MY FAVORITE NUMBER:

John Baez

September 19, 2008

The Rankin Lectures
Supported by the Glasgow Mathematical Journal Trust

People have liked the numbers 12 and 24 for a long time:

But their real magic is only becoming clear now.

Around 1735, Leonhard Euler gave a bizarre 'proof' that

$$\mathbf{1 + 2 + 3 + 4 + \cdots \ = \ -\frac{1}{12}}$$

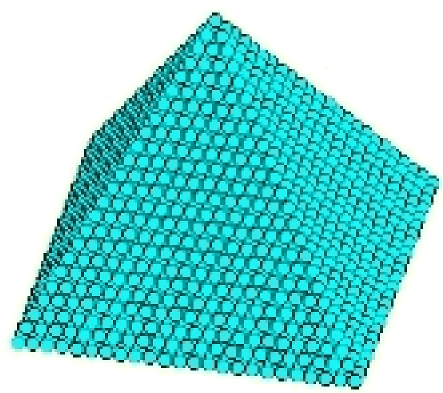
Later the mathematician Abel wrote:

The divergent series are
the invention of the devil,
and it is a shame to base on them
any demonstration whatsoever.

But Euler's equation has been made rigorous —
and it explains why bosonic string theory works well
only in $\mathbf{24 + 2 = 26}$ dimensions!

In 1875, Édouard Lucas challenged his readers to prove this:

A square pyramid of cannon balls contains a square number of cannon balls only when it has 24 cannon balls along its base.



$$1^2 + 2^2 + \cdots + 24^2 = 70^2$$

Indeed, the only integer solution of

$$1^2 + 2^2 + \cdots + n^2 = m^2$$

not counting the silly ones $n = 0$ and $n = 1$, is $n = 24$!

It looks like a curiosity, but this solution gives the densest lattice packing of spheres in 24 dimensions! And when we combine this idea with bosonic string theory in 26 dimensions, all heaven breaks loose!

But we're getting ahead of ourselves....

EULER'S CRAZY CALCULATION

Euler started with this:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

He differentiated both sides:

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$$

He set $x = -1$ and got this:

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

Then Euler considered this function:

$$\zeta(s) \;=\; 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$$

He multiplied by 2^{-s}:

$$2^{-s}\zeta(s) \;=\; 2^{-s} + 4^{-s} + 6^{-s} + 8^{-s} + \cdots$$

Then he subtracted twice the second equation from the first:

$$(1 - 2 \cdot 2^{-s})\zeta(s) \;=\; 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

Taking this result:

$$(1 - 2 \cdot 2^{-s})\zeta(s) \;=\; 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

and setting $s = -1$, he got:

$$-3(1 + 2 + 3 + 4 + \cdots) = 1 - 2 + 3 - 4 + \cdots$$

Since he already knew the right-hand side equals 1/4, he concluded:

$$1 + 2 + 3 + 4 + \cdots \;=\; -\frac{1}{12}$$

Euler's calculation looks crazy, but it can be made rigorous! The sum

$$1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$$

converges for $\mathrm{Re}(s) > 1$ to an analytic function: the *Riemann zeta function*, $\zeta(s)$.

This function can be analytically continued to $s = -1$, and one can prove

$$\zeta(-1) = -\frac{1}{12}$$

But why does this make bosonic string theory work well in 26 dimensions?

GROUND STATE ENERGY

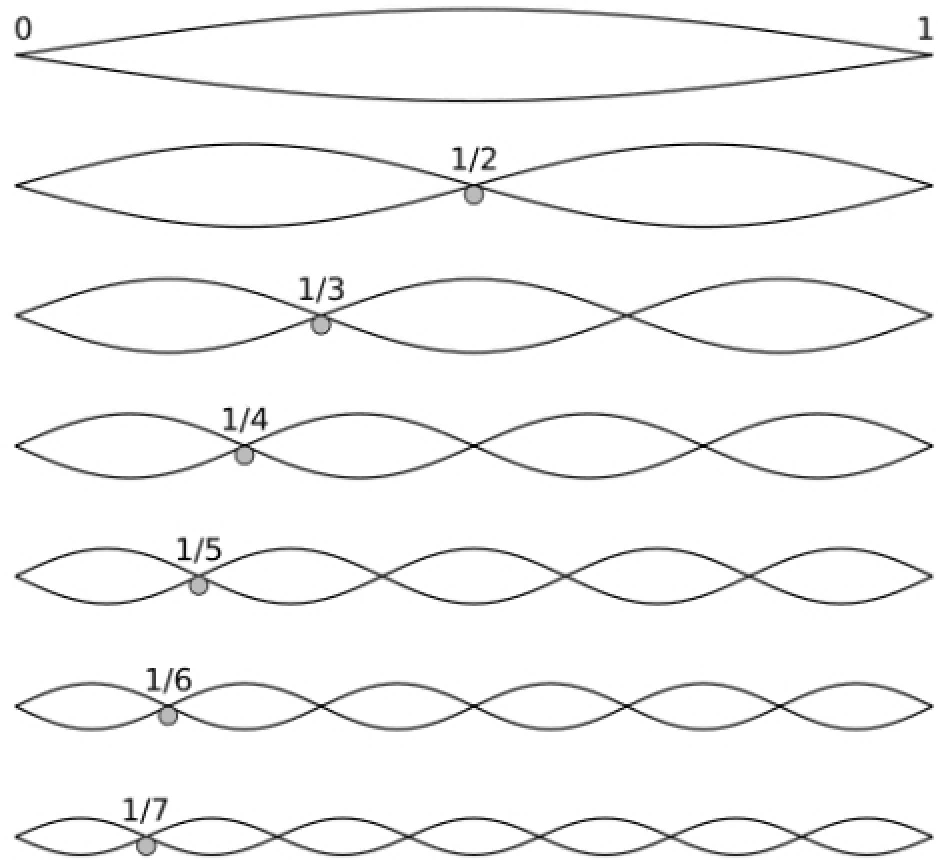
According to quantum mechanics, a harmonic oscillator that vibrates with frequency ω can have energy

