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NEW CONJECTURES ON REPRESENTATIONS OF INTEGERS (I)

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ABSTRACT. In this paper we study representations of natural numbers by weighted sums of powers and pose some new conjectures in this direction. For example, we show that there are essentially only 9 polynomials of the form $aw^h + bx^i + cy^j + dz^k$ with a, b, c, d positive integers, $h, i, j, k \in \{2, 3, 4, \ldots\}$ and at most one of h, i, j, k equal to two, for which it seems that each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as $aw^h + bx^i + cy^j + dz^k$ with $w, x, y, z \in \mathbb{N}$. One of our conjectures states that $\{f(w, x, y, z) :\ w, x, y, z \in \mathbb{N}\} = \mathbb{N}$ whenever $f(w, x, y, z)$ is among the 9 polynomials $w^2 + x^3 + 2y^3 + cz^3$ ($c = 3, 4, 5, 6$), $w^2 + x^3 + 2y^3 + dz^4$ ($d = 1, 3, 6$), $2w^2 + x^3 + 4y^3 + z^4$ and $w^2 + x^3 + y^4 + 2z^4$. We also conjecture that any integer $n > 1$ can be written as $x^4 + y^3 + z^2 + 2^k$ with $x, y, z \in \mathbb{N}$ and $k \in \{1, 2, 3, \ldots\}$

1. INTRODUCTION

For a polynomial $f(x_1, \ldots, x_k)$ with coefficients in $\mathbb{N} = \{0, 1, 2, \ldots\}$, we define

$$\mathrm{Ran}(f) := \{f(x_1, \ldots, x_k) :\ x_1, \ldots, x_k \in \mathbb{N}\}.$$

If $\mathrm{Ran}(f) = \mathbb{N}$, then we say that f is *universal over* $\mathbb{N}$.

Lagrange's four-square theorem proved in 1770 states that any nonnegative integer can be written as the sum of four squares, i.e., $w^2+x^2+y^2+z^2$ is universal over $\mathbb{N}$. In 1917 S. Ramanujan [3] conjectured that for his 55 listed quadruples (a, b, c, d) with $a, b, c, d \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $a \leqslant b \leqslant c \leqslant d$ the polynomial $aw^2 + bx^2 + cy^2 + dz^2$ is universal over $\mathbb{N}$. In 1927 L. E. Dickson [1] confirmed this for 54 of them and pointed our that the remaining one is wrong.

Now we present our first theorem.

Theorem 1.1. *Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c \leqslant d$, and let $h, i, j, k \in \{2, 3, \ldots\}$ with at most one of h, i, j, k equal to two. Assume that $h \leqslant i$*

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if $a = b$, $i \leqslant j$ if $b = c$, and $j \leqslant k$ if $c = d$. Suppose that

$$f(w,x,y,z) = aw^h + bx^i + cy^j + dz^k$$

is universal over $\mathbb{N}$. Then $f(w,x,y,z)$ must be among the following 9 polynomials

$$\begin{gathered} w^2+x^3+y^4+2z^3,\ w^2+x^3+y^4+2z^4,\ w^2+x^3+2y^3+3z^3, \\ w^2+x^3+2y^3+3z^4,\ w^2+x^3+2y^3+4z^3,\ w^2+x^3+2y^3+5z^3, \\ w^2+x^3+2y^3+6z^3,\ w^2+x^3+2y^3+6z^4,\ w^3+x^4+2y^2+4z^3. \end{gathered} \tag{1.1}$$

Remark 1.1. Let $a, b, c \in \mathbb{Z}^+$ and $i, j, k \in \{2, 3, \ldots\}$. By Theorem 1.1, the polynomial $ax^i + by^j + cz^k$ is not universal over $\mathbb{N}$ if at most one of i, j, k is two. By the way we prove Theorem 1.1, we are also able to show that $ax^2 + by^2 + cz^k$ is not universal over $\mathbb{N}$.

Conjecture 1.1. *If $f(w,x,y,z)$ is among the 9 polynomials listed in* (1.1)*, then it is universal over $\mathbb{N}$, i.e., any $n \in \mathbb{N}$ can be written as $f(w,x,y,z)$ with $w,x,y,z \in \mathbb{N}$.*

Remark 1.2. This conjecture looks quite challenging. We have verified that if $f(w,x,y,z)$ is among the 9 polynomials listed in (1.1) then every $n = 0, \ldots, 10^7$ can be written as $f(w,x,y,z)$ with $w,x,y,z \in \mathbb{N}$. See [4, A262827] for the number of ways to write $n \in \mathbb{N}$ as $w^2+x^3+y^4+2z^4$ with $w,x,y,z \in \mathbb{N}$.

Now we consider universal weighted sums of five nonnegative cubes. Clearly, if a_1, a_2, a_3, a_4, a_5 are integers greater than one then 1 cannot be written as $\sum_{i=1}^5 a_i x_i^3$ with $x_i \in \mathbb{N}$.

Theorem 1.2. *Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c \leqslant d$. Suppose that the polynomial $u^3 + av^3 + bx^3 + cy^3 + dz^3$ is universal over $\mathbb{N}$. Then (a,b,c,d) must be among the following 32 quadruples:*

$$\begin{gathered} (1,2,2,3),(1,2,2,4),(1,2,3,4),(1,2,4,5),(1,2,4,6),(1,2,4,9), \\ (1,2,4,10),(1,2,4,11),(1,2,4,18),(1,3,4,6),(1,3,4,9),(1,3,4,10), \\ (2,2,4,5),(2,2,6,9),(2,3,4,5),(2,3,4,6),(2,3,4,7),(2,3,4,8), \\ (2,3,4,9),(2,3,4,10),(2,3,4,12),(2,3,4,15),(2,3,4,18),(2,3,5,6), \\ (2,3,6,12),(2,3,6,15),(2,4,5,6),(2,4,5,8),(2,4,5,9),(2,4,5,10), \\ (2,4,6,7),(2,4,7,10). \end{gathered} \tag{1.2}$$

Motivated by Theorem 1.2, we pose the following conjectures based on our computation.

Conjecture 1.2. *If (a,b,c,d) is among the 32 quadruples listed in (1.2), then the polynomial $u^3+av^3+bx^3+cy^3+dz^3$ is universal over $\mathbb{N}$, i.e., any $n\in\mathbb{N}$ can be written as $u^3+av^3+bx^3+cy^3+dz^3$ with $u,v,x,y,z\in\mathbb{N}$.*

Remark 1.3. See [4, A271237] for the number of ways to write $n\in\mathbb{N}$ as $u^3+2v^3+3x^3+4y^3+5z^3$ with $u,v,x,y,z\in\mathbb{N}$.

We also have the following conjecture.

Conjecture 1.3. *Let*

$$P(u,v,x,y,z)=au^h+bv^i+cx^j+dy^k+ez^l$$

with $a,b,c,d,e\in\mathbb{Z}^+$, $h,i,j,k,l\in\{3,4,5,\ldots\}$, $h\leqslant i\leqslant j\leqslant k\leqslant l$ and $j\geqslant 4$. Then $P(u,v,x,y,z)$ is universal over $\mathbb{N}$ if and only if $h=i=3$, $j=k=l=4$, and one of the following (i)-(iii) *holds.*

(i) $\{a,b\}=\{1,2\}$ *and* $\{c,d,e\}=\{1,2,3\}$,

(ii) $\{a,b\}=\{1,2\}$ *and* $\{c,d,e\}=\{1,2,4\}$,

(iii) $\{a,b\}=\{1,5\}$ *and* $\{c,d,e\}=\{1,2,4\}$.

Remark 1.4. We have verified that if $P(u,v,x,y,z)$ is among the three polynomials

$$u^3+2v^3+x^4+2y^4+3z^4,\ u^3+2v^3+x^4+2y^4+4z^4,\ u^3+5v^3+x^4+2y^4+4z^4$$

then $\mathrm{Ran}(P(u,v,x,y,z))\supseteq\{0,\ldots,10^6\}$. The "only if" part of Conjecture 1.3 might be proved in the way we show Theorem 1.1, but we have not done that in details.

Our following conjecture is of particular interest.

Conjecture 1.4. (i) *Any $n\in\mathbb{N}$ can be written as $x^3+2y^2+5^kz^2$ with $k\in\{0,1\}$ and $x,y,z\in\mathbb{N}$.*

(ii) *For any $n\in\mathbb{Z}^+$, we can write $12n+1$ as $8x^4+4y^2+z^2$ with $x\in\mathbb{N}$ and $y,z\in\mathbb{Z}^+$. Also, for any $n\in\mathbb{N}$, we can write $12n+5$ as $4x^4+4y^2+z^2$ with $x\in\mathbb{N}$ and $y,z\in\mathbb{Z}^+$, and write $16n+5$ as $x^4+4y^2+z^2$ with $x\in\mathbb{N}$ and $y,z\in\mathbb{Z}^+$.*

Remark 1.5. See [4, A290491, A260418, A275150] for related data. It is well known that $\{2n+1:\ n\in\mathbb{N}\}\subseteq\mathrm{Ran}(2x^2+y^2+z^2)$ and $\{4n+1:\ n\in\mathbb{N}\}\subseteq\mathrm{Ran}(x^2+y^2+z^2)$ (see, e.g., [2, pp. 112-113]).

The next section is devoted to our proofs of Theorems 1.1 and 1.2. In Section 3 we pose more conjectures on polynomials in 4 or 5 variables which are universal over $\mathbb{N}$. In Section 4 we consider weighted sums of k-th powers and formulated some conjectures upgrading Waring's problem. In Section 5 we investigate for what positive integers a,b,c

we have $\{x^a+y^b-z^c:\ x,y,z\in\mathbb{Z}^+\}=\mathbb{Z}$. In Section 6 we pose several conjectures on writing $n\in\mathbb{Z}^+$ in the form $a^k+bx^h+cy^j+dz^k$ with $k,x,y,z\in\mathbb{N}$.

Note that the conjectures in this paper are different from the 100 conjectures in [6].

2. Proofs of Theorems 1.1 and 1.2

As Theorem 1.2 is relatively easy than Theorem 1.1, we first prove Theorem 1.2 and then show Theorem 1.1.

