Spans in Quantum Theory

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for *Deep Beauty*
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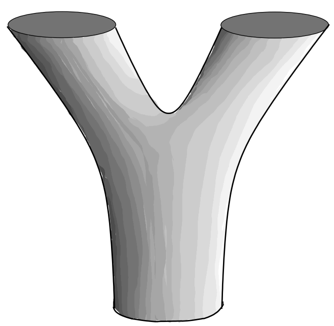
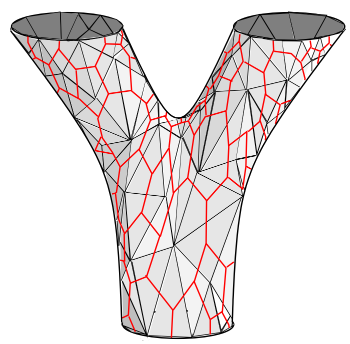
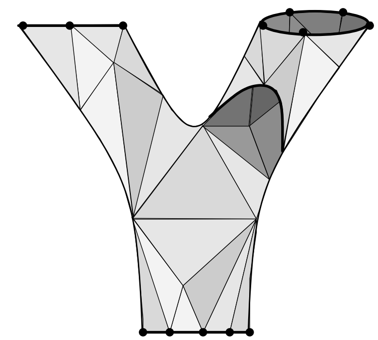




figures by Aaron Lauda

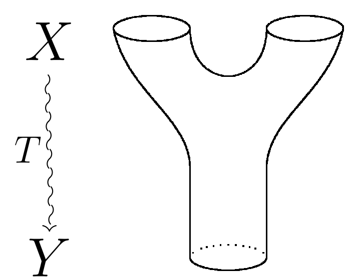
for more, see:

http://math.ucr.edu/home/baez/span/

Quantizing gravity is tough. But:

Quantum theory will make more sense when seen as part of a theory of spacetime.

Why? Quantum theory seems 'weird' because — seen from the eyes of category theory — *Hilbert spaces* and *operators* behave less like *sets* and *functions* than *manifolds describing choices of 'space'* and *cobordisms describing choices of 'spacetime'*:



	object $\bullet$	**morphism** $\bullet \to \bullet$
SET THEORY Set	**set**	**function between sets**
QUANTUM THEORY Hilb	**Hilbert space** **(state)**	**operator between Hilbert spaces** **(process)**
GENERAL RELATIVITY nCob	**manifold** **(space)**	**cobordism between manifolds** **(spacetime)**

The category Set **is 'cartesian', while** Hilb **and** $n\mathrm{Cob}$ **are 'dagger compact'.**

This means that Set **is a mathematical universe in which classical logic applies, while a very different sort of logic reigns in** $n\mathrm{Cob}$ **and** Hilb**. In a cartesian category:**

- **We can freely duplicate and delete information.**
- **We cannot reverse most processes.**
- **We do not have generalized Bell states.**

In a dagger compact category:

- **We cannot freely duplicate and delete information.**
- **Every process** T **has a 'reverse version',** $T^\dagger$**.**
- **We have generalized Bell states.**

How can we bridge the chasm between the classical world of cartesian categories and the quantum world of dagger compact categories?

We can bridge it with the concept of a 'span':