$$\frac{1}{2}\omega,\ (1+\frac{1}{2})\omega,\ (2+\frac{1}{2})\omega,\ (3+\frac{1}{2})\omega, \ldots$$

in units where Planck's constant equals 1.

The lowest energy is not zero! It's $\frac{1}{2}\omega$. This is called the *ground state energy* of the oscillator.

A violin string that only wiggles up and down can still vibrate with different frequencies:



In suitable units, these are $\omega = 1, 2, 3, \ldots$

So, a string that can only wiggle in one direction is the same as an infinite collection of oscillators with frequencies $\omega = 1, 2, 3, \ldots$.

When we have a bunch of oscillators, their ground state energies add. So, the ground state energy of the string seems to be:

$$\frac{1}{2}(1 + 2 + 3 + \cdots) \;=\; \infty$$

This is the simplest example of how quantum theory is plagued by infinities.

But Euler's crazy calculation gives a different answer!

$$\frac{1}{2}(1+2+3+\cdots) \;=\; -\frac{1}{24}$$

Indeed, this seems to be the *correct* ground state energy for a string that can wiggle in one direction!

Experiments have confirmed equally crazy-looking calculations.

A string that can wiggle in n directions has n times as much ground state energy.

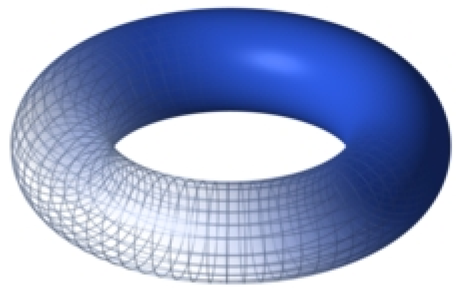
A string in 26-dimensional spacetime can wiggle in all 24 directions perpendicular to its 2-dimensional surface.

So, its ground state energy is simply

$$-1$$

But what's so great about this?

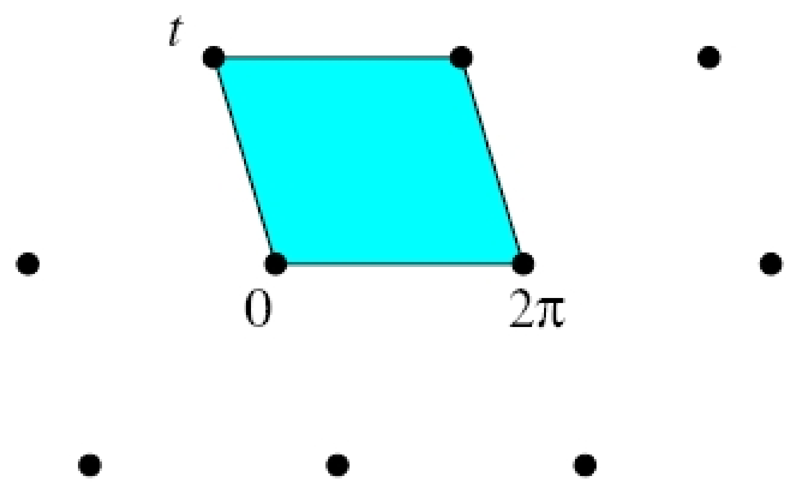
If a string loops around in time, its surface will be a torus:



Different shapes of torus happen with different probabilities, or more precisely, *amplitudes*.

Let's see the formula for these amplitudes — and see why it's only consistent when the string can wiggle in 24 directions perpendicular to its surface!