Proof of Theorem 1.2. Since $2\in\mathrm{Ran}(u^3+av^3+bc^3+cy^3+dz^3)$, we cannot have $a>2$. As $a+2\not\in\mathrm{Ran}(u^3+av^3)$, we must have $b\leqslant a+2\leqslant 4$. Clearly, $b+3\not\in\mathrm{Ran}(u^3+v^3+bx^3)$ and so $c\leqslant b+3$ if $a=1$. If $a=b=2$, then $c\leqslant 6$ since $6\not\in\mathrm{Ran}(u^3+2v^3+2x^3)$. If $a=2$ and $b=3$, then $c\leqslant 7$ since $7\not\in\mathrm{Ran}(u^3+2v^3+3x^3)$. If $a=2$ and $b=4$, then $c\leqslant 9$ since $9\not\in\mathrm{Ran}(u^3+2v^3+4x^3)$. Thus (a,b,c) is among the following triples

$$\begin{gathered}(1,1,1),\ (1,1,2),\ (1,1,3),\ (1,1,4),\ (1,2,2),\ (1,2,3),\\ (1,2,4),\ (1,2,5),\ (1,3,3),\ (1,3,4),\ (1,3,5),\ (1,3,6),\\ (2,2,i)\ (2\leqslant i\leqslant 6),\ (2,3,j)\ (3\leqslant j\leqslant 7),\ (2,4,k)\ (4\leqslant k\leqslant 9).\end{gathered}$$

Via a computer we find that $\mathrm{Ran}(u^3+av^3+bx^3+cy^3)$ cannot contain all the numbers $0,\ldots,19$, so $d\leqslant 19$. As $a\in\{1,2\}$, $a\leqslant b\leqslant a+2$, $b\leqslant c\leqslant b+5$ and $c\leqslant d\leqslant 19$, via a computer we see that if (a,b,c,d) is not among the 32 quadruples in (1.2) then

$$\{0,1,\ldots,350\}\not\subseteq\mathrm{Ran}(u^3+av^3+bx^3+cy^3+dz^3).$$

For example, 350 is the least nonnegative integer not in the form $u^3+2v^3+4x^3+9y^3+18z^3$ with $u,v,x,y,z\in\mathbb{N}$. This concludes the proof. $\square$

Proof of Theorem 1.1. Since $1,2\in\mathrm{Ran}(f)$, we must have $a=1$ and $b\leqslant 2$. As $3\not\in\mathrm{Ran}(w^h+x^i)$, we have $c\leqslant 3$ if $b=1$. Similarly, $5\not\in\mathrm{Ran}(w^h+2x^i)$, and hence $c\leqslant 5$ if $b=2$.

Case 1. $b=c=1$.

Suppose that $h>2$. Then $4\not\in\mathrm{Ran}(w^h+x^i+y^j)$. Since $4\in\mathrm{Ran}(f)$, we must have $d\leqslant 4$. Note that $w^h+x^i+y^j+4z^k\neq 11$ for all $w,x,y,z\in\mathbb{N}$. So $d\leqslant 3$ and hence $w^h+x^i+y^j+dz^k\neq 7$ for all $w,x,y,z\in\mathbb{N}$, which contradicts that $7\in\mathrm{Ran}(f)$.

By the above, $h=2$ and hence $i,j,k\geqslant 3$. As $7\not\in\mathrm{Ran}(w^2+x^i+y^j)$, we have $d\leqslant 7$. If $d=1$, then $8\in\mathrm{Ran}(w^2+x^i+y^j+z^k)$ and hence $i=3$, but $w^2+x^3+y^j+z^k\neq 15$ for any $w,x,y,z\in\mathbb{N}$. So

$2 \leqslant d \leqslant 7$ and hence $d2^k \geqslant 8d > d+7$. If $w^2+x^i+y^j+dz^k = d+7$ with $w,x,y,z \in \mathbb{N}$, then $z=0$ and $w^2+x^i+y^j = d+7$. Clearly $w^2+x^i+y^j \neq 7+7$ for any $w,x,y \in \mathbb{N}$, so $d \neq 7$. If $d \in \{5,6\}$ and $d+7 \in \mathrm{Ran}(w^2+x^i+y^j)$, then $i=3$. Note that $w^2+x^3+y^j+5z^k \neq 19$ for any $w,x,y,z \in \mathbb{N}$. If $20 \in \mathrm{Ran}(w^2+x^3+y^j+6z^k)$, then $j=3$ but $21 \notin \mathrm{Ran}(w^2+x^3+y^3+6z^k)$. Therefore $2 \leqslant d \leqslant 4$.

If $i>3$, then $w^2+x^i+y^j \neq 12$ for all $w,x,y \in \mathbb{N}$, and hence by $d+12 \in \mathrm{Ran}(f)$ and $2^k d \geqslant 8d > d+12$ we obtain $d+12 \in \mathrm{Ran}(w^2+x^i+y^j)$ which is impossible for $d=2,3$. Note also that when $i>3$ we have $w^2+x^i+y^j+4z^k \neq 12$ for any $w,x,y,z \in \mathbb{N}$. Therefore $i=3$.

Since $22 \notin \mathrm{Ran}(w^2+x^3+y^j+3z^k)$ and $19 \notin \mathrm{Ran}(w^2+x^3+y^j+4z^k)$, we must have $d=2$.

As $w^2+x^3+y^3+2z^k \neq 23$ for any $w,x,y,z \in \mathbb{N}$, we have $j>3$. If $j>4$, then by $22 \in \mathrm{Ran}(w^2+x^3+y^j+2z^k)$ we have $k=3$, but $w^2+x^3+y^j+2z^3 \neq 23$ for any $w,x,y,z \in \mathbb{N}$. Thus $j=4$. If $k>4$, then $w^2+x^3+y^4+2z^k \neq 48$ for all $w,x,y,z \in \mathbb{N}$. Therefore $k \in \{3,4\}$, and hence $f(w,x,y,z)$ is $w^2+x^3+y^4+2z^3$ or $w^2+x^3+y^4+2z^4$.

Case 2. $b=1$ and $c=2$.

Suppose that $h>2$. Then $i \geqslant h \geqslant 3$ and $5 \notin \mathrm{Ran}(w^h+x^i+2y^j)$. As $5 \in \mathrm{Ran}(f)$, we must have $2 \leqslant d \leqslant 5$. As $d2^k \geqslant 4d > d+5$, if $f(w,x,y,z) = w^h+x^i+2y^j+dz^k = d+5$ with $w,x,y,z \in \mathbb{N}$ then $z=0$ and $w^h+x^i+2y^j = d+5$. By $d+5 \in \mathrm{Ran}(w^h+x^i+2y^j)$, we have $3 \leqslant d \leqslant 5$, and also $h=3$ or $j=2$. If $w^h+x^i+2y^j+3z^k = 15$ with $w,x,y,z \in \mathbb{N}$, then $z \geqslant 2$ and $k=2$. Thus, when $d=3$ we have $k=2$ and $h=3$. If $w^3+x^i+2y^j+3z^2 = 17$ with $w,x,y,z \in \mathbb{N}$, then $i=4$ or $j=3$. If $w^3+x^i+2y^j+3z^2 = 34$ with $w,x,y,z \in \mathbb{N}$, then $i=5$ or $j=4$. Thus $d=3$ is impossible. As $4 \leqslant d \leqslant 5$ and $11 \in \mathrm{Ran}(f)$, if $j=2$ then we get $h=3$. So we always have $h=3$. As $w^3+x^i+2y^j+5z^k \neq 12$ for any $w,x,y,z \in \mathbb{N}$, we must have $d=4$. If $j=2$, then $25 \in \mathrm{Ran}(w^3+x^i+2y^2+4z^k)$ and hence $i=4$, thus by $105 \in \mathrm{Ran}(w^3+x^4+2y^2+4z^k)$ we get $k \in \{3,4\}$. Note that $w^3+x^4+2y^2+4z^4 \neq 57$ for any $w,x,y,z \in \mathbb{N}$, and that $w^3+x^4+2y^2+4z^3$ is a polynomial listed in (1.1). If $k=2$, then

$$\{(2z)^2+w^3+x^i+2y^j:\ z,w,x,y \in \mathbb{N}\} = \mathrm{Ran}(f) = \mathbb{N}$$

and hence by the last passage in our discussion of Case 1 we must have $i=4$ and $j \in \{3,4\}$. Note that 51 cannot be written as $w^3+x^4+2y^3+4z^2$ or $w^3+x^4+2y^4+4z^2$ with $w,x,y,z \in \mathbb{N}$. So $k>2$. When $j>2$, by $25 \in \mathrm{Ran}(w^3+x^i+2y^j+4z^k)$ we get $j=3$, and by $26 \in \mathrm{Ran}(w^3+x^i+2y^3+4z^k)$ we obtain $i=4$. If $w^3+x^4+2y^3+$

$4z^k = 35$ with $w,x,y,z \in \mathbb{N}$, then $z \geqslant 2$ and hence $k \leqslant 3$. Note that $w^3 + x^4 + 2y^3 + 4z^3 \neq 38$ for any $w,x,y,z \in \mathbb{N}$.

Below we assume $h = 2$ and hence $i,j,k \geqslant 3$.

Suppose that $i > 3$. Then $w^2 + x^i + 2y^j \neq 8$ for all $w,x,y \in \mathbb{N}$. As $8 \in \mathrm{Ran}(f)$, we must have $d \leqslant 8$. If $5 \leqslant d \leqslant 7$, then $w^2+x^i+2y^j+dz^k \neq d+8$ for any $w,x,y,z \in \mathbb{N}$. Note that $w^2 + x^i + 2y^j + 2z^k \neq 15$ for all $w,x,y,z \in \mathbb{N}$, and also $w^2+x^i+2y^j+8z^k \neq 23$ for any $w,x,y,z \in \mathbb{N}$. So $3 \leqslant d \leqslant 4$. When $d = 3$, by $23, 24 \in \mathrm{Ran}(w^2+x^i+2y^j+3z^k)$ we get that $i = 4$ or $j = 3$, and that $\min\{j,k\} = 3$. As $w^2 + x^4 + 2y^j + 3z^3 \neq 47$ for any $w,x,y,z \in \mathbb{N}$, we have $j = 3$ if $d = 3$. Note that $w^2+x^i+2y^3+3z^k \neq 47$ for any $w,x,y,z \in \mathbb{N}$. So $d \neq 3$ and hence $d = 4$. If $k > 3$, then by $35 \in \mathrm{Ran}(w^2+x^i+2y^j+4z^k)$ we get $i = 5$, but $w^2+x^5+2y^j+4z^k \neq 44$ for any $w,x,y,z \in \mathbb{N}$. If $w^2 + x^i + 2y^j + 4z^3 = 47$ with $w,x,y,z \in \mathbb{N}$, then $x \geqslant 2$ and hence $i \in \{4,5\}$. If $w^2 + x^i + 2y^j + 4z^3 = 46$ with $w,x,y,z \in \mathbb{N}$, then $y \geqslant 2$ and hence $j \in \{3,4\}$. As $i \in \{4,5\}$ and $j \in \{3,4\}$, we can easily see that $w^2 + x^i + 2y^j + 4z^3 \neq 76$ for any $w,x,y,z \in \mathbb{N}$. This leads a contradiction.