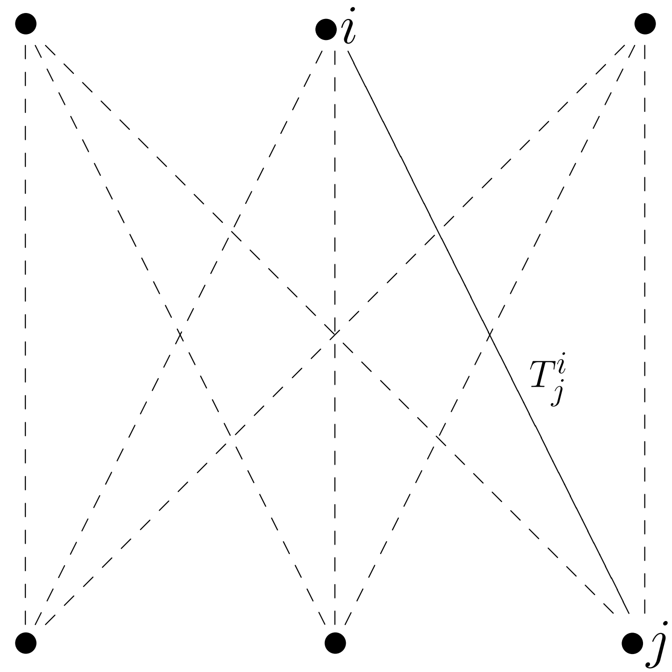
$$\begin{tikzcd} & T \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ X & & Y \end{tikzcd}$$

We'll see that any cartesian category C **with pullbacks gives a dagger compact category** $\mathrm{Span}(C)$**.**

This construction is a modern version of *matrix mechanics!*

But first, recall the original idea of matrix mechanics...

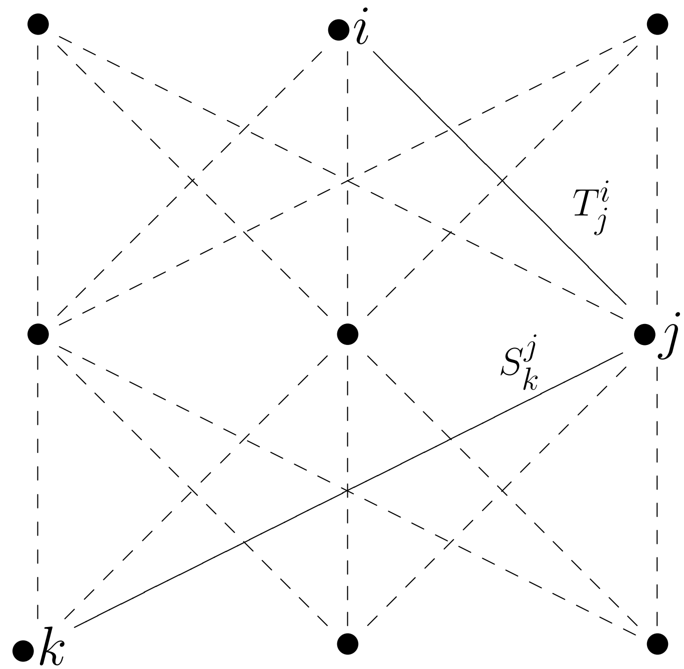
Heisenberg used matrices with complex entries to describe processes in quantum mechanics:



For each input state i and output state j, the process T gives a complex number, the *amplitude* to go from i to j:

$$T^i_j \in \mathbb{C}$$

To compose processes, we sum over paths:



$$(ST)^i_k = \sum_j S^j_k \times T^i_j$$

In the continuum limit, such sums become *path integrals.*

Matrix mechanics also works with other rigs (= rings without negatives) replacing the complex numbers:

- $[0, \infty)$ **with its usual** $+$ **and** $\times$

  Now T^i_j **gives the *probability* to go from** i **to** j**.**

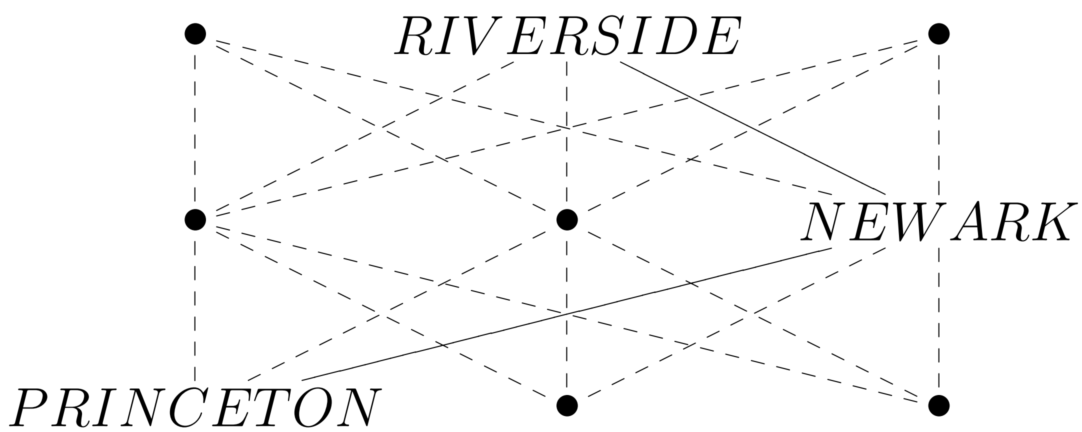
- $\{T, F\}$ **with** `OR` **as** $+$ **and** `AND` **as** $\times$

  Now T^i_j **gives the *possibility* to go from** i **to** j**.**

- $\mathbb{R}^{\mathtt{MIN}} = \mathbb{R} \cup \{+\infty\}$ **with** `MIN` **as** $+$ **and** $+$ **as** $\times$

  Now T^i_j **gives the *cost*, or *action*, to go from** i **to** j**.**

I did matrix multiplication in $\mathbb{R}^{\mathtt{MIN}}$ to find the cheapest flight here:



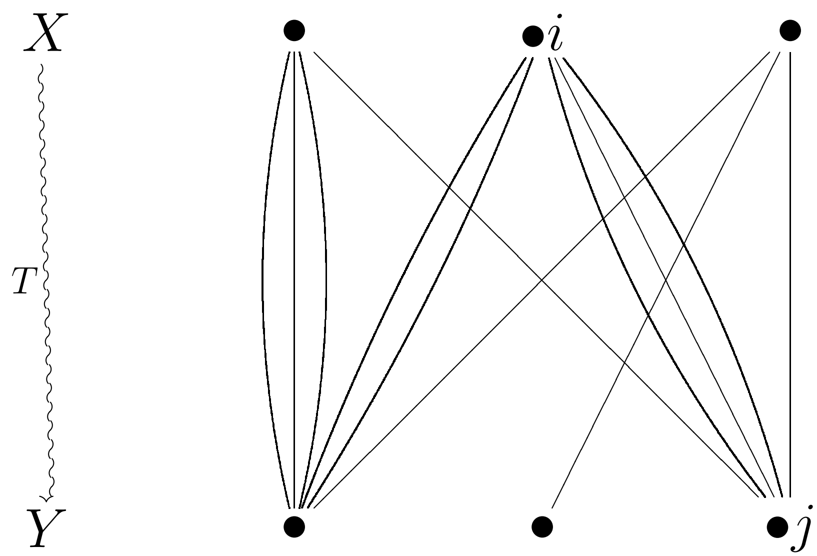
In nature, the continuum limit of this sort of calculation gives *the principle of least action*.

For more on the analogy

$$\mathbb{R}^{\mathtt{MIN}} : \mathbb{C} :: \text{classical mechanics} : \text{quantum mechanics}$$

read about 'idempotent analysis'.

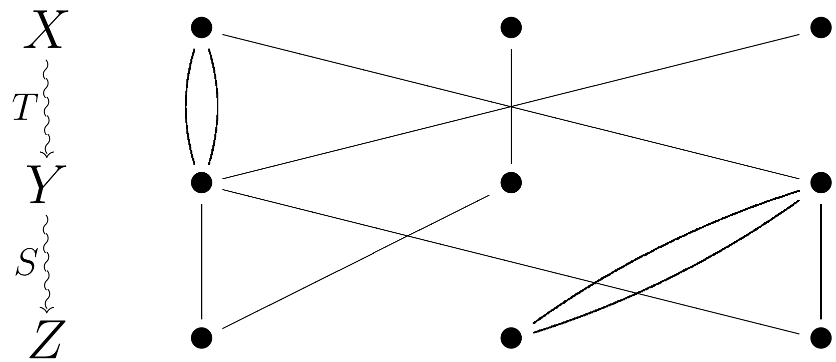
Perhaps the most fundamental example is when T^i_j is the *set of ways* to go from $i \in X$ to $j \in Y$:



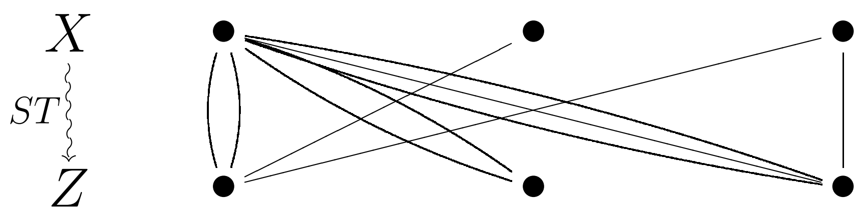
However, now the matrix T has entries taking values not in a rig but in a category: the category Set.

So, in this example we have 'categorified' matrix mechanics!

Set **is a categorified rig, with disjoint union as $+$ and cartesian product as $\times$. We use this to multiply matrices of sets. If we multiply these:**



we get this:



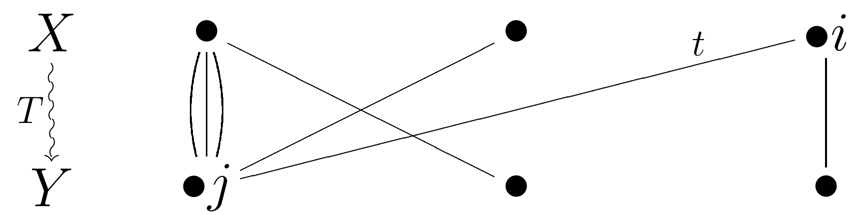
where

$$(ST)^i_k = \bigsqcup_j S^j_k \times T^i_j$$

A Set**-valued matrix is the same as a 'span' of sets:**

$$\begin{tikzcd} & T \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ X & & Y \end{tikzcd}$$

Think of T **as a set of 'paths' from points of** X **to points of** Y**. Each path** $t \in T$ **goes from some point** $p(t) = i$ **to some point** $p'(t) = j$**:**



To get a matrix of sets from a span of sets, define:

$$T^i_j = \{t \in T : \; p(t) = i, \; p'(t) = j\}$$

Multiplying matrices of sets is the same as 'composing spans' in Set**. Given spans like this:**

$$\begin{tikzcd}
 & T \arrow[dl, "p"'] \arrow[dr, "p'"] & & S \arrow[dl, "q"'] \arrow[dr, "q'"] & \\
X & & Y & & Z
\end{tikzcd}$$

we compose them as follows:

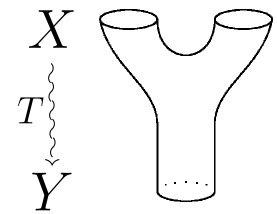
$$\begin{tikzcd}
 & & ST \arrow[dl] \arrow[dr] & & \\
 & T \arrow[dl, "p"'] \arrow[dr, "p'"] & & S \arrow[dl, "q"'] \arrow[dr, "q'"] & \\
X & & Y & & Z
\end{tikzcd}$$

where ST **is defined using a 'pullback':**

$$ST = \{(s,t) \in S \times T : \; q(s) = p'(t)\}$$

In words: a path in ST **is a path** $s \in S$ **and a path** $t \in T$**, such that** s **starts where** t **ends.**

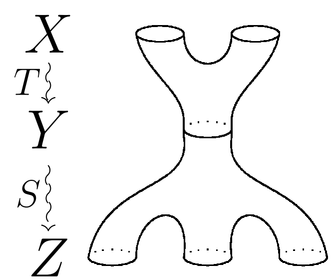
Cobordism are examples of spans in Diff^{op}**, the opposite of the category of manifolds with boundary, since a cobordism:**



is really a diagram like this in Diff**:**

$$
\begin{tikzcd}
 & T & \\
X \arrow[ur, "p"] & & Y \arrow[ul, "p'"']
\end{tikzcd}
$$