If our torus is formed by curling up this parallelogram:



then its amplitude is

$$Z(t) = \sum_k e^{-iE_k t}$$

where E_k are the energies the string can have, summed over all states with a definite energy.

This function:

$$Z(t) = \sum_k e^{-iE_k t}$$

is called the *partition function* of the string.

To calculate it quickly, we'll use two facts:

1) The partition function makes sense for any system with discrete energy levels.

2) When we combine a bunch of systems, we can multiply their partition functions to get the partition function of the combined system.

First: what's the partition function of a harmonic oscillator? An oscillator with frequency ω can have energies

$$\frac{1}{2}\omega,\ (1+\frac{1}{2})\omega,\ (2+\frac{1}{2})\omega,\ (3+\frac{1}{2})\omega, \ldots$$

So, its partition function is:

$$\sum_{k=0}^{\infty} e^{-i(k+\frac{1}{2})\omega t}$$

Summing the geometric series, we get:

$$\frac{e^{-\frac{i}{2}\omega t}}{1-e^{-i\omega t}}$$

Next, what's the partition function of a string that can wiggle in one direction? It's just like a bunch of oscillators with frequencies $1, 2, 3, \ldots$, so its partition function is a product:

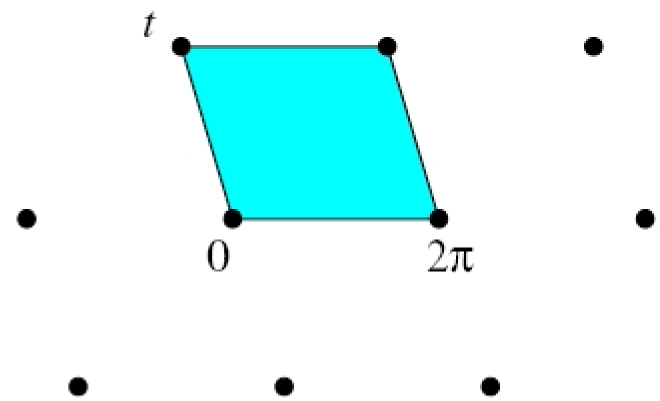
$$\prod_{n=1}^{\infty} \frac{e^{-\frac{i}{2}nt}}{1 - e^{-int}} = e^{-\frac{i}{2}(1+2+3+\cdots)t} \prod_{n=1}^{\infty} \frac{1}{1 - e^{-int}}$$

According to Euler's crazy calculation, this equals

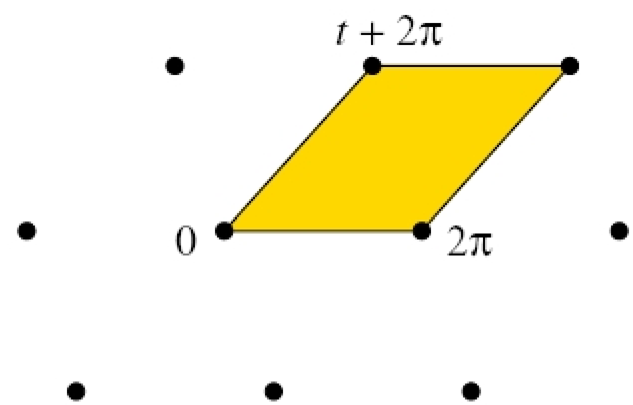
$$e^{\frac{i}{24}t} \prod_{n=1}^{\infty} \frac{1}{1 - e^{-int}}$$

This is the reciprocal of the *Dedekind eta function* — which was introduced in 1877, long before string theory!

However, the torus coming from this parallelogram:



is the same as the torus coming from this one:



So: our calculation only gives consistent answers if the partition function

$$Z(t) = e^{\frac{i}{24}t} \prod_{n=1}^{\infty} \frac{1}{1 - e^{-int}}$$

is unchanged when we add 2π **to** t**. Alas, it *does* change:**

$$Z(t + 2\pi) = e^{\frac{2\pi i}{24}} Z(t)$$

But $Z(t)^{24}$ **does *not* change! This is the partition function of 24 strings with one direction to wiggle — which is just like one string with 24 directions to wiggle.**

So, bosonic string theory works best
when spacetime has $24 + 2 = 26$ ***dimensions!***

The function $Z(t)^{24}$ was famous long before string theory. Its reciprocal is called *the discriminant of an elliptic curve.*

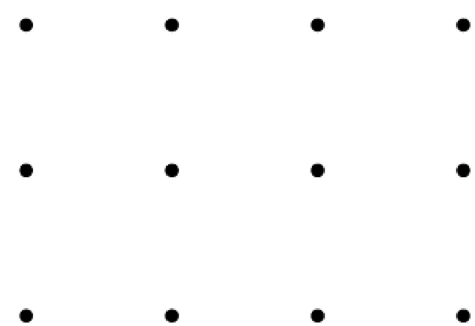
An *elliptic curve* is a torus formed by curling up a parallelogram.

