $m < p_n < m + 4.6\sqrt{m}$ if $2 \nmid m$ and $2m - 3.6\sqrt{m+1} < p_n < 2m + 2.2\sqrt{m}$ if $2 \mid m$.)

Examples.

$$10 = 17 - 13 + 11 - 7 + 5 - 3;$$

$$20 = 41 - 37 + 31 - 29 + 23 - 19 + 17 - 13 + 11 - 7 + 5 - 3;$$

$$303 = p_{76} - p_{75} + \cdots + p_{52},$$

$$p_{76} = 383 = \lfloor 303 + 4.6\sqrt{303} \rfloor, \ p_{52} = 239;$$

$$2382 = p_{652} - p_{651} + \cdots + p_{44} - p_{43},$$

$$p_{652} = 4871 = \lfloor 2 \cdot 2382 + 2.2\sqrt{2382} \rfloor, \ p_{43} = 191.$$

The conjecture has been verified for m up to 10^9.

Prize. I would like to offer 1000 US dollars for the first proof.

Practical numbers

A positive integer n is called a *practical* number if every $m = 1, \ldots, n$ can be written as a sum of some distinct divisors of n, i.e., there are distinct divisors $d_1, \ldots, d_k$ of n such that

$$\frac{m}{n} = \sum_{i=1}^{k} \frac{1}{d_i}.$$

For example, 6 is practical since $1, 2, 3, 6$ divides 6, and also $4 = 1 + 3$ and $5 = 2 + 3$. As any positive integer has a unique representation in base 2 with digits in $\{0, 1\}$, powers of 2 are all practical. **1 is the only odd practical number**.

Practical numbers below 50:

1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48.

I view prime numbers $2, 3, 5, 7, 11, \ldots,$ as *men* and practical numbers $1, 2, 4, 6, 8, 12, \ldots$ as *women*. If one of n and $n + 1$ is prime and the other is practical, then I call $\{n, n + 1\}$ a *couple*.

Goldbach-type results for practical numbers

Theorem (Stewart [Amer. J. Math., 76(1954)]). If $p_1 < \cdots < p_r$ are distinct primes and $a_1, \ldots, a_r$ are positive integers then $m = p_1^{q_1} \cdots p_r^{a_r}$ is practical if and only if $p_1 = 2$ and

$$p_{s+1} - 1 \leqslant \sigma(p_1^{a_1} \cdots p_s^{a_s}) \quad \text{for all } \ 0 < s < r,$$

where $\sigma(n)$ stands for the sum of all divisors of n.

The behavior of practical numbers is quite similar to that of primes. G. Melfi proved the following Goldbach-type conjecture of M. Margenstern.

Theorem (G. Melfi [J. Number Theory 56(1996)]). Each positive even integer is a sum of two practical numbers, and there are infinitely many practical numbers m with $m-2$ and $m+2$ also practical.

Conjecture (Sun, 2013). Any even integer $2n > 4$ can be written as $p + q = (p+1) + (q-1)$, where p and q are primes with $p+1$ and $q-1$ both practical.

Two kinds of sandwiches

In 2013, I introduced two kinds of sandwiches.

First kind of sandwiches: $\{p-1, p, p+1\}$ with p prime and $p \pm 1$ practical.

Second kind of sandwiches: $\{q-1, q, q+1\}$ with q practical and $q \pm 1$ prime.

Conjecture (Sun, 2013).
(i) Each $n = 4, 5, \ldots$ can be written as $p + q$, where $\{p-1, p, p+1\}$ is a sandwich of the first kind, and q is either prime or practical.

(ii) Each even number $n > 8$ can be written as $p + q + r$, where $\{p-1, p, p+1\}$ and $\{q-1, q, q+1\}$ are sandwiches of the first kind, and $\{r-1, r, r+1\}$ is a sandwich of the second kind.

Two conjectures on Euler's totient function

For any positive integer m, define

$$\varphi(m) = |\{1 \le a \le m : \ (a, m) = 1\}|.$$

The function φ is called *Euler's totient function*. If $m = p_1^{a_1} \dots p_k^{a_k}$ with $p_1, \dots, p_k$ distinct primes and $a_1, \dots, a_k$ positive integers, then

$$\varphi(m) = m \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \prod_{i=1}^{k} p_i^{a_i - 1}(p_i - 1).$$

Conjecture 1 (Sun, 2013). Any integer $n > 5$ can be written as $k + m$ with k and m positive integers such that $(\varphi(k) + \varphi(m))/2$ is prime.

Conjecture 2 (Sun, 2014). Any integer $n > 8$ can be written as $k + m$ with k and m *distinct* positive integers such that $\varphi(k)\varphi(m)$ is a square.