By the above, we must have $i = 3$. Since $w^2 + x^3 + 2y^j \neq 13$ for any $w,x,y \in \mathbb{N}$, by $13 \in \mathrm{Ran}(f)$ we must have $2 \leqslant d \leqslant 13$. Note that $d2^k \geqslant 8d > d+13$. If $w^2+x^3+2y^j+dz^k = d+13$ with $w,x,y,z \in \mathbb{N}$, then $z = 0$ and hence $w^2+x^3+2y^j = d+13$. Since $w^2+x^3+2y^j \neq 15, 22, 23$ for any $w,x,y \in \mathbb{N}$, we have $d \neq 2, 9, 10$. If $7+13 \in \mathrm{Ran}(w^2+x^3+2y^j)$, then $j = 3$ but $w^2 + x^3 + 2y^3 + 7z^k \neq 22$ for any $w,x,y,z \in \mathbb{N}$. If $8+13 \in \mathrm{Ran}(w^2 + x^3 + 2y^j)$, then $j = 3$ but $w^2 + x^3 + 2y^3 + 8z^k \neq 23$ for any $w,x,y,z \in \mathbb{N}$. If $34 \in \mathrm{Ran}(w^2 + x^3 + 2y^j + 11z^k)$, then $j = 4$ but $w^2+x^3+2y^4+11z^k \neq 109$ for any $w,x,y,z \in \mathbb{N}$. If $34 \in \mathrm{Ran}(w^2 + x^3 + 2y^j + 12z^k)$, then $j = 4$. If $98 \in \mathrm{Ran}(w^2 + x^3 + 2y^4 + 12z^k)$, then $k = 3$. Note that $w^2 + x^3 + 2y^4 + 12z^3 \neq 111$ for any $w,x,y,z \in \mathbb{N}$. If $61 \in \mathrm{Ran}(w^2+x^3+2y^j+13z^k)$, then $j = 4$ but $w^2+x^3+2y^4+13z^k \neq 90$ for any $w,x,y,z \in \mathbb{N}$. Therefore $3 \leqslant d \leqslant 6$.

Case 2.1. $d = 3$.

Since $23 \in \mathrm{Ran}(f) = \{w^2 + x^3 + 2y^j + 3z^k : w,x,y,z \in \mathbb{N}\}$, we must have $j = 3$. If $w^2 + x^3 + 2y^3 + 3z^k = 112$ with $w,x,y,z \in \mathbb{N}$, then $z \geqslant 2$ and hence $k \leqslant 5$. Note that $w^2 + x^3 + 2y^3 + 3z^5 \neq 247$ for any $w,x,y,z \in \mathbb{N}$. Thus $f(w,x,y,z)$ is $w^2 + x^3 + 2y^3 + 3z^3$ or $w^2 + x^3 + 2y^3 + 3z^4$, and hence $f(w,x,y,z)$ is among the polynomials listed in (1.1).

Case 2.2. $d = 4$.

If $w^2+x^3+2y^j+4z^4 = 60$ with $w,x,y,z \in \mathbb{N}$, then $y \neq 0, 1$ and hence $j \leqslant 4$. If $k \neq 4$ and $w^2 + x^3 + 2y^j + 4z^k = 90$ with $w,x,y,z \in \mathbb{N}$, then $y \geqslant 2$ and hence $j \leqslant 5$. If $w^2 + x^3 + 2y^5 + 4z^k = 60$ with $w,x,y,z \in \mathbb{N}$, then $z \geqslant 2$ and hence $k = 3$. Note that $w^2 + x^3 + 2y^5 + 4z^3 \neq 207$ for any $w, x, y, z \in \mathbb{N}$. Whether $k = 4$ or not, we always have $j \in \{3, 4\}$.

If $w^2 + x^3 + 2y^3 + 4z^k = 99$ with $w, x, y, z \in \mathbb{N}$, then $z \geqslant 2$ and hence $k \in \{3, 4\}$. Note that $308 \notin \mathrm{Ran}(w^2 + x^3 + 2y^3 + 4z^4)$ and $w^2 + x^3 + 2y^3 + 4z^3$ is among the polynomials listed in (1.1).

If $w^2 + x^3 + 2y^4 + 4z^k = 92$ with $w, x, y, z \in \mathbb{N}$, then $z \geqslant 2$ and hence $k \in \{3, 4\}$. Note that $94 \notin \mathrm{Ran}(w^2 + x^3 + 2y^4 + 4z^3)$ and $111 \notin \mathrm{Ran}(w^2 + x^3 + 2y^4 + 4z^4)$.

Case 2.3. $d = 5$.

Observe that $w^2 + x^3 + 2y^j \neq 94, 99$ for all $w, x, y \in \mathbb{N}$. If $w^2 + x^3 + 2y^j + 5z^k = 99$ with $w, x, y, z \in \mathbb{N}$, then $z \geqslant 2$ and hence $k \leqslant 4$. Note that $18, 20 \notin \mathrm{Ran}(w^2 + x^3 + 5z^k)$. If $w^2 + x^3 + 2y^j + 5z^k = 20$ with $w, x, y, z \in \mathbb{N}$, then $y \geqslant 2$ and hence $j = 3$. Note that $w^2+x^3+2y^3+5z^3$ is among the polynomial listed in (1.1) and $w^2 + x^3 + 2y^3 + 5z^4 \neq 114$ for any $w, x, y, z \in \mathbb{N}$.

Case 2.3. $d = 6$.

As $21 \in \mathrm{Ran}(f) = \{w^2 + x^3 + 2y^j + 6z^k : w, x, y, z \in \mathbb{N}\}$, we must have $j = 3$. If $w^2+x^3+2y^3+6z^k = 120$ with $w, x, y, z \in \mathbb{N}$, then $z \geqslant 2$ and hence $k \leqslant 4$. Both $w^2 + x^3 + 2y^3 + 6z^3$ and $w^2 + x^3 + 2y^3 + 6z^4$ are among the polynomials listed in (1.1).

Case 3. $b = 1$ and $c = 3$.

Since $6 \in \mathrm{Ran}(f)$ but $w^h + x^i + 3y^j \neq 6$ for all $w, x, y \in \mathbb{N}$, we must have $d \leqslant 6$. Note that $d2^k \geqslant 4d > d + 6$ since $d \geqslant 3$. If $w^h + x^i + 3y^j + dz^k = d + 6$ with $w, x, y, z \in \mathbb{N}$, then $z = 0$ and $w^h + x^i + 3y^j = d + 6$.

Case 3.1. $d = 3$.

As $3 + 6 \in \mathrm{Ran}(w^h + x^i + 3y^j)$, we have $h \leqslant 3$. If $h = 2$, then by $14 \in \mathrm{Ran}(w^2+x^i+3y^j+3z^k)$ we get $i = 3$, but $w^2+x^3+3y^j+3z^k \neq 21$ for any $w, x, y, z \in \mathbb{N}$. If $h = 3$, then $i \geqslant h = 3$ and $w^3+x^i+3y^j+3z^k \neq 10$ for any $w, x, y, z \in \mathbb{N}$.

Case 3.2. $d = 4$.

As $4+6 \in \mathrm{Ran}(w^h + x^i + 3y^j)$, we have $h = 2$. Since $15 \in \mathrm{Ran}(w^2 + x^i + 3y^j + 4z^k)$, we must have $i = 3$. Note that $w^2+x^3+3y^j+4z^k \neq 18$ for any $w, x, y, z \in \mathbb{N}$.

Case 3.3. $d = 5$.

As $5 + 6 \in \mathrm{Ran}(w^h + x^i + 3y^j)$, h or i is 3. If $h = 2$ and $i = 3$, then $w^2 + x^3 + 3y^j + 5z^k \neq 23$ for any $w, x, y, z \in \mathbb{N}$. So $i \geqslant h = 3$. Note that $w^3 + x^i + 3y^j + 5z^k \neq 15$ for any $w, x, y, z \in \mathbb{N}$.

Case 3.3. $d = 6$.

As $6+6\in\mathrm{Ran}(w^h+x^i+3y^j)$, if $j\neq 2$ then $h\leqslant 3$. If $j=2$, then by $15\in\mathrm{Ran}(w^h+x^i+3y^2+6z^k)$ we get $h\leqslant 3$. So we always have $h\leqslant 3$.

Suppose that $h=2$. By $21\in\mathrm{Ran}(w^2+x^i+3y^j+6z^k)$ we get $i=3$, and by $62\in\mathrm{Ran}(w^2+x^3+3y^j+6z^k)$ either $j=4$ or $k=3$. As $41\notin\mathrm{Ran}(w^2+x^3+3y^4+6z^k)$, we have $j\neq 4$ and hence $k=3$. By $38\in\mathrm{Ran}(w^2+x^3+3y^j+6z^3)$ we get $j=3$. Note that $62\notin\mathrm{Ran}(w^2+x^3+3y^3+6z^3)$. So we get a contradiction.

Now we assume $h=3$. As $i\geqslant h>2$, by $13\in\mathrm{Ran}(w^3+x^i+3y^j+6z^k)$ we get $j=2$. As $23\in\mathrm{Ran}(f)=\mathrm{Ran}(w^3+x^i+3y^2+6z^k)$, we have $i=4$. As $32\notin\mathrm{Ran}(w^3+x^4+3y^2+6z^k)$, we get a contradiction.

Case 4. $b=c=2$.