The *discriminant* is the simplest function of elliptic curves that vanishes when the parallelogram gets infinitely skinny.

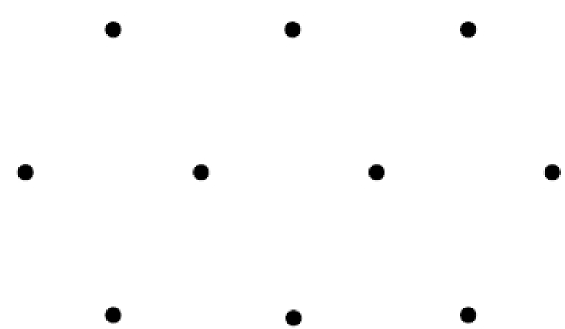
Why does the number 24 automatically show up when we study these ideas?
The answer is easy to state...
but not easy to grasp.

Two types of lattice in the plane have more symmetry than the rest!

One has 4-fold symmetry:



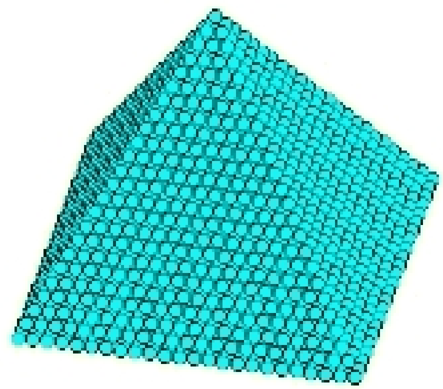
One has 6-fold symmetry:



...and believe it or not, it all boils down to this:

$$\mathbf{4 \times 6 = 24}$$

A full explanation would be quite long... so instead, let's see how Lucas' cannonball puzzle gets involved!

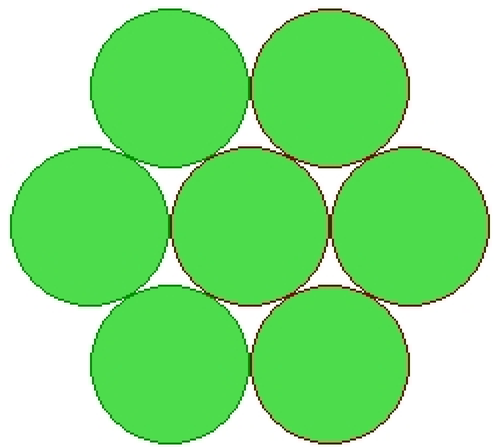


$$1^2 + 2^2 + \cdots + 24^2 = 70^2$$

THE LEECH LATTICE

A *lattice* is a discrete subset $L \subset \mathbb{R}^n$ that's closed under addition and subtraction.

We can use any lattice as a recipe for packing spheres.



The lattice that gives the densest sphere packing in 24 dimensions is called the *Leech lattice*.

In the Leech lattice, each sphere touches $196,560$ others!

How can we get this lattice?

Start with this dot product for vectors in $\mathbb{R}^{26}$:

$$a \cdot b = a_1 b_1 + \cdots + a_{25} b_{25} - a_{26} b_{26}$$

Sitting inside $\mathbb{R}^{26}$ there's the lattice $\mathrm{II}_{25,1}$, where the a_i are integers or all integers plus 1/2, and $a_1 + a_2 + \cdots + a_{25} - a_{26}$ is even. This contains the vector

$$v = (0, 1, 2, 3, \ldots, 24, 70) \in \mathbb{Z}^{26}$$

Let

$$v^\perp = \{a \in \mathrm{II}_{25,1} \colon \; a \cdot v = 0\}$$

This is a 25-dimensional lattice. Since

$$v \cdot v = 0^2 + 1^2 + \cdots + 24^2 - 70^2 = 0$$

we have $v \in v^\perp$. The quotient $v^\perp / v$ is a 24-dimensional lattice: the *Leech lattice*, Λ.

THE MONSTER

Bosonic strings like 24 extra dimensions — besides their own 2 — to wiggle around in! So, let's cook up a theory where these extra directions form a 24-dimensional torus built from the Leech lattice:

$$\boldsymbol{T} = \mathbb{R}^{24}/\boldsymbol{\Lambda}$$

Even better, let's use $\boldsymbol{T}/\mathbb{Z}_2$, where we count points $\boldsymbol{x}$ and $-\boldsymbol{x}$ as the same. In 1986, Richard Borcherds showed the resulting string theory has a colossal group of symmetries.

This group is called the *Monster*.

The Monster has

808017424794512875886459904961710757005754368000000000

or approximately

8×10^{53}

elements.