Part II. Representations involving powers and polygonal numbers

Universal sums over $\mathbb{N}$

Let $f(x_1,\ldots,x_k)\in\mathbb{Z}[x_1,\ldots,x_k]$. If any $n\in\mathbb{N}$ can be written as $f(x_1,\ldots,x_k)$ with $x_1,\ldots,x_k$ in $\mathbb{N}$ (or $\mathbb{Z}$), then we say that f is *universal over* $\mathbb{N}$ (or $\mathbb{Z}$).

Suppose that $a_1x_1^{n_1}+\ldots+a_kx_k^{n_k}$ (with $a_1,\ldots,a_k\in\mathbb{Z}^+$) is universal over $\mathbb{N}$. For any positive integer N, each $n=1,\ldots,N$ can be written as $\sum_{i=1}^k a_ix_i^{n_i}$ with $x_i\in\mathbb{N}$, thus

$$|\{(x_1,\ldots,x_k)\in\mathbb{N}^k:\ a_1x_1^{n_1}\leqslant N,\ \ldots,\ a_kx_k^{n_k}\leqslant N\}|\geqslant N$$

and hence

$$N\leqslant\prod_{i=1}^k\left(1+\left(\frac{N}{a_i}\right)^{1/n_i}\right).$$

As this holds for any $N\in\mathbb{Z}^+$, we must have

$$\sum_{i=1}^k\frac{1}{n_i}\geqslant 1.$$

Lagrange's Four-square Theorem

Four-Square Theorem. Each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as the sum of four squares.

Examples. $3 = 1^2 + 1^2 + 1^2 + 0^2$ and $7 = 2^2 + 1^2 + 1^2 + 1^2$.

A. Diophantus (AD 299-215, or AD 285-201) was aware of this theorem as indicated by examples given in his book *Arithmetica*.

In 1621 Bachet translated Diophantus' book into Latin and stated the theorem in the notes of his translation.

In 1748 L. Euler found the four-square identity

$$\begin{aligned}&(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)\\ =&(x_1y_1+x_2y_2+x_3y_3+x_4y_4)^2+(x_1y_2-x_2y_1-x_3y_4+x_4y_3)^2\\ &+(x_1y_3-x_3y_1+x_2y_4-x_4y_2)^2+(x_1y_4-x_4y_1-x_2y_3+x_3y_2)^2.\end{aligned}$$

and hence reduced the theorem to the case with n prime.

On the basis of Euler's work, in 1770 J. L. Lagrange first completed the proof of the four-square theorem. The celebrated theorem is now known as *Lagrange's Four-square Theorem*.

Ramanujan's Observation

S. Ramanujan's Observation (confirmed by L.E. Dickson in 1927). There are totally 54 quadruples $(a,b,c,d)\in(\mathbb{Z}^+)^4$ with $a\leqslant b\leqslant c\leqslant d$ such that each $n\in\mathbb{N}$ can be written as $aw^2+bx^2+cy^2+dz^2$ with $w,x,y,z\in\mathbb{Z}$. The 54 quadruples are

(1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2), (1, 2, 2, 2), (1, 1, 1, 3), (1, 1, 2, 3),
(1, 2, 2, 3), (1, 1, 3, 3), (1, 2, 3, 3), (1, 1, 1, 4), (1, 1, 2, 4), (1, 2, 2, 4),
(1, 1, 3, 4), (1, 2, 3, 4), (1, 2, 4, 4), (1, 1, 1, 5), (1, 1, 2, 5), (1, 2, 2, 5),
(1, 1, 3, 5), (1, 2, 3, 5), (1, 2, 4, 5), (1, 1, 1, 6), (1, 1, 2, 6), (1, 2, 2, 6),
(1, 1, 3, 6), (1, 2, 3, 6), (1, 2, 4, 6), (1, 2, 5, 6), (1, 1, 1, 7), (1, 1, 2, 7),
(1, 2, 2, 7), (1, 2, 3, 7), (1, 2, 4, 7), (1, 2, 5, 7), (1, 1, 2, 8), (1, 2, 3, 8),
(1, 2, 4, 8), (1, 2, 5, 8), (1, 1, 2, 9), (1, 2, 3, 9), (1, 2, 4, 9), (1, 1, 5, 9),
(1, 1, 2, 10), (1, 2, 3, 10), (1, 2, 4, 10), (1, 2, 5, 10), (1, 1, 2, 11), (1, 2, 4, 11),
1, 1, 2, 12), (1, 2, 4, 12), (1, 1, 2, 13), (1, 2, 4, 13), (1, 1, 2, 14), (1, 2, 4, 14).

Universal sums of four mixed powers

If any $n \in \mathbb{N}$ can be written as $f(x_1, \ldots, x_n)$ with $x_1, \ldots, x_n$ in $\mathbb{N}$ (or $\mathbb{Z}$), then we say that f is *universal over* $\mathbb{N}$ (or $\mathbb{Z}$).

Theorem (Z.-W. Sun [JNT 175(2017)]) For any $a \in \{1, 4\}$ and $k \in \{4, 5, 6\}$, $aw^k + x^2 + y^2 + z^2$ is universal over $\mathbb{N}$.

