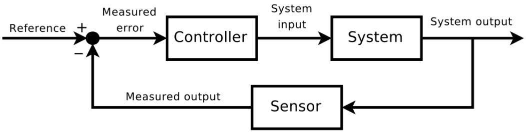
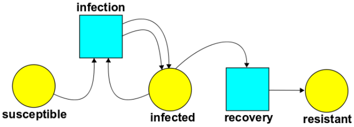
Network Theory

1. Tuesday 25 February, 3:30 pm: electrical circuits and signal-flow graphs.

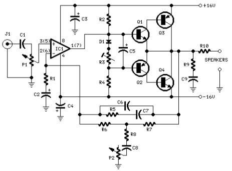


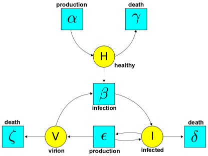
2. Tuesday 4 March, 3:30 pm: stochastic Petri nets, chemical reaction networks and Feynman diagrams.

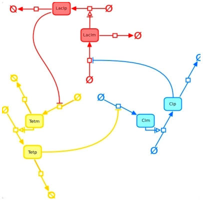


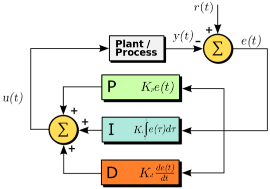
3. Tuesday 11 March, 3:30 pm: Bayesian networks, information and entropy.

In many areas of science and engineering, people use *diagrams of networks*, with boxes connected by wires:





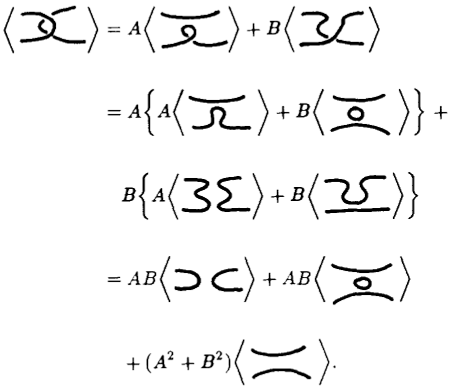




We need a good mathematical theory of these.

Categories must be part of the solution. This became clear in the 1980s, at the interface of knot theory and quantum physics:

Proof. (a)

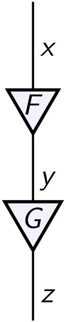


Part (b) is left for the reader.

Categories are great for describing processes of all kinds. A process with input x and output y is called a **morphism** $F\colon x \to y$, and we draw it like this:

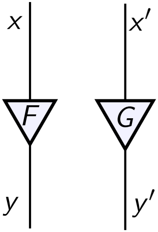
The input and output are called **objects**.

We can do one process after another if the output of the first equals the input of the second:



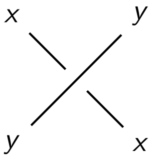
Here we are **composing** morphisms $F\colon x \to y$ and $G\colon y \to z$ to get a morphism $GF\colon x \to z$.

In a **monoidal** category, we can also do processes 'in parallel':

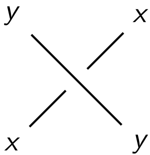


Here we are **tensoring** $F\colon x \to y$ and $G\colon x' \to y'$ to get a morphism $F \otimes G\colon x \otimes x' \to y \otimes y'$.

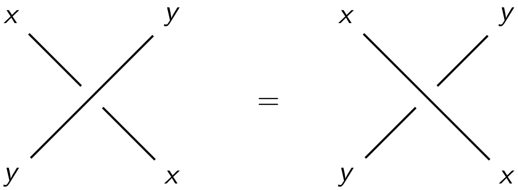
In a **braided** monoidal category, we have a process of switching:



This is called the **braiding** $B_{x,y}\colon x \otimes y \to y \otimes x$. It has an inverse:



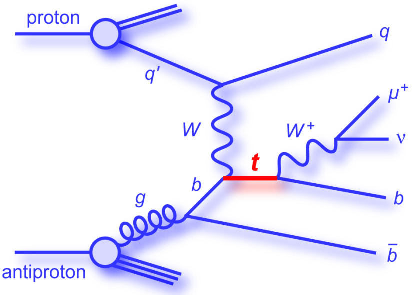
In a **symmetric** monoidal category it doesn't matter which wire goes over which:



All these kinds of categories obey some axioms, which are easy to find.