$\mathrm{Diff}^{\mathrm{op}}$ **doesn't have *all* pullbacks, but it has enough to compose cobordisms:**



So, we've seen:

- **Spans are a categorified version of matrix mechanics.**
- **A cobordism is a specially nice span in** $\mathrm{Diff}^{\mathrm{op}}$**.**

This suggests that the common features of $n\mathrm{Cob}$ **and** Hilb **are** ***features of categories of spans.***

This is true!

To make this precise, we'll use the concept of a 'dagger compact category'.

Suppose C is any category with pullbacks. Then we can form a category $\mathrm{Span}(C)$, in which:

- **an object is an object of C;**
- **a morphism $T\colon X \rightsquigarrow Y$ is an isomorphism class of spans**

$$\begin{tikzcd} & T \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ X & & Y \end{tikzcd}$$

where two spans T_1, T_2 are isomorphic if there's a commutative diagram:

$$\begin{tikzcd} & T_1 \arrow[r, "\sim"] \arrow[dl, "p_1"'] \arrow[drr, "p'_1"' near end] & T_2 \arrow[dll, "p_2" near end] \arrow[dr, "p'_2"] & \\ X & & & Y \end{tikzcd}$$

We compose morphisms in $\mathrm{Span}(C)$ via pullback, as sketched already.

In this situation $\mathrm{Span}(C)$ **is always a 'dagger category'. In other words, each span** $T\colon X \rightsquigarrow Y$**:**

$$\begin{tikzcd} & T \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ X & & Y \end{tikzcd}$$

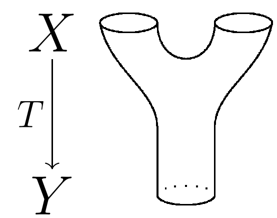
has an 'adjoint' $T^\dagger\colon Y \rightsquigarrow X$ **given by switching input and output:**

$$\begin{tikzcd} & T \arrow[dl, "p'"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

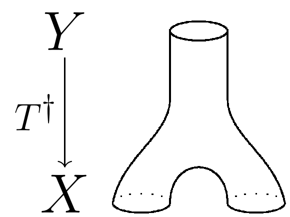
The axioms of a dagger category hold:

$$1_X^\dagger = 1_X \qquad (ST)^\dagger = T^\dagger S^\dagger \qquad T^{\dagger\dagger} = T$$

The dagger operation on spans makes $n\mathrm{Cob}$ into a dagger category where the adjoint of



is



When $C = \mathrm{Set}$, spans are Set-valued matrices and $T^\dagger\colon X \rightsquigarrow Y$ is just the *transpose* of $T\colon Y \rightsquigarrow X$.

Next suppose C is also 'cartesian', i.e. has cartesian products and a terminal object 1. Then $\mathrm{Span}(C)$ is a symmetric monoidal category, where the tensor product of two spans:

$$\begin{tikzcd} & T_1 \arrow[dl, "p_1"'] \arrow[dr, "p_1'"] & \\ X_1 & & Y_1 \end{tikzcd}$$

and

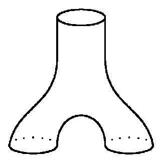
$$\begin{tikzcd} & T_2 \arrow[dl, "p_2"'] \arrow[dr, "p_2'"] & \\ X_2 & & Y_2 \end{tikzcd}$$

is given by 'doing them in parallel':

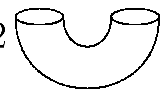
$$\begin{tikzcd} & T_1 \times T_2 \arrow[dl, "p_1 \times p_2"'] \arrow[dr, "p_1' \times p_2'"] & \\ X_1 \times X_2 & & Y_1 \times Y_2 \end{tikzcd}$$

When $C = \mathrm{Diff}^{\mathrm{op}}$**, this tensor product on** $\mathrm{Span}(C)$ **includes the usual tensor product of cobordisms as a special case:**