Theorem (Z.-W. Sun [Nanjing Univ. J. Math. Biquarterly 34(2017)]) Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c \leqslant d$, and let $h, i, j, k \in \{2, 3, \ldots\}$ with at most one of h, i, j, k equal to two. Suppose that $h \leqslant i$ if $a = b$, $i \leqslant j$ if $b = c$, and $j \leqslant k$ if $c = d$. If $f(w, x, y, z) = aw^h + bx^i + cy^j + dz^k$ is universal over $\mathbb{N}$, then $f(w, x, y, z)$ must be among the following 9 polynomials

$$w^2 + x^3 + y^4 + 2z^3,\ w^2 + x^3 + y^4 + 2z^4,\ w^2 + x^3 + 2y^3 + 3z^3,$$

$$w^2 + x^3 + 2y^3 + 3z^4,\ w^2 + x^3 + 2y^3 + 4z^3,\ w^2 + x^3 + 2y^3 + 5z^3,$$

$$w^2 + x^3 + 2y^3 + 6z^3,\ w^2 + x^3 + 2y^3 + 6z^4,\ w^3 + x^4 + 2y^2 + 4z^3.$$

Conjecture (Sun [Nanjing Univ. J. Math. Biquarterly 34(2017)]) All the 9 polynomials are universal over $\mathbb{N}$.

1-3-5-Conjecture (1350 US dollars for the first solution)

1-3-5-Conjecture (Z.-W. Sun, April 9, 2016): Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.

Examples.

$$\begin{aligned} 7 =& 1^2 + 1^2 + 1^2 + 2^2 \text{ with } 1 + 3 \times 1 + 5 \times 1 = 3^2, \\ 8 =& 0^2 + 2^2 + 2^2 + 0^2 \text{ with } 0 + 3 \times 2 + 5 \times 2 = 4^2, \\ 31 =& 5^2 + 2^2 + 1^2 + 1^2 \text{ with } 5 + 3 \times 2 + 5 \times 1 = 4^2, \\ 43 =& 1^2 + 5^2 + 4^2 + 1^2 \text{ with } 1 + 3 \times 5 + 5 \times 4 = 6^2. \end{aligned}$$

The conjecture was proved by António Machiavelo and his PhD student Nikolaos Tsopanidis in 2020.

We guess that, if a, b, c are positive integers with $\gcd(a, b, c)$ squarefree such that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $ax + by + cz$ a square, then we must have $\{a, b, c\} = \{1, 3, 5\}$.

The 24-conjecture with \$2400 prize

24-Conjecture (Z.-W. Sun, Feb. 4, 2017). Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $x + 24y$ are squares.

Remark. Qing-Hu Hou has verified this for $n \leqslant 10^{10}$. I would like to offer 2400 US dollars as the prize for the first proof.

$$\begin{aligned}
12 =& 1^2 + 1^2 + 1^2 + 3^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 1 = 5^2,\\
23 =& 1^2 + 2^2 + 3^2 + 3^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 2 = 7^2,\\
24 =& 4^2 + 0^2 + 2^2 + 2^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 0 = 2^2,\\
47 =& 1^2 + 1^2 + 3^2 + 6^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 1 = 5^2,\\
71 =& 1^2 + 5^2 + 3^2 + 6^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 5 = 11^2,\\
168 =& 4^2 + 4^2 + 6^2 + 10^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 4 = 10^2,\\
344 =& 4^2 + 0^2 + 2^2 + 18^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 0 = 2^2,\\
632 =& 0^2 + 6^2 + 14^2 + 20^2 \text{ with } 0 = 0^2 \text{ and } 0 + 24 \times 6 = 12^2,\\
1724 =& 25^2 + 1^2 + 3^2 + 33^2 \text{ with } 25 = 5^2 \text{ and } 25 + 24 \times 1 = 7^2.
\end{aligned}$$

Unify the four-square theorem and the twin prime conjecture

The following conjecture implies the twin prime conjecture.

Conjecture (Z.-W. Sun, August 23, 2017). Any positive odd integer can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $p = x^2 + 3y^2 + 5z^2 + 7w^2$ and $p - 2$ are twin prime.

Example.

$$39 = 1^2 + 3^2 + 5^2 + 2^2$$

with $1^2 + 3 \cdot 3^2 + 5 \cdot 5^2 + 7 \cdot 2^2 = 181$ and $181 - 2 = 179$ twin prime. Also,

$$123 = 7^2 + 3^2 + 7^2 + 4^2$$

with $7^2 + 3 \cdot 3^2 + 5 \cdot 7^2 + 7 \cdot 4^2 = 433$ and $433 - 2 = 431$ twin prime.

Four-square Conjecture and 1-2-3 Conjecture

Four-square Conjecture (Z.-W. Sun, June 21, 2019). Any integer $n > 1$ can be written as $x^2 + y^2 + (2^a 3^b)^2 + (2^c 5^d)^2$ with $x, y, a, b, c, d \in \mathbb{N}$.