The category with vector spaces as objects and linear maps between these as morphisms becomes a symmetric monoidal category with the usual $\otimes$.

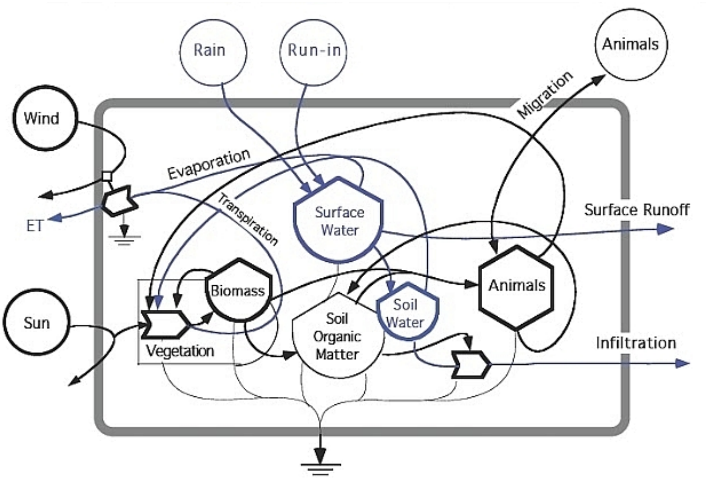
In particle physics, 'Feynman diagrams' are pictures of morphisms in this category:



But why should particle physicists have all the fun? This is the century of biology.

Now is our chance to understand the biosphere, and stop destroying it! We should use everything we can — even mathematics — to do this.

Back in the 1950's, Howard Odum introduced an Energy Systems Language for ecology:



Nowadays, biologists use networks *of many kinds* to describe the complex processes they find in life.

The Systems Biology Graphical Notation project is trying to standardize these network languages. They have developed three:

- activity flow diagrams
- entity relationship diagrams
- process diagrams

Activity Flow Diagrams show the flow of information between entities:

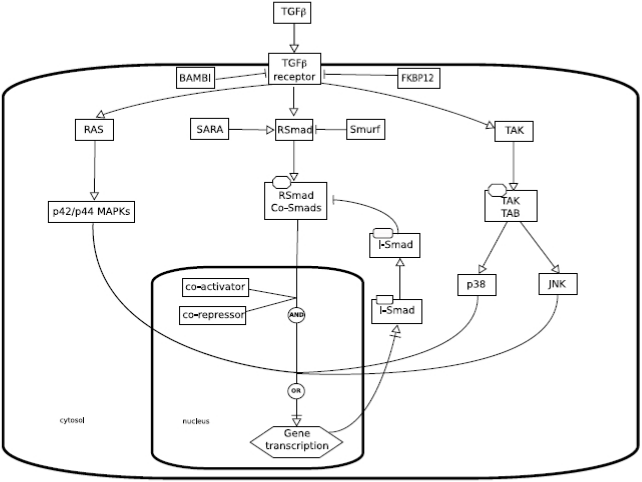


Figure A.3: *Transforming Growth Factor beta signaling pathway.*

Entity Relationship Diagrams show how entities influence the behavior of each other:

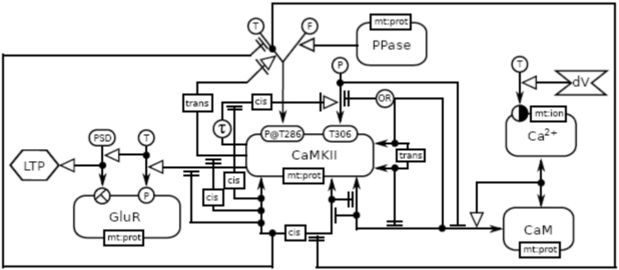


Figure A.2: *Regulation of calcium/calmoduline kinase II effect on synaptic plasticity.*