Since $7\notin\mathrm{Ran}(w^h+2x^i+2y^j)$, we have $2\leqslant d\leqslant 7$ by $7\in\mathrm{Ran}(f)$. When $d>2$ or $k>2$, if $w^h+2x^i+2y^j+dz^k=d+7$ with $w,x,y,z\in\mathbb{N}$, then $z=0$ and $w^h+2x^i+2y^j=d+7$. If $d=2$, then $k>2$ since $k\geqslant j\geqslant i$, and by $2+7\in\mathrm{Ran}(w^h+2x^i+2y^j)$ we get $\min\{h,i\}=2$. If $d=h=2$, then $i,j,k\geqslant 3$ and $w^2+2x^i+2y^j+2z^k\neq 12$ for any $w,x,y,z\in\mathbb{N}$. If $d=i=2$, then by $14\in\mathrm{Ran}(w^h+2x^2+2y^j+2z^k)$ we get $h=3$, but $w^3+2x^2+2y^j+2z^k\neq 15$ for any $w,x,y,z\in\mathbb{N}$. So $2<d\leqslant 7$ and $d+7\in\mathrm{Ran}(w^h+2x^i+2y^j)$.

Case 4.1. $d=3$.

Since $3+7\in\mathrm{Ran}(w^h+2x^i+2y^j)$, either $h=3$ or $i=2$. If $i>2$, then $h=3$, and by $14\in\mathrm{Ran}(w^3+2x^i+2y^j+3z^k)$ we get $k=2$, but $w^3+2x^i+2y^j+3z^2\neq 9$ for any $w,x,y,z\in\mathbb{N}$. So $i=2$ and $h,j,k\geqslant 3$. If $h\geqslant 4$, then $w^h+2x^2+2y^j+3z^k\neq 15$ for any $w,x,y,z\in\mathbb{N}$. Note also that $w^3+2x^2+2y^j+3z^k\neq 39$ for any $w,x,y,z\in\mathbb{N}$. So we get a contradiction.

Case 4.2. $d=4$.

Since $4+7\in\mathrm{Ran}(w^h+2x^i+2y^j)$, either $h=2$ or $i=2$. If $h=2$, then $i,j,k\geqslant 3$ and hence $w^2+2x^i+2y^j+4z^k\neq 14$ for any $w,x,y,z\in\mathbb{N}$. So $i=2$ and $h,j,k\geqslant 3$. By $17\in\mathrm{Ran}(w^h+2x^2+2y^j+4z^k)$ we get $j=3$. As $69\in\mathrm{Ran}(f)=\mathrm{Ran}(w^h+2x^2+2y^3+4z^k)$, either $k=4$ or $k=h=3$. Note that $w^3+2x^2+2y^3+4z^3\neq 123$ for all $w,x,y,z\in\mathbb{N}$. So $k=4$. By $27\in\mathrm{Ran}(w^h+2x^2+2y^3+4z^4)$ we get $h=3$. Note that $w^3+2x^2+2y^3+4z^4\neq 188$ for any $w,x,y,z\in\mathbb{N}$. So we get a contradiction.

Case 4.3. $d=5$.

As $5+7\in\mathrm{Ran}(w^h+2x^i+2x^j)$, we have $h=3$. By $1\in\mathrm{Ran}(w^3+2x^i+2y^j+5z^k)$ we get $i=2$, and by $41\in\mathrm{Ran}(w^3+2x^2+2y^j+5z^k)$ we obtain $j=4$ or $k=3$. As $22\notin\mathrm{Ran}(w^3+2x^2+2y^4+5z^k)$, we have $j\neq 4$ and hence $k=3$. By $22\in\mathrm{Ran}(f)=\mathrm{Ran}(w^3+2x^2+2y^j+5z^3)$

we get $j=3$. Note that $w^3+2x^2+2y^3+5z^3\neq 46$ for any $w,x,y,z\in\mathbb{N}$. So we get a contadiction.

Case 4.3. $d\in\{6,7\}$.

By $d+7\in\mathrm{Ran}(w^h+2x^i+2y^j)$, we get $d=6$ and $h=2$. Since $i,j,k\geqslant 3$ and $w^2+2x^i+2y^j+6z^k\neq 21$ for any $w,x,y,z\in\mathbb{N}$, we get a contradiction.

Case 5. $b=2$ and $c=3$.

If $h>2$, then $7\notin\mathrm{Ran}(w^h+2x^i+3y^j)$ and hence $d\leqslant 7$ by $7\in\mathrm{Ran}(f)$. If $h=2$, then $i,j,k\geqslant 3$, $8\notin\mathrm{Ran}(w^h+2x^i+3y^j)$ and hence $d\leqslant 8$, also $w^2+2x^i+3y^j+8z^k\neq 37$ for any $w,x,y,z\in\mathbb{N}$. So we have $3\leqslant d\leqslant 7$. Note that $d2^k\geqslant 4d>d+8$ and

$$d+7,d+8\in\mathrm{Ran}(f)=\mathrm{Ran}(w^h+2x^i+3y^j+dz^k).$$

If $d+7\notin\mathrm{Ran}(w^h+2x^i+3y^j)$, then $7\in\mathrm{Ran}(w^h+2x^i+3y^j)$ and hence $h=2$. If $d+8\notin\mathrm{Ran}(w^h+2x^i+3y^j)$, then $8\in\mathrm{Ran}(w^h+2x^i+3y^j)$, hence $h=3$ or $i=2$.

Case 5.1. $d=3$.

If $h>2$, then $3+7\in\mathrm{Ran}(w^h+2x^i+3y^j)$ and hence $h=3$. If $h=3$ and $j>2$, then by $12\in\mathrm{Ran}(w^3+2x^i+3y^j+3z^k)$ we get $i=2$, but $w^3+2x^2+3y^j+3z^k\neq 17$ for any $w,x,y,z\in\mathbb{N}$. When $h=3$ and $j=2$, we have $w^3+2x^i+3y^2+3z^k\neq 21$ for any $w,x,y,z\in\mathbb{N}$. So we must have $h=2$ and $i,j,k\geqslant 3$. As $w^2+2x^i+3y^j+3z^k\neq 13$ for any $w,x,y,z\in\mathbb{N}$, we get a contradiction.

Case 5.2. $d=4$.

If $h>2$, then $4+7\in\mathrm{Ran}(w^h+2x^i+3y^j)$ and hence $h=3$ or $i=2$. If $h=3$ and $i\geqslant 3$, then by $25\in\mathrm{Ran}(w^3+2x^i+3y^j+4z^k)$ we get $j=3$, and by $18\in\mathrm{Ran}(w^3+2x^i+3y^3+4z^k)$ we obtain $k=2$, but $123\notin\mathrm{Ran}(w^3+2x^3+3y^3+4z^2)$ and $23\notin\mathrm{Ran}(w^3+2x^i+3y^3+4z^2)$ for $i>3$. So $h=2$ or $i=2$.

If $h=2$, then by $17\in\mathrm{Ran}(w^2+2x^i+3y^j+4z^k)$ we get $i=3$, but $w^2+2x^3+3y^j+4z^k\neq 47$ for any $w,x,y,z\in\mathbb{N}$. Thus $i=2$ and $h,j,k\geqslant 3$. By $14\in\mathrm{Ran}(w^h+2x^2+3y^j+4z^k)$ we obtain $h=3$. Also, by $24\in\mathrm{Ran}(w^3+2x^2+3y^j+4z^k)$ we get $j=3$, and by $41\in\mathrm{Ran}(w^3+2x^2+3y^3+4z^k)$ we obtain $k=3$. Since $w^3+2x^2+3y^3+4z^3\neq 176$ for any $w,x,y,z\in\mathbb{N}$, we get a contradiction.

Case 5.2. $d=5$.

If $h=2$, then $i,j\geqslant 3$ and $5+8\notin\mathrm{Ran}(w^2+2x^i+3y^j)$, hence we get a contradiction. So $h>2$ and hence $5+7\in\mathrm{Ran}(w^h+2x^i+3y^j)$. It follows that $i=2$ or $j=2$. If $i=2$, then $w^h+2x^2+3y^j+5z^k\neq 20$

for any $w,x,y,z\in\mathbb{N}$. So $j=2$. As $23\in\mathrm{Ran}(w^h+2x^i+3y^2+5z^k)$, we must have $h=4$. Note that $w^4+2x^i+3y^2+5z^k\neq 31$ for any $w,x,y,z\in\mathbb{N}$. So we get a contradiction.

Case 5.3. $d=6$.

If $h>2$, then $6+7\in\mathrm{Ran}(w^h+2x^i+3y^j)$ and hence $h=3$ or $j=2$. If $h=3$ and $j>2$, then by $15\in\mathrm{Ran}(w^3+2x^i+3y^j+6z^k)$ we get $i=2$, but $w^3+2x^2+3y^j+5z^k\neq 20$ for any $w,x,y,z\in\mathbb{N}$. So $h=2$ or $j=2$.

If $h=2$, then by $23\in\mathrm{Ran}(w^2+2x^i+3y^j+6z^k)$ we get $i=3$, but $w^2+2x^3+3y^j+6z^k\neq 37$ for any $w,x,y,z\in\mathbb{N}$. So $j=2$ and hence $h,i,k\geqslant 3$. Since $w^h+2x^i+3y^2+6z^k\neq 31$ for any $w,x,y,z\in\mathbb{N}$, we get a contradiction.

Case 5.4. $d=7$.

If $h>2$, then $7+7\in\mathrm{Ran}(w^h+2x^i+3y^j)$ and hence $j=2$. If $h=2$, then $h\neq 3$ and $i\neq 2$, hence $7+8\in\mathrm{Ran}(w^2+2x^i+3y^j)$ which is impossible. So $j=2$ and $h,i,k\geqslant 3$. By $25\in\mathrm{Ran}(w^h+2x^i+3y^2+7z^k)$ we get $h=4$. Since $w^4+2x^i+3y^2+7z^k\neq 31$ for any $w,x,y,z\in\mathbb{N}$, we obtain a contradiction.

Case 6. $b=2$ and $c=4$.