Remark. See `http://oeis.org/A308734` for related data. In 2019 G. Resta verified the conjecture for n up to 10^{10}.

Conjecture (1-2-3 Conjecture, Z.-W. Sun, Oct. 10, 2020).

(i) (Weak version) Any positive odd integer can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x + 2y + 3z \in \{2^a : a \in \mathbb{Z}^+\}$.

(ii) (Strong version) Any integer $m > 4627$ with $m \not\equiv 0, 2 \pmod 8$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x + 2y + 3z \in \{4^a : a \in \mathbb{Z}^+\}$.

Triangular numbers

Triangular numbers are those

$$T_n = \sum_{r=0}^{n} r = \frac{n(n+1)}{2} \quad (n \in \mathbb{N}).$$

Note that

$$T_{-n-1} = \frac{(-n-1)(-n)}{2} = T_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Theorem (conjectured by Fermat and proved by Gauss). Each $n \in \mathbb{N}$ can be written as $T_x + T_y + T_z$ with $x, y, z \in \mathbb{N}$.

Liouville's Theorem (Liouville, 1862). Let $a, b, c \in \mathbb{Z}^+$ and $a \leqslant b \leqslant c$. Then any $n \in \mathbb{N}$ can be written in the form $aT_x + bT_y + cT_z$ if and only if (a, b, c) is among

$$(1,1,1),\ (1,1,2),\ (1,1,4),\ (1,1,5),\ (1,2,2),\ (1,2,3),\ (1,2,4).$$

Mixed Sums of Squares and Triangular Numbers

Gauss-Legendre Theorem:

$$\{x^2+y^2+z^2:\ x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{4^k(8m+7):\ k,m\in\mathbb{N}\}.$$

Euler's Observation:

$$8n+1=(2x)^2+(2y)^2+(2z+1)^2$$
$$\Longrightarrow n=\frac{x^2+y^2}{2}+T_z=\left(\frac{x+y}{2}\right)^2+\left(\frac{x-y}{2}\right)^2+T_z.$$

Lionnet's Assertion (proved by Lebesgue & Réalis in 1872). Any $n\in\mathbb{N}$ is the sum of two triangular numbers and a square.

B. W. Jone and G. Pall [Acta Math. 1939]. Every $n\in\mathbb{N}$ is the sum of a square, an *even* square and a triangular number.

Theorem. (i) (Z. W. Sun, Acta Arith. 2007) Any $n\in\mathbb{N}$ is the sum of an *even* square and two triangular numbers.

(ii) (Conjectured by Z. W. Sun and proved by B. K. Oh and Sun [JNT, 2009]) Any positive integer n can be written as the sum of a square, an *odd* square and a triangular number.

Mixed Sums of Squares and Triangular Numbers

In 2005 Z. W. Sun [Acta Arith. 2007] investigated what kind of mixed sums $ax^2 + by^2 + cT_z$ or $ax^2 + bT_y + cT_z$ (with $a, b, c \in \mathbb{Z}^+$) are universal (i.e., all natural numbers can be so represented). This project was completed via three papers: Z. W. Sun [Acta Arith. 2007], S. Guo, H. Pan & Z. W. Sun [Integers, 2007], and B. K. Oh & Sun [JNT, 2009].

List of all universal $ax^2 + by^2 + cT_z$ **or** $ax^2 + bT_y + cT_z$:

$T_x + T_y + z^2,\ T_x + T_y + 2z^2,\ T_x + T_y + 4z^2,\ T_x + 2T_y + z^2,$
$T_x + 2T_y + 2z^2,\ T_x + 2T_y + 3z^2,\ T_x + 2T_y + 4z^2,\ 2T_x + T_y + z^2,$
$2T_x + 4T_y + z^2,\ 2T_x + 5T_y + z^2,\ T_x + 3T_y + z^2,\ T_x + 4T_y + z^2,$
$T_x + 4T_y + 2z^2,\ T_x + 6T_y + z^2,\ T_x + 8T_y + z^2,\ T_x + y^2 + z^2,$
$T_x + y^2 + 2z^2,\ T_x + y^2 + 3z^2,\ T_x + y^2 + 4z^2,\ T_x + y^2 + 8z^2,$
$T_x + 2y^2 + 2z^2,\ T_x + 2y^2 + 4z^2,\ 2T_x + y^2 + z^2,\ 2T_x + y^2 + 2y^2,$
$4T_x + y^2 + 2z^2.$

Two conjectures involving cubes or fourth powers

Conjecture (Z.-W. Sun, 2015). Each positive integer can be written as $x^3 + y^2 + T_z$ with $x, y \in \mathbb{N}$ and $z \in \mathbb{Z}^+$.