It's the largest 'sporadic finite simple group'. Fischer and Griess predicted its existence in 1973, and Griess constructed it in 1981.

The Monster has profound but puzzling connections to elliptic curves. People call this subject *Monstrous Moonshine*.

It may take another century to fully understand this.

So: different numbers have different 'personalities'.

Some seem boring.

Others take us on amazing journeys.

My favorite is

APPENDIX: QUATERNIONS, THE TETRAHEDRON, AND THE NUMBER 24

In the second appendix of my talk on the number 8, we saw how the unit quaternions link the dodecahedron to the E_8 lattice. Now let's see how unit quaternions link the tetrahedron to the number 24 and elliptic curves.

Rotational symmetries of a regular tetrahedron give all *even* permutations of its 4 vertices, so these symmetries form what is called the 'alternating group' A_4, with

$$4!/2 = 12$$

elements.

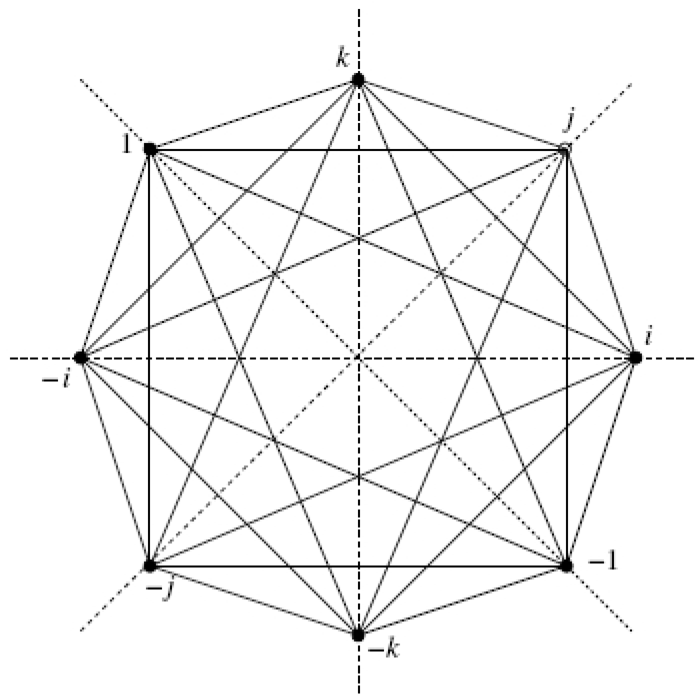
Since the group of unit quaternions is the double cover of the 3d rotation group, there are

$$2 \times 12 = 24$$

unit quaternions that give rotational symmetries of the tetrahedron. These form a group usually called the *binary tetrahedral group*.

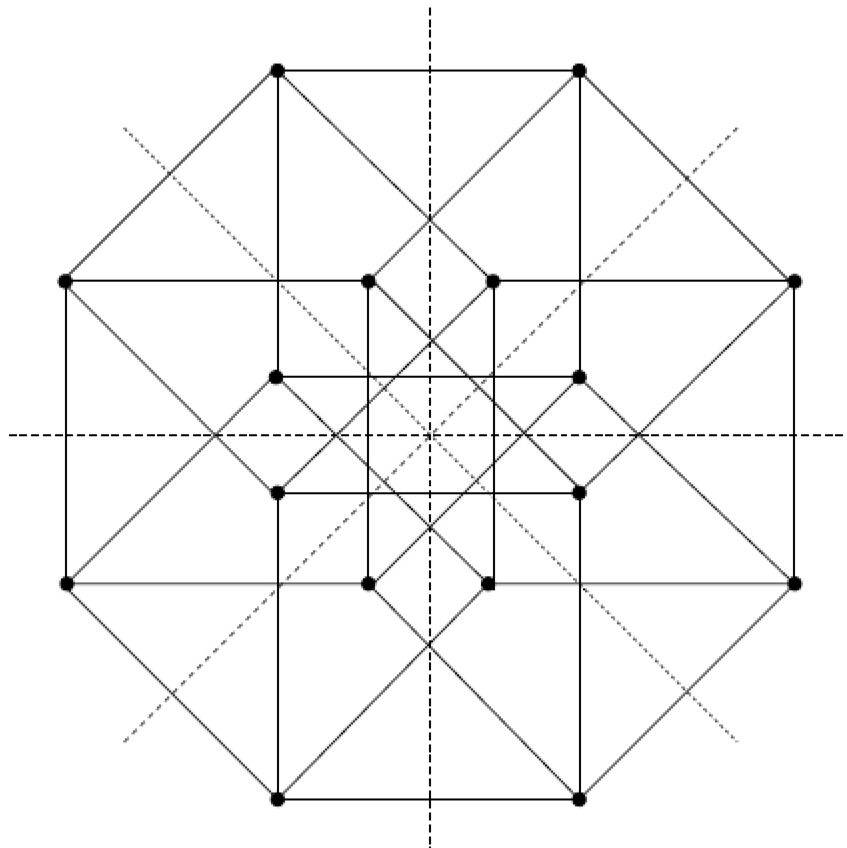
(In case you're wondering, the binary tetrahedral group is *not* isomorphic to the 24-element permutation group S_4, just as the binary icosahedral group is not isomorphic to the 120-element permutation group S_5.)