If $h=2$, then $12\notin\mathrm{Ran}(w^2+2x^i+4y^j)$ and hence $d\leqslant 12$ by $12\in\mathrm{Ran}(f)$. For $i,j,k\geqslant 3$ we have $w^2+2x^i+4y^j+12z^k\neq 26$ for any $w,x,y,z\in\mathbb{N}$. If $h>2$, then $11\notin\mathrm{Ran}(w^h+2x^i+4y^j)$ and hence $d\leqslant 11$. So we always have $4\leqslant d\leqslant 11$.

If $h>2$ and $i>2$, then $9\notin\mathrm{Ran}(w^h+2x^i+4y^j)$ and hence $d\leqslant 9$ by $9\in\mathrm{Ran}(f)$, also $d+9\in\mathrm{Ran}(w^h+2x^i+4y^j)$ since $d+9\in\mathrm{Ran}(f)$ and $d2^k\geqslant 4d>d+9$.

Case 6.1. $d\in\{4,6\}$.

If $h>2$ and $i>2$, then $d+9\in\mathrm{Ran}(f)=\mathrm{Ran}(w^h+2x^i+4y^j)$ which is actually impossible. So $h=2$ or $i=2$.

If $h=2$ and $i,j,k\geqslant 3$, then $23\notin\mathrm{Ran}(w^2+2x^i+4y^j+4z^k)$ and hence $d=6$, thus $i=3$ by $23\in\mathrm{Ran}(f)=\mathrm{Ran}(w^2+2x^i+4y^j+6z^k)$. If $j,k\geqslant 3$ and $w^2+2x^3+4y^j+6z^k=77$ with $w,x,y,z\in\mathbb{N}$, then $z\geqslant 2$ and hence $k=3$. When $j>2$, clearly $w^2+2x^3+4y^j+6z^3\neq 78$ for any $w,x,y,z\in\mathbb{N}$. Thus $h>2$, hence $i=2$ and $j,k\geqslant 3$. Whether d is 4 or 6, we have $w^h+2x^2+4y^j+dz^k\neq d+11$ for any $w,x,y,z\in\mathbb{N}$, This contradicts that $d+11\in\mathrm{Ran}(f)$.

Case 6.2. $d=5$.

If $h>2$ and $i>2$, then $5+9\in\mathrm{Ran}(w^h+2x^i+4y^j)$ and hence $h=3$, also by $18\in\mathrm{Ran}(w^3+2x^i+4y^j+5z^k)$ we get $j=2$, but $w^3+2x^i+4y^2+5z^k\neq 30$ for any $w,x,y,z\in\mathbb{N}$. So $h=2$ or $i=2$.

If $h=2$, then $i,j,k\geqslant 3$, and by $17\in \mathrm{Ran}(w^2+2x^i+4y^j+5z^k)$ we get $i=3$, but $w^2+2x^3+4y^j+5z^k\neq 19$ for any $w,x,y,z\in\mathbb{N}$. If $i=2$, then by $15\in\mathrm{Ran}(w^h+2x^2+4y^j+5z^k)$ we get $h=3$, but $w^3+2x^2+4y^j+5z^k\neq 57$ for any $w,x,y,z\in\mathbb{N}$. So we get a contradiction.

Case 6.3. $d=7$.

By $15\in\mathrm{Ran}(w^h+2x^i+4y^j+7z^k)$, we have $h\leqslant 3$ or $i=2$. If $h=3$ and $i>2$, then $w^3+2x^i+4y^j+7z^k\neq 22$ for any $w,x,y,z\in\mathbb{N}$. So $h=2$ or $i=2$. If $i,j,k\geqslant 3$, then $w^2+2x^i+4y^j+7z^k\neq 19$ for any $w,x,y,z\in\mathbb{N}$. Thus $h\neq 2$ and hence $i=2$. By $17\in\mathrm{Ran}(w^h+2x^2+4y^j+7z^k)$, we get $h=3$. Since $w^3+2x^2+4y^j+7z^k\neq 24$ for any $w,x,y,z\in\mathbb{N}$, we get a contradiction.

Case 6.4. $d=8$.

Since $23\in\mathrm{Ran}(f)=\mathrm{Ran}(w^h+2x^i+4y^j+8z^k)$, either $h=2$ or $i=2$. If $i,j,k\geqslant 3$, then $w^2+2x^i+4y^j+8z^k\neq 47$ for any $w,x,y,z\in\mathbb{N}$. So $h>2$ and hence $i=2$. As $h,j,k\geqslant 3$, we see that $w^h+2x^2+4y^j+8z^k\neq 25$ for any $w,x,y,z\in\mathbb{N}$. This leads a contradiction.

Case 6.5. $d=9$.

Since $23\in\mathrm{Ran}(f)=\mathrm{Ran}(w^h+2x^i+4y^j+9z^k)$, either $h=3$ or $i=2$. If $h=3$ and $i>2$, then by $20\in\mathrm{Ran}(w^3+2x^i+4y^j+9z^k)$ we get $i=3$, but $w^3+2x^3+4y^j+9z^k\neq 22$ for any $w,x,y,z\in\mathbb{N}$. Thus $i=2$ and $h,j,k\geqslant 3$. Since $w^h+2x^2+4y^j+9z^k\neq 26$ for any $w,x,y,z\in\mathbb{N}$, we get a contradiction.

Case 6.6. $d=10$.

By $9\in\mathrm{Ran}(w^h+2x^i+4y^j+10z^k)$, we have $h=2$ or $i=2$. If $h=2$, then $i,j,k\geqslant 3$, and by $24\in\mathrm{Ran}(w^2+2x^i+4y^j+10z^k)$ we get $i=3$, but $w^2+2x^3+4y^j+10z^k\neq 47$ for any $w,x,y,z\in\mathbb{N}$. So $i=2$ and $h,j,k\geqslant 3$. As $w^h+2x^2+4y^j+10z^k\neq 21$ for any $w,x,y,z\in\mathbb{N}$, we get a contradiction.

Case 6.7. $d=11$.

By $9\in\mathrm{Ran}(w^h+2x^i+4y^j+11z^k)$, we have $h=2$ or $i=2$. If $i,j,k\geqslant 3$, then $w^2+2x^i+4y^j+11z^k\neq 23$ for any $w,x,y,z\in\mathbb{N}$. So $h\neq 2$ and hence $i=2$. By $10\in\mathrm{Ran}(w^h+2x^2+4y^j+11z^k)$ we get $h=3$. Note that $w^3+2x^2+4y^j+11z^k\neq 28$ for any $w,x,y,z\in\mathbb{N}$. This leads a contradiction.

Case 7. $b=2$ and $d=5$.

As $d\geqslant 5$ and $4\in\mathrm{Ran}(f)=\mathrm{Ran}(w^h+2x^i+5y^j+dz^k)$, we have $4\in\mathrm{Ran}(w^h+2x^i)$, hence $h=2$ and $i,j,k\geqslant 3$. Since $10\notin\mathrm{Ran}(w^2+$

$2x^i+5y^j$) and $d+10 \in \mathrm{Ran}(w^2+2x^i+5y^j+dz^k)$, we must have $d \leqslant 10$ and $d+10 \in \mathrm{Ran}(w^2+2x^i+5y^j)$. As $5\times 2^j \geqslant 5\times 2^3 > d+10$, $\mathrm{Ran}(w^2+2x^i)$ contains $d+5$ or $d+10$.

If $d \in \{5,9\}$, then $\mathrm{Ran}(w^2+2x^i)\cap\{d+5,d+10\}=\emptyset$. Note also that $w^2+2x^i+5y^j+6z^k \neq 19$ for any $w,x,y,z\in\mathbb{N}$. If $\mathrm{Ran}(w^2+2x^i)\cap\{10+5,10+10\}\neq\emptyset$, then $i=3$, but $w^2+2x^3+5y^j+10z^k\neq 29$ for any $w,x,y,z\in\mathbb{N}$. If $d\in\{7,8\}$, then by $d+12\in\mathrm{Ran}(w^2+2x^i+5y^j+dz^k)$ we get $i=3$ and $d=8$, but $w^2+2x^3+5y^j+8z^k\neq 39$ for any $w,x,y,z\in\mathbb{N}$. So we get a contradiction.

The proof of Theorem 1.1 is now complete. $\square$

3. More conjectures on polynomials in 4 or 5 variables universal over $\mathbb{N}$

Based on our computation, we pose here several conjectures on polynomials in 4 or 5 variables universal over $\mathbb{N}$.

Conjecture 3.1. (i) *Any $n\in\mathbb{N}$ can be written as $P(\delta,x,y,z)$ with $\delta\in\{0,1\}$ and $,x,y,z\in\mathbb{N}$, whenever $P(\delta,x,y,z)$ is among*

$$2\delta+x^2+4y^2+5z^2,\ c\delta+x^2+2y^2+7z^2\ (c=2,3,4,5),$$
$$4\delta+x^2+2y^2+cz^2\ (c=9,11,12).$$

(ii) *We have $\{P(w,x,y,z):\ w,x,y,z\in\mathbb{N}\}=\mathbb{N}$ if $P(w,x,y,z)$ is among the polynomials*

$$w^3+x^2+3y^2+5z^2,\ w^3+2x^2+2y^2+5z^2,\ w^3+2x^2+3y^2+4z^2,$$
$$w^3+2x^2+3y^2+5z^2,\ w^3+2x^2+4y^2+cz^2\ (c=5,7,9,11),$$
$$w^3+x^2+2y^2+cz^2\ (c=12,\ldots,15),\ 2w^3+x^2+y^2+cz^2\ (c=11,13),$$
$$2w^3+x^2+3y^2+cz^2\ (c=5,7),\ 2w^3+x^2+4y^2+cz^2\ (c=7,10),$$
$$3w^3+x^2+2y^2+9z^2,\ 4w^3+x^2+2y^2+cz^2\ (c=13,14),$$

$$w^4+x^2+2y^2+cz^2\ (c=11,12,13),\ w^4+2x^2+3y^2+5z^2,$$
$$2w^4+x^2+y^2+13z^2,\ 2w^4+x^2+3y^2+5z^2,$$
$$3w^4+x^2+2y^2+9z^2,\ 4w^4+x^2+2y^2+13z^2,$$
$$w^5+2x^2+3y^2+5z^2,\ w^5+x^2+2y^2+cz^2\ (c=11,12,13).$$

Remark 3.1. The author [5] showed by induction that for each $a\in\{1,4\}$ and $k\in\{4,5,6\}$ any $n\in\mathbb{N}$ can be written as $aw^k+x^2+y^2+z^2$ with $w,x,y,z\in\mathbb{N}$. In a similar way, we can show $\mathrm{Ran}(aw^k+bx^2+cy^2+dz^2)=\mathbb{N}$ for some other $a,b,c,d\in\mathbb{Z}^+$ and $k\in\{3,4,\ldots\}$ with $bx^2+cy^2+dz^2$ regular. (For all the 102 regular diagonal ternary quadratic forms $bx^2+cy^2+dz^2$ listed in [2, pp. 112-113], the set $\mathrm{Ran}(bx^2+cy^2+dz^2)$ is known explicitly.) Note that none of the forms $x^2+y^2+cz^2$ ($c = 7, 10, 11, 13$) is regular.