Conjecture (Z.-W. Sun, 2015). Each $n = 0, 1, 2, \ldots$ can be written as

$$x^4 + \frac{y(3y+1)}{2} + \frac{z(7z+1)}{2}$$

with $x, y, z \in \mathbb{Z}$.

Polygonal Numbers

Polygonal numbers are nonnegative integers constructed geometrically from the regular polygons. For $m = 3, 4, 5, \ldots$, the m-gonal numbers are given by

$$p_m(n) = (m-2)\binom{n}{2} + n \quad (n = 0, 1, 2, \ldots).$$

Clearly

$$p_3(n) = T_n,\ p_4(n) = n^2,\ p_5(n) = \frac{3n^2 - n}{2},\ p_6(n) = 2n^2 - n = T_{2n-1}.$$

The larger m is, the more sparse m-gonal numbers are.

Fermat's Assertion (1638). Any natural number n can be written as the sum of m m-gonal numbers.

Confirmation: $m = 4$ (Lagrange 1770), $m = 3$ (Gauss 1796), $m \geqslant 5$ (Cauchy 1813).

Mixed Sums of Three Polygonal Numbers

Conjecture [Z. W. Sun, 2009]. Let $3 \leqslant i \leqslant j \leqslant k$ and $k \geqslant 5$. Then each $n \in \mathbb{N}$ can be written as the sum of an i-gonal number, a j-gonal number and a k-gonal number, if and only if (i, j, k) is among the following 31 triples:

$$(3,3,5),\ (3,3,6),\ (3,3,7),\ (3,3,8),\ (3,3,10),\ (3,3,12),\ (3,3,17),$$
$$(3,4,5),\ (3,4,6),\ (3,4,7),\ (3,4,8),\ (3,4,9),\ (3,4,10),\ (3,4,11),$$
$$(3,4,12),\ (3,4,13),\ (3,4,15),\ (3,4,17),\ (3,4,18),\ \underline{(3,4,27)},$$
$$(3,5,5),\ (3,5,6),\ \underline{(3,5,7)},\ (3,5,8),\ (3,5,9),\ (3,5,11),\ (3,5,13),$$
$$(3,7,8),\ (3,7,10),\ (4,4,5),\ \underline{(4,5,6)}.$$

Remark. Sun proved the 'only if' part. The 'if' part is difficult!

Sun [Sci. China Math. 58(2015)] also showed that there are only 95 candidates for universal sums over $\mathbb{N}$ of the form $ap_i(x) + bp_j(y) + cp_k(z)$.

On $x(ax+b)+y(ay+c)+z(az+d)$ with $x,y,z \in \mathbb{Z}$

If any $n \in \mathbb{N}$ can be written as $f(x_1,\ldots,x_n)$ with $x_1,\ldots,x_n \in \mathbb{Z}$, then we say that f is *universal over* $\mathbb{Z}$.

As $T_n = T_{-n-1}$ for all $n \in \mathbb{N}$, we see that

$$\{T_n:\ n \in \mathbb{N}\} = \{T_{2x} = x(2x+1):\ x \in \mathbb{Z}\}$$

and hence $x(2x+1)+y(2y+1)+z(2z+1)$ is universal over $\mathbb{Z}$.

Theorem (Z.-W. Sun [JNT 171(2017)]) Let $a,b,c,d \in \mathbb{N}$ with $a>2$ and $b \leqslant c \leqslant d \leqslant a$. Then $x(ax+b)+y(ay+c)+z(az+d)$ is universal over $\mathbb{Z}$ if and only if the quadruple (a,b,c,d) is among

$$(3,0,1,2),\ (3,1,1,2),\ (3,1,2,2),\ (3,1,2,3),\ (4,1,2,3).$$

On $x(ax+1)+y(by+1)+z(cz+1)$ with $x,y,z\in\mathbb{Z}$

Theorem (Z.-W. Sun [JNT 171(2017)]) (i) Let $a,b,c\in\mathbb{Z}^+$ with $a\leqslant b\leqslant c$. If $f_{a,b,c}(x,y,z):=x(ax+1)+y(by+1)+z(cz+1)$ is universal over $\mathbb{Z}$, then (a,b,c) is among the following 17 triples:

$$(1,1,2),\ (1,2,2),\ (1,2,3),\ (1,2,4),\ (1,2,5),$$
$$(2,2,2),\ (2,2,3),\ (2,2,4),\ (2,2,5),\ (2,2,6),$$
$$(2,3,3),\ (2,3,4),\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10).$$

(ii) $f_{a,b,c}(x,y,z)$ is universal over $\mathbb{Z}$ if (a,b,c) is among

$$(1,2,3),\ (1,2,4),\ (1,2,5),\ (2,2,4),\ (2,2,5),\ (2,3,3),\ (2,3,4).$$

Conjecture (Sun). $f_{a,b,c}(x,y,z)$ is universal over $\mathbb{Z}$ if (a,b,c) is among (2,2,6), (2,3,5), (2,3,7), (2,3,8), (2,3,9), (2,3,10).