Process Diagrams show how entities change from one type to another over time:

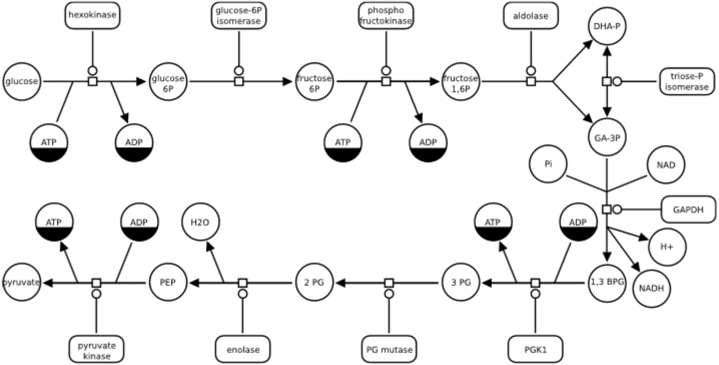
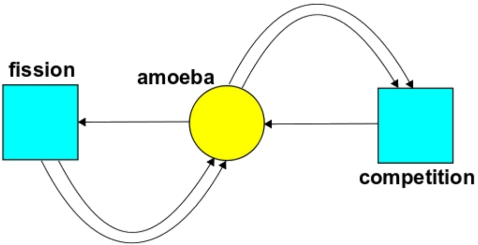


Figure A.1: *Glycolysis. This example illustrates how SBGN can be used to describe metabolic pathways.*

With numbers for rates, we get a system of differential equations!

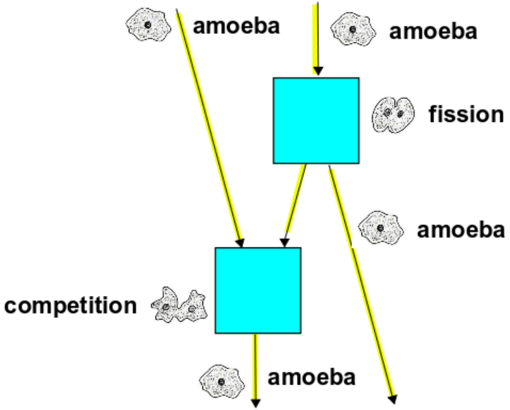
Process diagrams generalize ‘stochastic Petri nets’, which are mathematically well-understood — I’ll talk about them in Part 2.

A **Petri net** is a presentation of a symmetric monoidal category that is freely generated by some objects and morphisms:



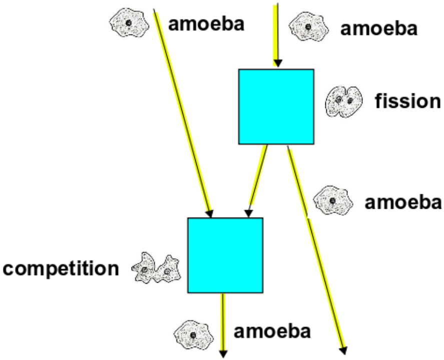
$$\text{fission: amoeba} \to \text{amoeba} \otimes \text{amoeba}$$
$$\text{competition: amoeba} \otimes \text{amoeba} \to \text{amoeba}$$

From these 'generators' we can build all the morphisms in our category by composition, tensoring and the braiding:



In a **stochastic** Petri net, each generating morphism is assigned a **rate constant** $r > 0$.

Using this we can calculate a probability for any morphism to occur in any given amount of time $t > 0$.