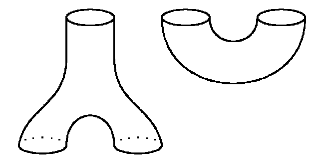
$$\begin{array}{c} X_1 \\ {\scriptstyle T_1}\big\downarrow \\ Y_1 \end{array}$$



$$\begin{array}{c} X_2 \\ {\scriptstyle T_2}\big\downarrow \\ Y_2 \end{array}$$



$$\begin{array}{c} X_1 \times X_2 \\ {\scriptstyle T_1 \times T_2}\big\downarrow \\ Y_1 \times Y_2 \end{array}$$



Note that $\times$ **in** $\mathrm{Diff}^{\mathrm{op}}$ **is disjoint union!**

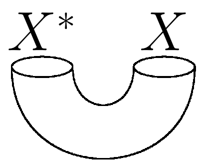
When $C = \mathrm{Set}$**, the tensor product on** $\mathrm{Span}(C)$ **is the more or less obvious 'tensor product' of set-valued matrices.**

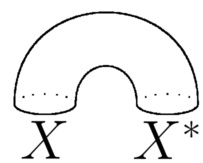
A symmetric monoidal category is 'compact' if every object X has a dual object X^* with counit and unit

$$e_X \colon X^* \otimes X \to 1, \quad i_X \colon 1 \to X \otimes X^*$$

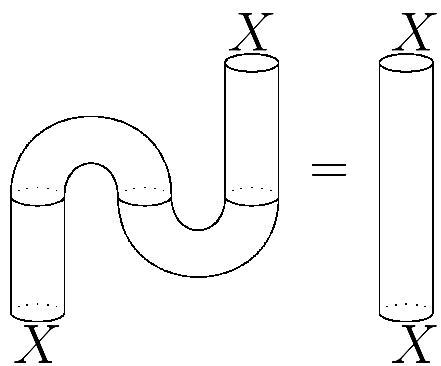
satisfying the 'zig-zag identities'.

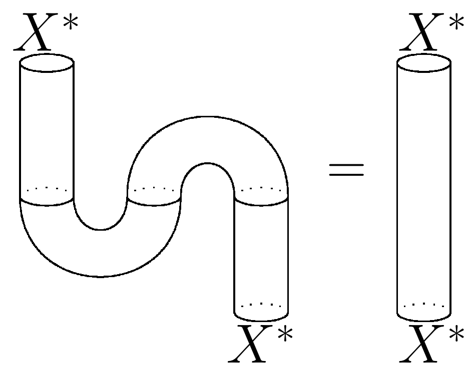
In $n\mathrm{Cob}$, X^* is X with its orientation reversed. We have:

e_X = 

i_X = 

and zig-zag identities look like this:





In Hilb**,** X^* **is the dual Hilbert space. We have:**

$$\begin{array}{rrcl} e_X\colon & X^* \otimes X & \to & \mathbb{C} \\ & \ell \otimes \psi & \mapsto & \ell(\psi) \end{array} \qquad \begin{array}{rrcl} i_X\colon & \mathbb{C} & \to & X \otimes X^* \\ & c & \mapsto & c\,1_X \end{array}$$

and the zig-zag identities say familiar things about linear algebra.

In quantum theory, the unit i_X **describes a 'generalized Bell state'. The zig-zag identities are fundamental in Abramsky and Coecke's treatment of quantum teleportation.**

A 'dagger compact category' is a dagger category that is also compact, with some extra equations relating the two structures — see the papers by Abramsky & Coecke and Selinger for the precise definition. One can show:

THEOREM. If C is a cartesian category with pullbacks, $\mathrm{Span}(C)$ is a dagger compact category.

This result illustrates how the span construction, as a generalized version of matrix mechanics, relates 'classical' categories like Set to 'quantum' categories like $n\mathrm{Cob}$ and Hilb.

There's much more to say, but perhaps not now!