In 2017, Ju and Oh [arXiv:1701.02974] proved that

$$f_{2,2,6}(x,y,z)\ \text{ and }\ f_{2,3,c}(x,y,z)\ (c=5,7)$$

are universal over $\mathbb{Z}$. The universality of $f_{2,3,c}(x,y,z)$ over $\mathbb{Z}$ for $c=8,9,10$ remains open.

Write $n = a^2 + b^2 + 3^c + 5^d$

Conjecture (Z.-W. Sun, April 28, 2018). Any integer $n > 1$ can be written as $a^2 + b^2 + 3^c + 5^d$ with $a, b, c, d \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

Remark. I have verified this for n up to 2×10^{10}, and I'd like to offer 3500 US dollars as the prize for the first proof of this conjecture. I also conjecture that 5^d in the conjecture can be replaced by 2^d.

Example.

$2 = 0^2 + 0^2 + 3^0 + 5^0, \ 5 = 0^2 + 1^2 + 3^1 + 5^0, \ 25 = 1^2 + 4^2 + 3^1 + 5^1.$

Conjecture (Z.-W. Sun, April 26, 2018). Any integer $n > 1$ can be written as the sum of two squares and two central binomial coefficients.

Remark. I have verified this for n up to 10^{10}.

Example.

$$2435 = 32^2 + 33^2 + \binom{2 \times 4}{4} + \binom{2 \times 5}{5}.$$

Sums of two triangular numbers and two powers of 5

Recall that those $T_n = n(n+1)/2$ with $n \in \mathbb{N}$ are called triangular numbers. As claimed by Fermat and proved by Gauss, each $n \in \mathbb{N}$ is the sum of three triangular numbers.

Conjecture (Z.-W. Sun, April 23, 2018). Any integer $n > 1$ can be written as $T_a + T_b + 5^c + 5^d$ with $a, b, c, d \in \mathbb{N}$.

Remark. I have verified this for n up to 10^{10}.

Conjecture (Z.-W. Sun, April 23, 2018). Any integer $n > 1$ can be written as the sum of $p_5(a) + p_5(b) + 3^c + 3^d$ with $a, b, c, d \in \mathbb{N}$, where $p_5(k)$ denotes the pentagonal number $k(3k-1)/2$.

Remark. I have verified this for n up to 7×10^6.

Example.

$$285 = p_5(1) + p_5(11) + 3^3 + 3^4, \ 13372 = p_5(17) + p_5(65) + 3^4 + 3^8.$$

Representations involving tetrahedral numbers

Those numbers

$$t_n := \sum_{k=0}^{n} T_n = \sum_{k=0}^{n} \frac{k^2+k}{2} = \frac{n(n+1)(n+2)}{6} = \binom{n+2}{3} \ (n \in \mathbb{N})$$

are called tetrahedal numbers.

Pollock's Conjecture (Pollock, 1850). Each $n \in \mathbb{N}$ is the sum of five tetrahedal numbers.

Conjecture (Z.-W. Sun, Feb. 2019). (i) Any $n \in \mathbb{N}$ can be written as $2\binom{w}{3} + \binom{x}{3} + \binom{y}{3} + \binom{z}{3}$ with $w, x, y, z \in \mathbb{N}$.

(ii) Each $n \in \mathbb{N}$ can be written as $w^3 + \binom{x}{3} + \binom{y}{3} + \binom{z}{3}$ with $w, x, y, z \in \mathbb{N}$.

Remark. I verified parts (i) and (ii) for n up to 5×10^5 and 2×10^6. Later, a student of mine continued the verification for n up to 10^8.

Example. $1284 = 10^3 + \binom{7}{3} + \binom{9}{3} + \binom{11}{3}$.

2-4-6-8 Conjecture

2-4-6-8 Conjecture (Z.-W. Sun, Feb. 18, 2019). Any positive integer n can be written as

$$\binom{w}{2}+\binom{x}{4}+\binom{y}{6}+\binom{z}{8}$$

with $w,x,y,z\in\{2,3,\ldots\}$.

Remark. I verified this for n up to 3×10^7. Later, Max Alekseyev and Yaakov Baruch extended the verification for n up to 2×10^{11} and 2×10^{12} respectively.

Example.

$$23343989=\binom{365}{2}+\binom{76}{4}+\binom{40}{6}+\binom{34}{8}.$$

The 2-4-6-8 conjecture is very strong since

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}=\frac{25}{24}\approx 1.04.$$

Prize. 2468 US dollars for a proof, and 2468 RMB for a concrete counterexample.

Part IV. Some representations of integers or rational numbers

Write integers as $x^a + y^b - z^c$ with $x, y, z \in \mathbb{Z}^+$

Conjecture (Sun, Dec. 2015). If $\{a, b, c\}$ is among the multisets

$\{2, 2, p\}$ (p is prime or a product of some primes congruent to 1 mod 4),

$$\{2,3,3\},\ \{2,3,4\} \text{ and } \{2,3,5\},$$

then any integer m can be written as $x^a + y^b - z^c$, where x, y and z are positive integers.

