THE ARF-KERVAIRE INVARIANT PROBLEM IN ALGEBRAIC TOPOLOGY: INTRODUCTION

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ABSTRACT. This paper gives the history and background of one of the oldest problems in algebraic topology, along with a short summary of our solution to it and a description of some of the tools we use. More details of the proof are provided in our second paper in this volume, *The Arf-Kervaire invariant problem in algebraic topology: Sketch of the proof*. A rigorous account can be found in our preprint *The non-existence of elements of Kervaire invariant one* on the arXiv and on the third author's home page. The latter also has numerous links to related papers and talks we have given on the subject since announcing our result in April, 2009.

CONTENTS

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi^S_{2^{j+1}-2}$ *do not exist for* $j \geq 7$.

Here π_k^S denotes the kth stable homotopy group of spheres, which will be defined shortly.

The kth (for a positive integer k) homotopy group of the topological space X, denoted by $\pi_k(X)$, is the set of continuous maps to X from the k-sphere S^k, up to continuous deformation. For technical reasons we require that each map send a specified point in S^k (called a *base point*) to a specified point $x_0 \in X$. When X is path connected the choice of these two points is irrelevant, so it is usually omitted from the notation. When X is not path connected, we get different collections of maps depending on the path connected component of the base point.

This set has a natural group structure, which is abelian for $k > 1$. The word *natural* here means that a continuous base point preserving map $f : X \to Y$ induces a homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$, sometimes denoted by $\pi_k(f)$.

It is known that the group $\pi_{n+k}(S^n)$ is independent of n for $n > k$. There is a homomorphism $E : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ defined as follows. S^{n+1} $[S^{n+k+1}]$ can be obtained from S^n $[S^{n+k}]$ by a double cone construction known as *suspension*. The cone over S^n is an $(n+1)$-dimensional ball, and gluing two such balls together along there common boundary gives and $(n+1)$-dimensional sphere. A map $f : S^{n+k} \to S^n$ can be canonically extended (by suspending both its source and target) to a map $Ef : S^{n+k+1} \to S^{n+1}$, and this leads to the suspension homomorphism E. The Freudenthal Suspension Theorem [Fre38], proved in 1938, says that it is onto for $k = n$ and an isomorphism for $n > k$. For this reason the group $\pi_{n+k}(S^n)$ is said to be *stable* when $n > k$, and it is denoted by π_k^S and called the *stable k-stem*.

The Main Theorem above concerns the case $k = 2^{j+1} - 2$. The θ_j in the theorem is a hypothetical element related a geometric invariant of certain manifolds studied originally by Pontryagin starting in the 1930s, [Pon38], [Pon50] and [Pon55]. The problem came into its present form with a theorem of Browder [Bro69] published in 1969. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved, namely that ll of the θ_j exist.

The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. Browder's theorem says that such things can exist only in dimensions that are 2 less than a power of 2.

Some homotopy theorists, most notably Mahowald, speculated about what would happen if θ_j existed for all j. They derived numerous consequences about homotopy groups of spheres. The possible nonexistence of the θ_j for large j was known as the DOOMSDAY HYPOTHESIS.

After 1980, the problem faded into the background because it was thought to be too hard. In 2009, just a few weeks before we announced our theorm, Snaith published a book [Sna09] on the problem "to stem the tide of oblivion." On the difficulty of the problem, he wrote

> In the light of ... the failure over fifty years to construct framed manifolds of Arf-Kervaire invariant one this might turn out to be a book about things which do not exist. This [is] why the quotations which preface each chapter contain a preponderance of utterances from the pen of Lewis Carroll.

Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then.

1. BACKGROUND AND HISTORY

1.1. **Pontryagin's early work on homotopy groups of spheres.** The Arf-Kervaire invariant problem has its origins in Pontryagin's early work on a geometric approach to the homotopy groups of spheres, [Pon38], [Pon50] and [Pon55].

Pontryagin's approach to maps $f : S^{n+k} \to S^n$ is to assume that f is smooth and that the base point y_0 of the target is a regular value. (Any continuous f can be continuously deformed to a map with this property.) This means that $f^{-1}(y_0)$ is a closed smooth k-manifold M in S^{n+k}. Let D^n be the closure of an open ball around y_0. If it is sufficiently small, then $V^{n+k} = f^{-1}(D^n) \subset S^{n+k}$ is an $(n+k)$-manifold homeomorphic to $M \times D^n$ with boundary homeomorphic to $M \times S^{n-1}$. It is also a tubular neighborhood of M^k and comes equipped with a map $p : V^{n+k} \to M^k$ sending each point to the nearest point in M. For each $x \in M$, $p^{-1}(x)$ is homeomorphic to a closed n-ball B^n. The pair $(p, f|V^{n+k})$ defines an explicit homeomorphism

$$V^{n+k} \xrightarrow[\approx]{(p,f|V^{n+k})} M^k \times D^n.$$

This structure on M^k is called a *framing*, and M is said to be *framed in* $\mathbf{R}^{n+k}$. A choice of basis of the tangent space at $y_0 \in S^n$ pulls back to a set of linearly independent normal vector fields on $M \subset \mathbf{R}^{n+k}$. These will be indicated in Figures 1–3 and 6 below.