The details work *almost exactly like Feynman diagrams in particle physics.* But there's one big difference! We take the symmetric monoidal category used in quantum physics, where:

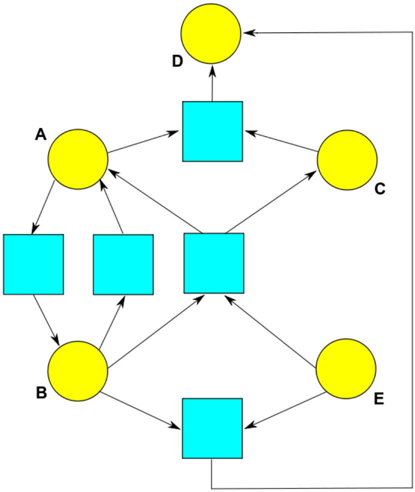
- objects are complex vector spaces,
- morphisms are linear maps,
- $\otimes$ is the usual tensor product of vector spaces

and everywhere replace the complex numbers, $\mathbb{C}$, by nonnegative real numbers, $[0, \infty)$.

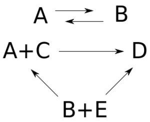
This lets us take quantum techniques and apply them to stochastic processes.

In Part 2 we'll see that this gives a new outlook on chemistry.

Instead of Petri nets, chemists use **reaction networks**, in which



becomes



But this is an equivalent formalism!

In Part 3 we'll see that closely related categories also give a new way to think about *entropy*.

For example, let `FinProb` be the category where:

- an object (X, p) is a finite set equipped with a probability distribution on it:

$$p_x \geq 0 \quad \text{and} \quad \sum_{x \in X} p_x = 1$$

- a morphism $f \colon (X, p) \to (Y, q)$ is a probability-preserving function:

$$q_y = \sum_{x \in X \colon f(x) = y} p_x$$

We can define **convex linear combinations** of objects in `FinProb`. For for any $0 \leq c \leq 1$, let

$$c(X, p) + (1 - c)(Y, q)$$

be the disjoint union of X and Y, with the probability distribution given by cp on X and $(1 - c)q$ on Y.

We can also define convex linear combinations of morphisms.

Any object in $\texttt{FinProb}$ has an **entropy**

$$S(X,p) = -\sum_{x \in X} p_x \ln p_x$$

But what's so great about this? Here's one answer.

There is a category $[0,\infty)$ with one object $*$, where morphisms $c\colon * \to *$ are numbers $c \geq 0$ and composition is addition.

Theorem (Baez, Fritz, Leinster). Any map between categories

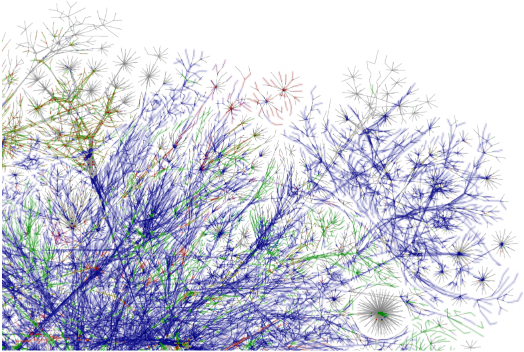
$$F\colon \texttt{FinProb} \to [0,\infty)$$

that is continuous and preserves convex linear combinations is a multiple of the change in entropy: for some $\alpha \geq 0$,

$$g\colon (X,p) \to (Y,q) \quad \Longrightarrow \quad F(g) = \alpha(S(X,p) - S(Y,q))$$

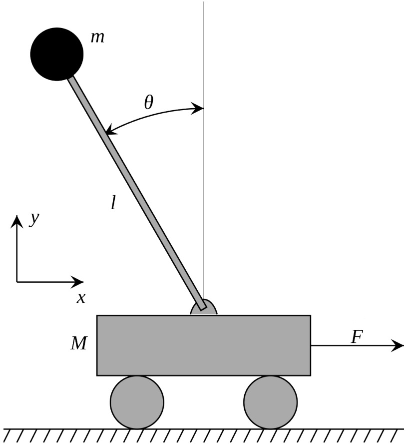
We can use networks to describe *processes*, and this makes us treat them as *morphisms*. But we can also use them to describe *things*, and this makes us treat them as *objects*.