The 24 unit quaternions in the binary tetrahedral group are also the Hurwitz integers of norm 1, which I described in my last talk. There are 8 like this...



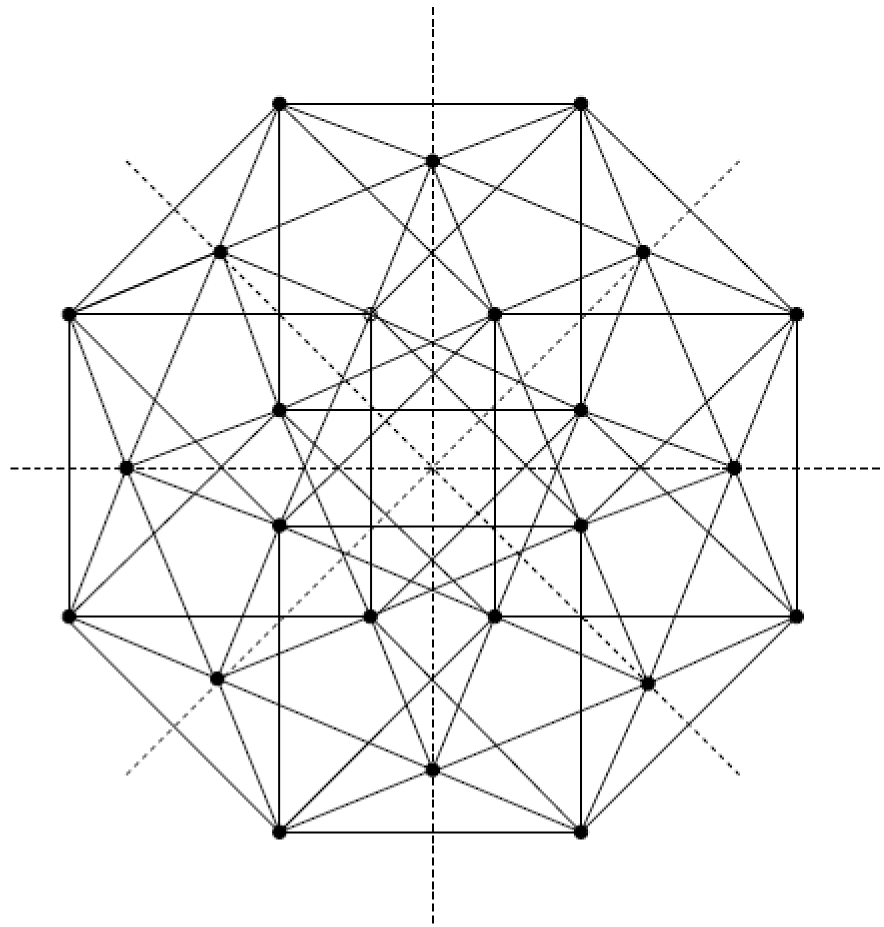
$$\pm 1,\ \pm i,\ \pm j,\ \pm k$$

and 16 like this...



$$\pm\frac{\mathbf{1}}{\mathbf{2}}\pm\frac{\boldsymbol{i}}{\mathbf{2}}\pm\frac{\boldsymbol{j}}{\mathbf{2}}\pm\frac{\boldsymbol{k}}{\mathbf{2}}$$

...for a total of 24:



These 24 points are also the vertices of a regular polytope called the *24-cell.*

But the really interesting thing is how the binary tetrahedral group is related to elliptic curves!

The space of all elliptic curves is $\boldsymbol{H}/\mathrm{SL}(2,\mathbb{Z})$, where $\boldsymbol{H}$ is the upper half of the complex plane, where our variable $\boldsymbol{t}$ lived. But $\mathrm{SL}(2,\mathbb{Z})$ doesn't act freely on $\boldsymbol{H}$, because there are elliptic curves with extra symmetries corresponding to the square and hexagonal lattices.

However, the subgroup $\Gamma(3)$ *does* act freely. This subgroup consists of integer matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant 1, such that each entry is congruent to the corresponding entry of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

modulo 3.

The quotient $H/\Gamma(3)$ is a nice space without any 'points of greater symmetry'. In the theory of elliptic curves it's called $X(3)$: it's the space of elliptic curves equipped with a basis for their 3-torsion.