Conjecture 3.2. (i) *Any $n \in \mathbb{N}$ can be written as $x^2+ay^2+z^3+b\delta$ with $x, y, z \in \mathbb{N}$ and $\delta \in \{0, 1\}$, provided that (a, b) is among the following pairs*

$$(1,2),\ (1,6),\ (2,4),\ (2,5),\ (2,11),\ (2,12),\ (3,3).$$

(ii) *$P(w, x, y, z)$ is universal over $\mathbb{N}$ whenever it is among the following polynomials*

$$
\begin{gathered}
w^3+x^3+2y^2+4z^2,\ w^3+2x^3+y^2+2z^2,\ w^3+2x^3+2y^2+3z^2,\\
w^3+2x^3+2y^2+4z^2,\ w^3+2x^3+2y^2+5z^2,\ w^3+2x^3+y^2+6z^2,\\
w^3+2x^3+y^2+13z^2,\ w^3+3x^3+y^2+2z^2,\ w^3+3x^3+2y^2+3z^2,\\
w^3+3x^3+2y^2+4z^2,\ w^3+3x^3+y^2+5z^2,\ w^3+3x^3+2y^2+5z^2,\\
w^3+3x^3+y^2+6z^2,\ 2w^3+3x^3+y^2+cz^2\ (c=2,3,4),\\
w^3+4x^3+y^2+z^2,\ w^3+4x^3+2y^2+cz^2\ (c=2,3,5),\\
2w^3+4x^3+y^2+cz^2\ (c=1,6,8,10),\ w^3+5x^3+y^2+z^2,\\
2w^3+5x^3+y^2+3z^2,\ 3w^3+5x^3+y^2+2z^2,\ 4w^3+5x^3+y^2+2z^2,\\
4w^3+6x^3+y^2+2z^2,\ w^3+7x^3+y^2+2z^2,\ 2w^3+7x^3+y^2+3z^2,\\
3w^3+7x^3+y^2+2z^2,\ 2w^3+8x^3+y^2+4z^2,\ 4w^3+8x^3+y^2+2z^2,\\
w^3+9x^3+2y^2+4z^2,\ 3w^3+9x^3+y^2+2z^2,\ w^3+10x^3+y^2+2z^2,\\
2w^3+10x^3+y^2+z^2,\ w^3+11x^3+2y^2+4z^2,\ 4w^3+11x^3+y^2+2z^2,\\
2w^3+13x^3+y^2+z^2,\ w^3+14x^3+y^2+2z^2,\\
w^4+x^3+cy^2+3z^2\ (c=1,2),\ w^4+x^3+2y^2+4z^2,\\
w^4+x^3+y^2+6z^2,\ w^4+2x^3+y^2+4z^2,\\
w^4+cx^3+y^2+2z^2\ (c=3,4,12),\ 2w^4+x^3+2y^2+4z^2,\\
2w^4+x^3+y^2+5z^2,\ 2w^4+x^3+3y^3+5z^2,\\
2w^4+x^3+y^2+11z^2,\ 2w^4+x^3+y^2+12z^2,\\
2w^4+4x^3+y^2+z^2,\ 3w^4+x^3+y^2+cz^2\ (c=1,2),\\
3w^4+4x^3+y^2+2z^2,\ 4w^4+x^3+cy^2+3z^2\ (c=1,2),\\
4w^4+2x^3+y^2+cz^2\ (c=1,8),\ 5w^4+x^3+y^2+z^2,
\end{gathered}
$$

$$5w^4+x^3+2y^2+3z^2,\ 5w^4+x^3+2y^2+4z^2,$$
$$5w^4+2x^3+y^2+3z^2,\ 6w^4+x^3+y^2+3z^2,$$
$$6w^4+2x^3+y^2+z^2,\ 8w^4+x^3+y^2+2z^2,$$
$$8w^4+4x^3+y^2+2z^2,\ 10w^4+x^3+y^2+2z^2,$$
$$10w^4+2x^3+y^2+z^2,\ 11w^4+x^3+2y^2+4z^2,$$
$$13w^4+2x^3+y^2+z^2,\ cw^4+x^3+y^2+2z^2\ (c=14,15),$$
$$w^5+x^3+y^2+2z^2,\ w^5+x^3+y^2+3z^2,\ w^5+4x^3+y^2+2z^2,$$
$$w^5+12x^3+y^2+2z^2,\ 2w^5+x^3+y^2+cz^2\ (c=2,3),$$
$$3w^5+x^3+y^2+cz^2\ (c=1,2),\ 3w^5+2x^3+y^2+z^2,$$
$$4w^5+x^3+y^2+3z^2,\ 4w^5+x^3+2y^2+cz^2\ (c=3,5),$$
$$4w^5+2x^3+y^2+cz^2\ (c=1,2),\ 4w^5+3x^3+y^2+2z^2,$$
$$6w^5+x^3+y^2+3z^2,\ 6w^5+2x^3+y^2+z^2,$$
$$8w^5+x^3+y^2+2z^2,\ 8w^5+4x^3+y^2+2z^2,$$
$$11w^5+x^3+2y^2+4z^2,\ 15w^5+x^3+y^2+2z^2,$$
$$w^5+x^4+y^2+3z^2,\ w^6+2x^3+y^2+z^2,$$
$$2w^6+x^3+y^2+3z^2,\ 2w^6+x^3+y^2+4z^2,$$
$$3w^6+x^3+y^2+z^2,\ 4w^6+x^3+y^2+3z^2,$$
$$4w^6+3x^3+y^2+2z^2,\ 6w^6+x^3+y^2+3z^2,$$
$$6w^6+2x^3+y^2+z^2,\ 8w^6+x^3+y^2+2z^2,$$
$$15w^6+x^3+y^2+2z^2,\ w^7+2x^3+y^2+z^2,$$
$$2w^7+x^3+y^2+cz^2\ (c=3,4),\ 3w^7+x^3+y^2+z^2,$$
$$4w^7+3x^3+y^2+2z^2,\ 6w^7+x^3+y^2+3z^2,$$
$$15w^7+x^3+y^2+2z^2,\ w^8+2x^3+y^2+z^2,$$
$$2w^8+x^3+y^2+cz^2\ (c=3,4),\ 4w^8+3x^3+y^2+2z^2,$$
$$w^9+2x^3+y^2+z^2,\ 2w^9+x^3+y^2+cz^2\ (c=3,4),$$
$$w^{10}+2x^3+y^2+z^2,\ 2w^{10}+x^3+y^2+4z^2.$$

Remark 3.2. See [4, A273917] for the number of ways to write $n\in\mathbb{Z}^+$ as $w^5+x^4+y^2+3z^2$ with $w,x,y,z\in\mathbb{N}$ and $y>0$.

By Conjecture 1.2 the polynomial $u^3+2v^3+3x^3+4y^3+8z^3$ should be universal over $\mathbb{N}$, we suggest below a further refinement of this.

Conjecture 3.3. *For each $n \in \mathbb{N}$, either n or $n-8$ can be written as $w^3+2x^3+3y^3+4z^3$ with $w,x,y,z \in \mathbb{N}$. Moreover, the only nonnegative integers not represented by $w^3+2x^3+3y^3+4z^3$ with $w,x,y,z \in \mathbb{N}$ are the following* 122 *numbers*

18, 22, 39, 60, 63, 74, 76, 77, 100, 103, 106, 107, 117, 126, 178, 180, 201, 215, 228, 230, 245, 271, 289, 291, 295, 315, 341, 356, 357, 393, 413, 419, 420, 480, 481, 523, 559, 606, 616, 671, 673, 705, 854, 855, 963, 980, 981, 998, 1103, 1121, 1130, 1298, 1484, 1510, 1643, 1729, 1849, 1916, 1934, 1946, 1950, 1951, 1975, 1991, 2070, 2168, 2506, 2621, 2799, 2836, 2848, 2850, 2965, 3092, 3171, 3182, 3323, 3339, 3393, 3414, 3667, 3788, 3846, 3938, 4094, 4184, 4606, 4981, 4988, 4990, 5310, 5338, 5822, 6130, 6398, 6502, 7419, 7420, 7522, 8119, 9297, 9411, 10185, 10305, 10580, 11267, 11722, 11871, 13465, 13860, 14844, 15928, 15951, 21015, 21053, 22149, 22688, 23884, 27093, 29137, 34974, 41405.