A large body of network theory does this, using ideas from graph theory. For example, the Internet is a thing worth studying:

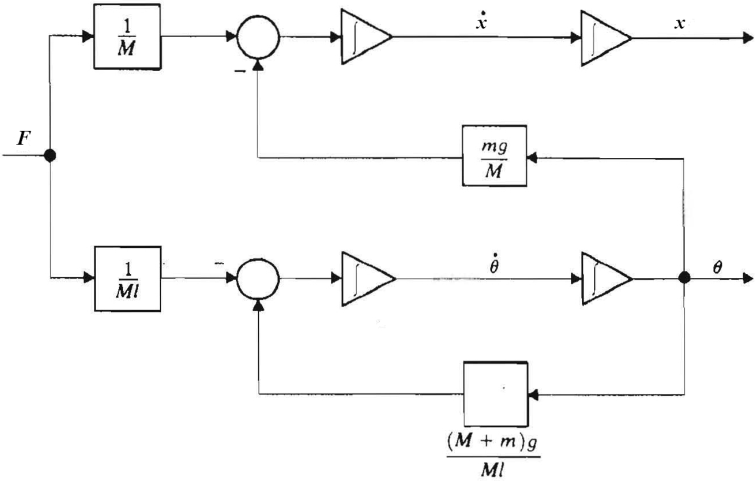


The interplay between *networks as things* and *networks as processes* is especially clear in control theory, which uses 'signal-flow graphs' to describe physical systems with inputs and outputs:

For example, an upside-down pendulum on a cart...



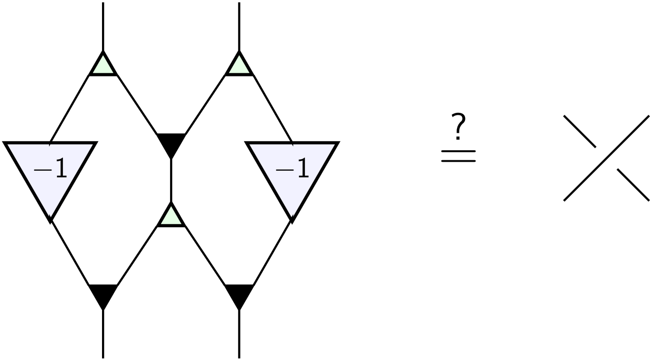
… has this signal-flow graph:



We'll see in Part 1 that signal-flow graphs describe morphisms in a certain symmetric monoidal category... which is similar to that used in particle physics, but also curiously *different*.

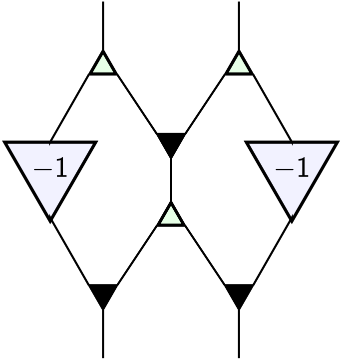
When we call them 'morphisms', we are treating signal-flow graphs as *processes*: ways of turning input signals into outputs.

But signal-flow graphs that implement the same *process* can be very different as *things:*

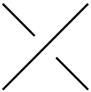


To deal with this more clearly we should use *bicategories*, which have

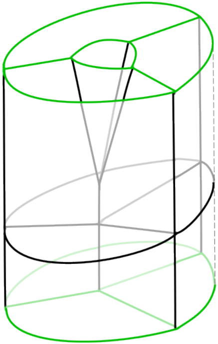
- objects
- morphisms between objects: $F\colon x \to y$
- 2-morphisms between morphisms $\alpha\colon F \Rightarrow G$



$\Rightarrow$



So, not just categories but *bicategories* pervade network theory. These should be especially important in the study of networks that change with time:



In the spin foam approach to quantum gravity, *space itself* is a changing network of this kind. But we should try to use these ideas for something more practical!