We have verified that $\{x^4 - y^3 + z^2 :\ x, y, z \in \mathbb{Z}^+\}$ contains all integers m with $|m| \leqslant 10^5$. For example,

$$0 = 4^4 - 8^3 + 16^2,\quad -1 = 1^4 - 3^3 + 5^2,$$

$$-20 = 32^4 - 238^3 + 3526^2,\quad 11019 = 4325^4 - 71383^3 + 3719409^2.$$

Other Examples.

$$394 = 2283^3 + 128^4 - 110307^2,\quad 570 = 546596^2 + 8595^3 - 983^4,$$

$$445 = 9345^3 + 34^5 - 903402^2,\quad 435 = 475594653^2 + 290845^3 - 3019^5.$$

Egyptian fractions

Unit fractions have the form $1/n$ with $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

A sum of finitely many distinct unit fractions is called a *Egyptian fraction* as it was first studied by the ancient Egyptians around 1650 B.C.

As

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)},$$

any positive rational number $r = m/n$ with $m, n \in \mathbb{Z}^+$ is an Egyptian fraction.

This easy fact was first proved by Fibonacci in 1202 and it implies that the series $\sum_{n=1}^{\infty} 1/n$ diverges.

For example,

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{3} = \frac{1}{3} + \left(\frac{1}{3+1} + \frac{1}{3 \times 4}\right) = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}.$$

On Egyptian fractions involving primes

Euler: $\sum_p 1/p$ diverges, where p runs over all primes.

Dirichlet's Theorem: If $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ are relatively prime, then there are infinitely many primes $p \equiv a \pmod{m}$.

For any $m \in \mathbb{Z}^+$ there are infinitely many primes p with $p-1$ (or $p+1$) a multiple of m.

Conjecture 1.3 `with $500 Prize` (S., Sept. 9-10, 2015). For any positive rational number r, there is a finite set P_r^- of primes such that

$$\sum_{p \in P_r^-} \frac{1}{p-1} = r,$$

also there is a finite set P_r^+ of primes such that

$$\sum_{p \in P_r^+} \frac{1}{p+1} = r.$$

Verification: Qing-Hu Hou at Tianjin Univ. has verified this for all rational numbers $r \in (0,1]$ with denominators not exceeding 100.

Examples:

$$
\begin{aligned}
1 =& \frac{1}{2-1} = \frac{1}{3-1} + \frac{1}{5-1} + \frac{1}{7-1} + \frac{1}{13-1}, \\
1 =& \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{5+1} + \frac{1}{7+1} + \frac{1}{11+1} + \frac{1}{23+1}, \\
\frac{1}{19} =& \frac{1}{37-1} + \frac{1}{137-1} + \frac{1}{191-1} + \frac{1}{229-1} \\
& + \frac{1}{331-1} + \frac{1}{397-1} + \frac{1}{761-1} + \frac{1}{1021-1} \\
=& \frac{1}{37+1} + \frac{1}{107+1} + \frac{1}{227+1} + \frac{1}{239+1} \\
& + \frac{1}{311+1} + \frac{1}{359+1} + \frac{1}{701+1} + \frac{1}{911+1} \text{ (Sun)}.
\end{aligned}
$$

In 2018, Prof. Guo-Niu Han found 2065 distinct primes $p_1 < \ldots < p_{2065}$ with $p_{2065} \approx 4.7 \times 10^{218}$ such that

$$\frac{1}{p_1+1} + \ldots + \frac{1}{p_{2065}+1} = 2.$$

A similar conjecture involving practical numbers

$n \in \mathbb{Z}^+$ is *practical* if each $m = 1, \ldots, n$ can be written as the sum of some distinct (positive) divisors of n. 1 is the only odd practical number, and all powers of two are practical numbers.

For $x > 0$ let $P(x) = |\{q \leqslant x : \ q \text{ is practical}\}|$. Then

$$P(x) \sim c\frac{x}{\log x} \quad \text{for some constant } c > 0,$$

(conjectured by M. Margenstern in 1991 and proved by A. Weingartner in 2014).

Conjecture 1.4 (Sun, Sept. 12, 2015). Any positive rational number r can be written as $\sum_{j=1}^{k} 1/q_j$, where $q_1, \ldots, q_k$ are distinct practical numbers.

Example.

$$\frac{10}{11} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{48} + \frac{1}{132} + \frac{1}{176}$$

with $2, 4, 8, 48, 132, 176$ all practical numbers.

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with $2, 4, 8, 48, 132, 176$ all practical numbers.

In Nov. 2016, David Eppstein proved the above conjecture.

Motivations from Pell's equations

If $d \in \mathbb{Z}^+$ is not a square, then the Pell equation

$$x^2 - dy^2 = 1$$

has infinitely many integral solutions.