The group

$$\mathbf{SL(2,\mathbb{Z})/\Gamma(3) = SL(2,\mathbb{Z}/3)}$$

acts on $\boldsymbol{H}/\Gamma(3)$. To get the moduli space of elliptic curves from $\boldsymbol{H}/\Gamma(3)$, we just need to mod out by the action of this group.

But in fact, this group has 24 elements.

It's the binary tetrahedral group!

CREDITS AND NOTES

1. SoftCurve 4-inch brass numbers 2 and 4, available from Home Depot, `http://www.polyvore.com/cgi/thing?id=464459` and `http://www.polyvore.com/cgi/thing?id=262780`
2. 12/24 hour wall clock made by Skilcraft, available from AbilityOne, `http://www.jwodcatalog.com/products.aspx?sid=15`
3. Text, John Baez
4. Cannonball puzzle by Édouard Lucas, Question 1180, *Nouv. Ann. Math. Ser.* 2 (1875), 336. Picture of stacked cannonballs, source unknown.
5. Information on the cannonball puzzle from Mathworld article Cannonball Problem.
6. Euler's argument as summarized by Jeffrey Stopple, *A Primer of Analytic Number Theory, from Pythagoras to Riemann*, Cambridge U. Press, Cambridge, 2003.

7. Euler's argument as summarized in ***A Primer of Analytic Number Theory***, ***op. cit.***

8. Euler's argument as summarized in ***A Primer of Analytic Number Theory***, ***op. cit.***

9. Rigorous proofs that $\zeta(-1) = -1/12$ can be found in many sources, including ***A Primer of Analytic Number Theory***, ***op. cit.*** For an easy introduction to the history of the Riemann zeta function, try K. Sabbagh, ***Dr. Riemann's Zeros***, Atlantic Books, 2002. Also try the Wikipedia article Riemann zeta function.

10. For an introduction to the quantum harmonic oscillator try any decent textbook on quantum mechanics, for example R. Shankar, ***Principles of Quantum Mechanics***, Springer, 1994. Also try the Wikipedia article Quantum harmonic oscillator.

11. **Picture of string vibrational modes by 'Qef' from Wikicommons.**

    The picture here is a bit misleading since it shows a violin string with its ends tied down, but we're really interested in 'closed strings' — that is, closed loops of string. However, both have vibrational modes with frequencies $\omega = 1, 2, 3, \ldots$. The closed string also has a mode with $\omega = 0$, but we ignore this in our simplified treatment. In fact, we only consider 'right-movers': modes corresponding to the functions $\exp(i\omega(t - x))$ for $\omega = 1, 2, 3, \ldots$. For more details, an easy starting-point is Barton Zwiebach, *A First Course in String Theory*, Cambridge U. Press, Cambridge, 2004.

    Also: a string with tension = 1 has integer frequencies if its length is 2π, not 1 as shown here. On page 15 we switch to a string of length 2π.

12. **Text, John Baez.**
13. **Text, John Baez.**
14. **Text, John Baez**

15. Picture of torus by 'Kieff' from Wikipedia article Torus.

16. Picture of parallelogram, John Baez.

17. Text, John Baez. For an introduction to partition functions see any decent introduction to statistical physics, for example F. Reif, *Fundamentals of Statistical and Thermal Physics*, McGraw Hill, New York, 1965. In quantum field theory the partition function is

$$\sum_k e^{-iE_k t} = \mathrm{tr}(\exp(-iHt))$$

where H is an operator called the *Hamiltonian*, whose eigenvalues are the energies E_k. In statistical physics we work instead with $\mathrm{tr}(\exp(-Ht))$ with $t > 0$. However, on page 18 we'll take t to lie in the upper half of the complex plane! When it's purely imaginary, we're doing statistical mechanics as usual.

18. This calculation (with t replacing it) is what convinced Planck that quantum mechanics was a good idea in the first place. It helped him solve the 'ultraviolet catastrophe'. For more, try M. S. Longair, *Theoretical Concepts in Physics*, Cambridge U. Press, Cambridge, 1986.

    Also see the Wikipedia article Planck's law of blackbody radiation. In the derivation there, vibrational modes are indexed by three numbers. That's because they're considering a 3d box of light, instead of a 1d string. They also use t where we have it, as explained above.

19. For basic facts on the Dedekind eta function see the Wikipedia article Dedekind eta function. Our variable t corresponds to the more commonly used variable $2\pi\tau$.

    For more, try Neal Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer, Berlin, 1993.

20. Pictures of parallelograms, John Baez. Similar pictures can be found in any book on elliptic curves, but with the number 1 replacing the number 2π, because we use $t = 2\pi\tau$ where others use τ. See for example Anthony W. Knapp, *Elliptic Curves*, Princeton University Press, Princeton 1992.