Remark 3.3. Conjecture 3.3 implies that we can write each $n \in \mathbb{Z}^+$ as $(p-1)^3+w^3+2x^3+3y^3+4z^3$ with p prime and $w,x,y,z \in \mathbb{N}$, and write $25n+8$ $(n=0,1,2,\ldots)$ as $w^3+2x^3+3y^3+4z^3$ with $w,x,y,z \in \mathbb{N}$. It also implies that $P(v)+w^3+2x^3+3y^3+4z^3$ is universal over $\mathbb{N}$ whenever $P(v)$ is among the polynomials

$$\begin{aligned}&av^3\,(a=1,5,6,7,9,10,12,15,18),\\&bv^4\,(b=1,2,3,5,6,12,18),\\&cv^5\,(c=1,2,5,12),\ 5v^6,\ 12v^6,\ 5v^7,\ 12v^7.\end{aligned}$$

Conjecture 3.4. (i) *Any $n \in \mathbb{N}$ can be written as $P(u,v,x,y,z)$ with $u,v,x,y,z \in \mathbb{N}$, provided that $P(u,v,x,y,z)$ is among the following polynomials:*

$$\begin{aligned}&u^6+v^3+2x^3+4y^3+5z^3,\ u^5+v^3+x^3+2y^3+4z^3,\\&u^5+v^3+2x^3+3y^3+az^3\,(a=6,9),\\&bu^5+v^3+2x^3+4y^3+5z^3\,(b=1,2,6),\\&cu^5+v^3+2x^3+4y^3+6z^3\,(c=2,3),\\&u^5+v^3+2x^3+4y^3+dz^3\,(d=9,13).\end{aligned}$$

(ii) *We have* $\{P(u,v,x,y,z):\ u,v,x,y,z\in\mathbb{N}\}=\mathbb{N}$ *if* $P(u,v,x,y,z)$ *is among the polynomials*

$$u^5+v^4+x^3+2y^3+3z^3,\ u^5+v^4+x^3+2y^3+5z^3,$$
$$u^5+2v^4+x^3+2y^3+3z^3,\ u^5+3v^4+x^3+2y^3+3z^3,$$
$$2u^5+v^4+x^3+y^3+4z^3,\ 2u^5+v^4+x^3+2y^3+4z^3,$$
$$3u^5+v^4+x^3+2y^3+4z^3,\ 5u^5+v^4+x^3+2y^3+4z^3,$$
$$u^4+2v^4+x^3+y^3+4z^3,\ u^4+2v^4+x^3+2y^3+3z^3,$$
$$u^4+2v^4+x^3+2y^3+4z^3,\ u^4+2v^4+x^3+2y^3+6z^3,$$
$$u^4+2v^4+x^3+3y^3+4z^3,\ u^4+2v^4+x^3+4y^3+5z^3,$$
$$u^4+2v^4+x^3+4y^3+6z^3,\ u^4+2v^4+x^3+4y^3+10z^3,$$
$$u^4+3v^4+x^3+2y^3+3z^3,\ u^4+3v^4+x^3+2y^3+4z^3,$$
$$u^4+3v^4+x^3+2y^3+6z^3,\ u^4+4v^4+x^3+y^3+2z^3,$$
$$u^4+4v^4+x^3+2y^3+3z^3,\ u^4+4v^4+x^3+2y^3+4z^3,$$
$$u^4+5v^4+x^3+2y^3+4z^3,\ u^4+6v^4+x^3+2y^3+3z^3,$$
$$u^4+7v^4+x^3+2y^3+3z^3,\ u^4+9v^4+x^3+2y^3+4z^3,$$
$$2u^4+4v^4+x^3+2y^3+3z^3,\ 2u^4+6v^4+x^3+2y^3+4z^3,$$
$$3u^4+6v^4+x^3+2y^3+4z^3.$$

Remark 3.4. See [4, A271076] for the number of ways to write $n\in\mathbb{Z}^+$ as $u^5+v^4+x^3+2y^3+3z^3$ with $u,v,x,y,z\in\mathbb{N}$ and $v>0$. There are also finitely many (but quite a lot) polynomials $P(u,v,x,y,z)$ of the form $mu^4+av^3+bx^3+cy^3+dz^3$ with $a,b,c,d,m\in\mathbb{Z}^+$ for which any $n\in\mathbb{N}$ should have the form $P(u,v,x,y,z)$ with $u,v,x,y,z\in\mathbb{N}$.

Conjecture 3.5. *The polynomial* $u^2+v^4+P(x,y,z)$ *is universal over* $\mathbb{N}$ *whenever* $P(x,y,z)$ *is among the following polynomials*

$$2x^4+3y^4+4z^4,\ 2x^4+3y^4+8z^4,\ 2x^4+4y^4+cz^4\ (c=6,7,8,11),$$
$$2x^4+2y^5+4z^5,\ 2x^4+4y^5+7z^5,\ 4x^4+y^5+2z^5,\ 2x^4+3y^4+8z^5,$$
$$2x^4+4y^4+cz^5\ (c=1,2,12,23),\ 2x^4+8y^4+dz^5\ (d=1,4),$$
$$3x^4+5y^4+z^5,\ 3x^4+6y^4+z^5,\ 3x^4+8y^4+2z^5,$$
$$2x^4+7y^4+4z^6,\ 2x^4+8y^4+4z^6.$$

Also, the polynomial $u^2+2v^4+4x^4+8y^4+z^5$ *is universal over* $\mathbb{N}$.

Remark 3.5. We also conjecture that for any $n \in \mathbb{N}$ either n or $n-4$ can be be written as $u^2+v^3+x^4+2y^8$ with $u,v,x,y \in \mathbb{N}$. Note that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{8}=\frac{29}{24}=1.208333\ldots.$$

4. Upgrading Waring's problem

Let $k>1$ be an integer. The classical Waring problem posed in 1770 by E. Waring asks for the least positive integer $g(k)=r$ such that each $n \in \mathbb{N}=\{0,1,2,\ldots\}$ can be written as $x_1^k+\ldots+x_r^k$ with $x_1,\ldots,x_s \in \mathbb{N}$. In 1909 D. Hilbert proved that $g(k)$ always exists. J. A. Euler (a son of Leonhard Euler), conjectured in about 1772

$$g(k)=2^k+\left\lfloor\left(\frac{3}{2}\right)^k\right\rfloor-2.$$

It is known that this holds if $k \in \{2,3,4,5,6\}$ or k is sufficiently large (cf. [7]).

We propose the following problem upgrading Waring's problem.

Problem 4.1. *Determine $s(k)$ and $t(k)$ for any integer $k>1$, where $s(k)$ is the smallest positive integer s such that*

$$\{a_1x_1^k+a_2x_2^k+\ldots+a_sx_s^k:\ x_1,\ldots,x_s \in \mathbb{N}\}=\mathbb{N}$$

for some $a_1,\ldots,a_s \in \mathbb{Z}^+$, and $t(k)$ is the smallest positive integer t such that

$$\{a_1x_1^k+a_2x_2^k+\ldots+a_tx_t^k:\ x_1,\ldots,x_t \in \mathbb{N}\}=\mathbb{N}$$

for some $a_1,\ldots,a_t \in \mathbb{Z}^+$ with $a_1+a_2+\ldots+a_t=g(k)$.

Clearly, $s(k) \leqslant t(k) \leqslant g(k)$ for all $k=2,3,4,\ldots$. It is easy to see that $s(2)=t(2)=4$. By Theorem 1.1 or 1.2, we have $s(3) \geqslant 5$. In view of Conjecture 1.2, $u^3+v^3+2x^3+2y^3+3z^3$ should be universal over $\mathbb{N}$ and this implies that $s(3)=t(3)=5$ since $1+1+2+2+3=9=g(3)$.

Conjecture 4.1. *We have $s(4)=t(4)=7$. Moreover,*

$$\{x_1^4+x_2^4+2x_3^4+2x_4^4+3x_5^4+3x_6^4+7x_7^4:\ x_1,\ldots,x_7 \in \mathbb{N}\}=\mathbb{N}$$

and also any $n \in \mathbb{N}$ can be written as

$$x_1^4+x_2^4+2x_3^4+2x_4^4+3x_5^4+4x_6^4+6x_7^4$$

with $x_3 \in \{0,1\}$ and $x_1,x_2,x_4,\ldots,x_7 \in \mathbb{N}$.

Remark 4.1. See [4, A267861] for related data. For $a_1,\ldots,a_7 \in \mathbb{Z}^+$ with $a_1 \leqslant \ldots \leqslant a_7$ and $\sum_{i=1}^7 a_i=g(4)=19$, if $\sum_{i=1}^7 a_ix_i^4$ is universal over $\mathbb{N}$ then the tuple $(a_1,\ldots,a_7)$ is either $(1,1,2,2,3,3,7)$ or $(1,1,2,2,3,4,6)$.

Conjecture 4.2. *We have* $s(5) = t(5) = 8$. *Moreover,*

$$\{x_1^5 + x_2^5 + 2x_3^5 + 3x_4^5 + 4x_5^5 + 5x_6^5 + 7x_7^5 + 14x_8^5 :\ x_1, \ldots, x_8 \in \mathbb{N}\} = \mathbb{N},$$
$$\{x_1^5 + x_2^5 + 2x_3^5 + 3x_4^5 + 4x_5^5 + 6x_6^5 + 8x_7^5 + 12x_8^5 :\ x_1, \ldots, x_8 \in \mathbb{N}\} = \mathbb{N}.$$

Remark 4.2. See [4, A271169] for related data. For $a_1, \ldots, a_8 \in \mathbb{Z}^+$ with $a_1 \leqslant \ldots \leqslant a_8$ and $\sum_{i=1}^8 a_i = g(5) = 37$, if $\sum_{i=1}^8 a_i x_i^5$ is universal over $\mathbb{N}$ then the tuple $(a_1, \ldots, a_8)$ is either $(1, 1, 2, 3, 4, 5, 7, 14)$ or $(1, 1, 2, 3, 4, 6, 8, 12)$.

Conjecture 4.3. *We have* $t(6) = 10$, *moreover the polynomial*

$$x_1^6 + x_2^6 + x_3^6 + 2x_4^6 + 3x_5^6 + 5x_6^6 + 6x_7^6 + 10x_8^6 + 18x_9^6 + 26x_{10}^6$$

is universal over $\mathbb{N}$.

Remark 4.3. Note that $1+1+1+2+3+5+6+10+18+26 = g(6) = 73$.

Conjecture 4.4. *For any integer* $k > 2$ *we have* $t(k) \leqslant 2k - 1$.

5. Write integers as $x^a + y^b - z^c$ with $x, y, z \in \mathbb{Z}^+$

As $m = ((m+1)/2)^2 - ((m-1)/2)^2$ for any odd integer m, for any integer $a \geqslant 2$ each integer can be written as $x^a + y^2 - z^2$ with $x, y, z \in \mathbb{Z}^+$ in infinitely many ways.

Theorem 5.1. *Let* $a > 2$ *be an integer. If* a *is even or* a *is composite with a prime divisor congruent to* 3 *mod* 4, *then*

$$\{x^2 + y^2 - z^a :\ x, y, z \in \mathbb{Z}\} \neq \mathbb{Z}.$$

Proof. If a is even, $m, z \in \mathbb{Z}$ and $m \equiv 6 \pmod 8$, then $m + z^a \equiv 6, 7 \pmod 8$ and hence $m + z^a$ is not the sum of two squares.