Let $r = a/b$ with $a, b \in \mathbb{Z}^+$ and $\gcd(a, b) = 1$. If r is not a square of rational numbers (i.e., ab is not a square), then there is a positive integer k such that $(ka)(kb) + 1 = abk^2 + 1$ is a square, i.e., we can write $r = m/n$ with $m, n \in \mathbb{Z}^+$ such that $mn + 1$ is a square.

For any $r = a/b$ with $a, b \in \mathbb{Z}^+$ and $\gcd(a, b) = 1$, by Schinzel's Hypothesis there is a positive integer x such that $p = ax + 1$ and $q = bx + 1$ are both prime and hence

$$r = \frac{ax}{bx} = \frac{p-1}{q-1}.$$

Motivated by the above, the speaker posed some conjectures on writing each positive rational number as a special ratio.

m/n with $p_m + p_n$ a square

Conjecture (i) (Sun, 2015-07-03) The set

$$\left\{ \frac{m}{n} : \ m, n \in \mathbb{Z}^+ \text{ and } p_m + p_n \text{ is a square} \right\}$$

contains any positive rational number r.

(ii) (Sun, 2015-08-20) Any positive rational number $r \neq 1$ can be written as m/n with $m, n \in \mathbb{Z}^+$ such that $p_{p_m} + p_{p_n}$ is a square.

We have verified part (i) of this conjecture for all those $r = a/b$ with $a, b \in \{1, \ldots, 200\}$, and part (ii) for all those $r = a/b \neq 1$ with $a, b \in \{1, \ldots, 60\}$. For example, $2 = 20/10$ with

$$p_{20} + p_{10} = 71 + 29 = 10^2,$$

and $2 = 92/46$ with

$$p_{p_{92}} + p_{p_{46}} = p_{479} + p_{199} = 3407 + 1217 = 68^2.$$

m/n with $\varphi(m)$ and $\sigma(n)$ both squares

Conjecture (Sun, 2015-07-08). Any positive rational number r can be written as m/n with $m, n \in \mathbb{Z}^+$ such that both $\varphi(m)$ and $\sigma(n)$ are both squares.

We have verified this for all $r = a/b$ with $a, b \in \{1, \ldots, 150\}$.

Examples:

$$\frac{4}{5} = \frac{136}{170} \text{ with } \varphi(136) = 8^2 \text{ and } \sigma(170) = 18^2,$$

and

$$\frac{5}{4} = \frac{1365}{1092} \text{ with } \varphi(1365) = 24^2 \text{ and } \sigma(1092) = 56^2.$$

Theorem (D. Krachun and Z.-W. Sun [Amer. Math. Monthly 127(2020)]). Each positive rational number can be written as $\varphi(m^2)/\varphi(n^2)$, where m and n are positive integers.

m/n with $\pi(m)\pi(n)$ a positive square

Conjecture (Sun, 2015-07-05). Any positive rational number r can be written as m/n with $m, n \in \mathbb{Z}^+$ such that $\pi(m)\pi(n)$ is a positive square, where $\pi(x) = |\{p \leqslant x : \ p \text{ is prime}\}|$.

Remark. We have verified this conjecture for all those rational numbers $r = a/b$ with $a, b \in \{1, \dots, 60\}$. For example,

$$\frac{49}{58} = \frac{1076068567}{1273713814}$$

with

$$\pi(1076068567)\pi(1273713814) = 54511776 \cdot 63975626 = 59054424^2.$$

m/n with $p(m)^2 + p(n)^2$ a square

Conjecture (Sun, 2015-07-02). Any positive rational number r can be written as m/n with $m, n \in \mathbb{Z}^+$ such that $p(m)^2 + p(n)^2$ is prime, where $p(\cdot)$ is the partition function.

Remark. This conjecture implies that there are infinitely many primes of the form $p(m)^2 + p(n)^2$ with $m, n \in \mathbb{Z}^+$. We have verified it for all those $r = a/b$ with $a, b \in \{1, \ldots, 100\}$.

Example. $4/5 = 124/155$ with

$$\begin{aligned} p(124)^2 + p(155)^2 =& 2841940500^2 + 66493182097^2 \\ =& 4429419891190341567409 \end{aligned}$$

prime.

References

For main sources of my conjectures mentioned here, you may look at:

1. Z.-W. Sun, *On functions taking only prime values*, J. Number Theory **133** (2013), 2794–2812.

2. Z.-W. Sun, *Conjectures on representations involving primes*, in: Combinatorial and Additive Number Theory II: CANT, New York, NY, USA, 2015 and 2016 (edited by M. Nathanson), Springer Proc. in Math. & Stat., Vol. 220, Springer, New York, 2017.

3. Z.-W. Sun, *Universal sums of three quadratic polynomials*, Sci. China Math. **63** (2020), 501–520.

Thank you!