We can think of an elliptic curve as a torus equipped with a base point and a conformal structure (a way of measuring oriented angles). Two elliptic curves count as 'the same' if there's a conformal mapping between them that preserves the basepoint. So, rescaling a parallelogram doesn't change the elliptic curve it defines. This is why we can assume without loss of generality that our parallelogram has 'width' equal to 1 — or 2π, if we prefer.

21. Here we are secretly saying that the 24th power of the Dedekind eta function is a modular form. See the Wikipedia article Modular form. For more, see *Introduction to Elliptic Curves and Modular Forms*, *op. cit.*

We've seen that any point in the upper half-plane H determines an elliptic curve, but different points of H can give the same elliptic curve. The space of all elliptic curves — the so-called *moduli space* of elliptic curves — can be identified with H modulo this action of $\mathrm{SL}(2,\mathbb{Z})$:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

Modular forms are functions on H that change in a specific way under these transformations. But for physics, what ultimately matters is that the partition function of the string be *invariant* under these transformations: only then is it a well-defined function on the moduli space of elliptic curves. The 24th power of the Dedekind eta function is *not* invariant. To get a partition function with this property, we would have to consider not just right-movers (see the note to page 11) but also left-movers and the mode with $\omega = 0$.

One might wonder whether we could simply *ignore* the string's ground state energy and leave out the pesky factor of $\exp(\frac{i}{24}t)$. However, the 24th power of that factor is ultimately *required* to obtain a partition function that is well-defined on the moduli space of elliptic curves.

For more on these issues see John Baez, *This Week's Finds in Mathematical Physics*, Week 126 and Week 127.

22. See the section on the Modular discriminant in the Wikipedia article Weierstrass's elliptic function. For more, try *Introduction to Elliptic Curves and Modular Forms* and also *Elliptic Curves*, *op. cit.*

23. Pictures of lattices, John Baez.

24. Here I'm secretly alluding to the 'moduli stack' of elliptic curves, and how the Gaussian and Eisenstein integers give points on this stack with extra symmetry, giving an interesting Picard group. For more details see John Baez, *This Week's Finds in Mathematical Physics*, Week 125.

25. Picture of stacked cannonballs, source unknown.

26. Pictures of hexagonal circle packing, John Baez.

27. Leech lattice information from John H. Conway and Neil J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer, Berlin, 1998.

    The Leech lattice was first announced by John Leech, Notes on sphere packings, *Canad. J. Math.* 19 (1967), 251–267. (Apparently Ernst Witt discovered it in 1940, but did not publish his discovery.)

28. Leech lattice information from *Sphere Packings, Lattices and Groups*, *op. cit.*

29. For a quick introduction to the Monster see the Wikipedia article Monster group. For details, see *Sphere Packings, Lattices and Groups*, *op. cit.* The works of Richard Borcherds can be found on his webpage, `http://math.berkeley.edu/~reb/papers/index.html`

    The first construction of the Monster appears in Robert L. Griess, Jr., The friendly giant, *Invent. Math.* 69 (1982), 1–102.

    For a nice history of Monstrous Moonshine, not assuming any knowledge of mathematics, try Mark Ronan, *Symmetry and the Monster*, Oxford University Press, Oxford, 2006.

    To dig deeper try Terry Gannon, Monstrous Moonshine: The first twenty-five years, available as arXiv:math/0402345.

    For even more try Terry Gannon's wonderful book *Moonshine beyond the Monster: The Bridge Connecting Algebra, Modular Forms and Physics*, Cambridge U. Press, Cambridge, 2006. Enjoy!

30. Text, John Baez.

31. SoftCurve 4-inch brass numbers 2 and 4, available from Home Depot, *op. cit.*

32. Text, John Baez

33. Text, John Baez

34. Picture from John Baez, ***On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry***, by John H. Conway and Derek A. Smith, review in ***Bull. Amer. Math. Soc.*** 42 (2005), 229–243. Also available at `http://math.ucr.edu/home/baez/octonions/conway_smith/`.

35. Picture from review of ***On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry***, ***op. cit.***

36. Picture from review of ***On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry***, ***op. cit.***

    For information on the 24-cell see the Wikipedia article 24-cell. For more on 4d regular polytopes and the quaternions see John Baez, Platonic solids in all dimensions, `http://math.ucr.edu/home/baez/platonic.html`.

37. See ***Elliptic Curves***, ***op. cit.***

38. The group $\Gamma(N)$ is called the *principal congruence subgroup of level* N. For a quick introduction see the Wikipedia article Modular curve.

39. **For a bit of information about $X(N)$ see the Wikipedia article Modular curve. For much more see F. Diamond and J. Shurman, *A First Course in Modular Forms*, Springer, Berlin, 2005. Also see Stephan Stolz's remarks in This Week's Finds in Mathematical Physics, Week 197.**