Now suppose that $a = pn$ with $n > 1$ odd and p a prime congruent to 3 mod 4. We assert that

$$(2p)^p \notin \{x^2 + y^2 - z^{pn} :\ x, y, z \in \mathbb{N}\}.$$

For any $z \in \mathbb{Z}$, the integer $(2z)^{pn} + (2p)^p = 2^p(2^{p(n-1)}z^{pn} + p^p)$ is not the sum of two squares since $2^{p(n-1)}z^{pn} + p^p \equiv p \equiv 3 \pmod 4$, and the number $(pz)^{pn} + (2p)^p = p^p(p^{p(n-1)}z^{pn} + 2^p)$ is not the sum of two squares since $p \equiv 3 \pmod 4$ and $p^{p(n-1)}z^{pn} + 2^p \not\equiv 0 \pmod p$.

Now assume that $z \in \mathbb{Z}$ is relatively prime to $2p$. If q is a prime dividing $z^n + 2p$, then $q \neq 2, p$ and

$$\frac{z^{pn} + (2p)^p}{z + 2p} = \sum_{k=0}^{p-1} (z^n)^k (-2p)^{p-1-k} \equiv p(-2p)^{p-1} \not\equiv 0 \pmod q.$$

So $z^n + 2p$ and $(z^{pn} + (2p)^p)/(z + 2p)$ are relatively prime. Note that $z^{pn} + (2p)^p \equiv z \pmod 4$. If the odd number $z^{pn} + (2p)^p$ is the sum of two squares, then $z \equiv 1 \pmod 4$ and $z^n + 2p \equiv 3 \pmod 4$, hence for some prime $r \equiv 3 \pmod 4$ we have

$$\mathrm{ord}_r(z^{pn} + (2p)^p) = \mathrm{ord}_r(z^n + 2p) \equiv 1 \pmod 2$$

which contradicts that $z^{pn} + (2p)^p \in \{x^2 + y^2 : \ x, y \in \mathbb{N}\}$.

The proof of Theorem 5.1 is now complete. $\square$

Motivated by Theorem 5.1 and certain heuristic arguments, we pose the following conjecture.

Conjecture 5.1. (i) *If q is an odd prime or a product of some primes congruent to* 1 *mod* 4*, then each $m \in \mathbb{Z}$ can be expressed as $x^2 + y^2 - z^q$ with $x, y, z \in \mathbb{Z}^+$ in infinitely many ways.*

(ii) *If $\{a, b, c\}$ is among the multisets*

$$\{2,3,3\}, \ \{2,3,4\} \text{ and } \{2,3,5\},$$

then any integer m can be written as $x^a + y^b - z^c$ with $x, y, z \in \mathbb{Z}^+$ in infinitely many ways.

Remark 5.1. We have verified that $\{x^4 - y^3 + z^2 : \ x, y, z \in \mathbb{Z}^+\}$ contains all integers m with $|m| \leqslant 10^5$. See [4, A266152, A266153, A266212, A266215, A266230, A266231, A266277, A266314, A266363, A266364, A266528, A266985] for some data related to Conjecture 5.1. Here are some concrete examples for Conjecture 5.1:

$$0 = 4^4 - 8^3 + 16^2, \quad -1 = 1^4 - 3^3 + 5^2,$$

$$-20 = 32^4 - 238^3 + 3526^2, \quad 11019 = 4325^4 - 71383^3 + 3719409^2,$$

$$394 = 2283^3 + 128^4 - 110307^2, \quad 570 = 546596^2 + 8595^3 - 983^4,$$

$$445 = 9345^3 + 34^5 - 903402^2, \quad 435 = 475594653^2 + 290845^3 - 3019^5.$$

Now we explain why part (ii) of Conjecture 5.1 seems reasonable via (not rigorous) heuristic arguments. Let $a, b, c \in \{2, 3, 4, \ldots\}$. As

$$|\{(x,y) \in (\mathbb{Z}^+)^2 : \ x^a \leqslant N \text{ and } y^b \leqslant N\}|$$
$$\sim N^{1/a+1/b} = \int_0^N \left(\frac{1}{a} + \frac{1}{b}\right) t^{1/a+1/b-1} dt,$$

we might think that $t \in \mathbb{Z}^+$ has the form $x^a + y^b$ ($x, y \in \mathbb{Z}^+$) with "probability" at least $C_\varepsilon t^{1/a+1/b-1}$ (where C_ε is a positive constant depending on ε). Note that the series $\sum_{z=1}^\infty (m + z^c)^{1/a+1/b-1-\varepsilon}$ diverges if $1 - 1/a - 1/b + \varepsilon < 1/c$. Thus, when $1/a + 1/b + 1/c > 1$, we might expect that there are infinitely many triples (x, y, z) of positive integers with $m = x^a + y^b - z^c$. If $2 \leqslant a < b \leqslant c$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1 \iff a = 2, \ b = 3, \text{ and } c \in \{3, 4, 5\}.$$

Conjecture 5.2. *We have*

$$\{x^2+y^3-p^3:\ x,y\in\mathbb{Z}^+,\ \text{and } p \text{ is prime}\}=\mathbb{Z}$$

and

$$\{x^2-y^3+p^3:\ x,y\in\mathbb{Z}^+,\ \text{and } p \text{ is prime}\}=\mathbb{Z}.$$

Remark 5.2. See [4, A266548] for related data.

6. Write positive integers as $a^k+bx^h+cy^i+dz^j$ with $k,x,y,z\in\mathbb{N}$

Conjecture 6.1. (i) *Any integer $n>1$ can be written as $x^4+y^3+z^2+2^k$ with $x,y,z\in\mathbb{N}$ and $k\in\mathbb{Z}^+$. Also, each $n\in\mathbb{Z}^+$ can be written as $3x^4+y^3+z^2+2^k$ with $k,x,y,z\in\mathbb{N}$, and any $n\in\mathbb{Z}^+$ can be written as $4x^4+y^3+z^2+3^k$ with $k,x,y,z\in\mathbb{N}$.*

(ii) *Each $n\in\mathbb{Z}^+$ can be written as $x^4+y^2+z^2+2^k$ with $x,y,z\in\mathbb{N}$ and $k\in\{0,1,2,3,4,5\}$, and any $n\in\mathbb{Z}^+$ can be written as $x^4+4y^2+z^2+3^k$ with $x,y,z\in\mathbb{N}$ and $k\in\{0,1,2,3,4\}$.*

(iii) *Any $n\in\mathbb{Z}^+$ can be written as $x^6+3y^2+z^2+2^k$ with $k,x,y,z\in\mathbb{N}$, and any $n\in\mathbb{Z}^+$ can be written as $2x^6+2y^2+z^2+5^k$ with $k,x,y,z\in\mathbb{N}$.*

(iv) *For each $m=6,7,8$, any $n\in\mathbb{Z}^+$ can be written as $4x^m+2y^2+z^2+3^k$ with $k,x,y,z\in\mathbb{N}$.*

(v) *Each $n\in\mathbb{Z}^+$ can be written as $4x^5+3y^2+2z^2+2^k$ with $k,x,y,z\in\mathbb{N}$. Also, any $n\in\mathbb{Z}^+$ can be written as $ax^5+by^2+z^2+2^k$ with $k,x,y,z\in\mathbb{N}$, whenever (a,b) is among the ordered pairs*

$$\begin{gathered}(1,2),\ (1,3),\ (2,1),\ (2,2),\ (2,6),\ (3,2),\ (3,3),\\ (4,2),\ (5,1),\ (5,3),\ (6,2),\ (6,5),\ (9,2),\ (10,1),\\ (10,2),\ (12,2),\ (13,1),\ (13,3),\ (19,2),\ (20,2).\end{gathered}$$

(vi) *Any $n\in\mathbb{Z}^+$ can be written as $ax^5+by^2+z^2+3^k$ with $k,x,y,z\in\mathbb{N}$, whenever (a,b) is among the ordered pairs $(1,4),\ (4,1),\ (4,2),\ (5,4)$.*

(vii) *For each $a=2,3$, any $n\in\mathbb{Z}^+$ can be written as $ax^5+2y^2+z^2+5^k$ with $k,x,y,z\in\mathbb{N}$.*

Remark 6.1. The author verified the first assertion in part (i) for n up to 2×10^7, and Qing-Hu Hou (at Tianjin Univ.) extended the verification for n up to 10^9; see [4, A280356] for related data. The author would like to offer US \$234 as the prize for the first correct solution to the first assertion in part (i) of Conjecture 6.1. We have verified all other assertions in parts (i)-(iv) and (v)-(vii) for n up to 10^7 and 10^6 respectively.

Conjecture 6.2. (i) *Let $a,b,c\in\mathbb{Z}^+$ with $b\leqslant c$. Then, any $n\in\mathbb{Z}^+$ can be written as $ax^3+by^2+cz^2+4^k$ with $k,x,y,z\in\mathbb{N}$ if and only if (a,b,c) is among the following triples*

$$(1,1,1),\ (1,1,2),\ (1,1,3),\ (1,1,5),\ (1,1,6),\ (1,2,3),\ (1,2,5),$$
$$(2,1,1),\ (2,1,2),\ (2,1,3),\ (2,1,6),\ (4,1,2),\ (5,1,2),\ (8,1,2),\ (9,1,2).$$

(ii) *For each $a\in\{3,4,6\}$, any $n\in\mathbb{Z}^+$ can be written as $x^3+2y^3+z^2+a^k$ with $k,x,y,z\in\mathbb{N}$. Also, for each $a\in\{5,8,9\}$, any $n\in\mathbb{Z}^+$ can be written as $x^2+2y^2+z^3+a^k$ with $k,x,y,z\in\mathbb{N}$.*

(iii) *For each $a=7,9,11,12,13$, any $n\in\mathbb{Z}^+$ can be written as $ax^2+2y^2+z^2+5^k$ with $k,x,y,z\in\mathbb{N}$. Also, for each $a=7,9$, any $n\in\mathbb{Z}^+$ can be written as $ax^2+2y^2+z^2+6^k$ with $k,x,y,z\in\mathbb{N}$.*

Remark 6.2. We have verified Conjecture 6.2 for n up to 10^6. See [4, A280153] for some data related to part (ii) of Conjecture 6.2.

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