Conversely, suppose we have a closed sub-k-manifold $M \subset \mathbf{R}^{n+k}$ with a closed tubular neighborhood V and a homeomorphism h to $M \times D^n$ as above. This is called a *framed sub-k-manifold* of $\mathbf{R}^{n+k}$. Some remarks are in order here.

- The existence of a framing puts some restrictions on the topology of M. All of its charactersitic classes must vanish. In particular it must be orientable.
- A framing can be twisted by a map $g : M \to SO(n)$, where $SO(n)$ denotes the group of orthogonal $n \times n$ matrices with determinant 1. Such matrices act on D^n in an obvious way. The twisted framing is the composite

$$\begin{array}{ccccc} V & \xrightarrow{h} & M^k \times D^n & \longrightarrow & M^k \times D^n \\ & & (m,x) & \longmapsto & (m, g(m)(x)). \end{array}$$

  We will say more about this later.
- If we drop the assumption that M is framed, then the tubular neighborhood V is a (possibly nontrivial) disk bundle over M. The map $M \to y_0$ needs to be replaced by a map to the classifying space for such bundles, $BO(n)$. This leads to unoriented bordism theory, which was analyzed by Thom in [Tho54]. Two helpful references for this material are the books by Milnor-Stasheff[MS74] and Stong[Sto68].

Pontryagin constructs a map $P(M,h) : S^{n+k} \to S^n$ as follows. We regard S^{n+k} as the one point compactification of $\mathbf{R}^{n+k}$ and S^n as the quotient $D^n/\partial D^n$. This leads to a diagram

$$\begin{tikzcd} & (V,\partial V) \arrow[r, "h"] \arrow[d, hook] & M \times (D^n,\partial D^n) \arrow[r, "p_2"] & (D^n,\partial D^n) \arrow[d] \\ & (\mathbf{R}^{n+k},\mathbf{R}^{n+k}-\mathrm{int}V) \arrow[r] & (S^{n+k},S^{n+k}-\mathrm{int}V) \arrow[r, dotted, "P(M,h)"] & (S^n,\{\infty\}) \\ & & S^{n+k}-\mathrm{int}V \arrow[r, mapsto, "P(M,h)"] & \{\infty\} \end{tikzcd}$$

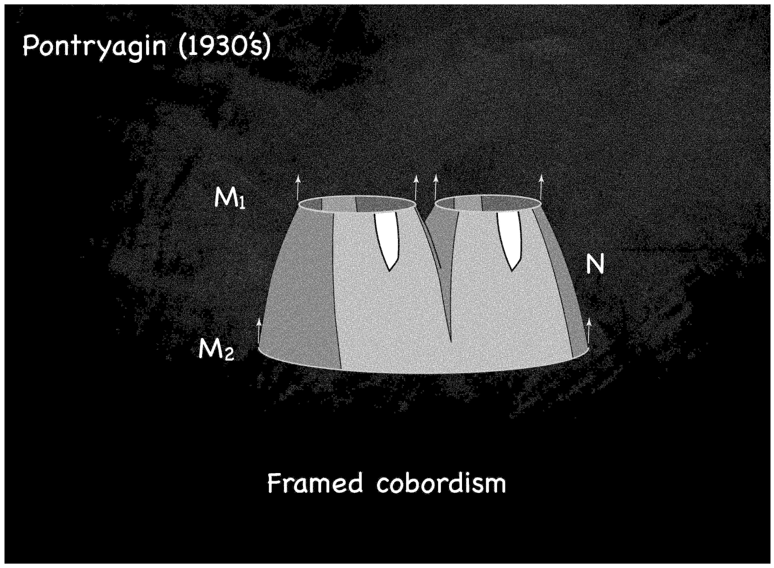


FIGURE 1. A framed cobordism between $M_1 = S^1 \coprod S^1 \subset \mathbf{R}^2$ and $M_2 = S^1 \subset \mathbf{R}^3$ with $N \subset [0,1] \times \mathbf{R}^2$. The normal framings on the circles can be chosen so they extend over N.

The map $P(M,h)$ is the extension of $p_2 h$ obtained by sending the compliment of V in S^{n+k} to the point at infinity in S^n. For $n > k$, the choice of the embedding (but not the choice of framing) of M into the Euclidean space is irrelevant. Any two embeddings (with suitably chosen framings) lead to the same map $P(M,h)$ up to continuous deformation.

To proceed further, we need to be more precise about what we mean by continuous deformation. Two maps $f_1, f_2 : X \to Y$ are *homotopic* if there is a continuous map $h : X \times [0,1] \to Y$ (called a *homotopy between* f_1 *and* f_2) such that

$$h(x,0) = f_1(x) \qquad \text{and} \qquad h(x,1) = f_2(x).$$

Now suppose $X = S^{n+k}$, $Y = S^n$, and the map h (and hence f_1 and f_2) is smooth with y_0 as a regular value. Then $h^{-1}(y_0)$ is a framed $(k+1)$-manifold N whose boundary is the disjoint union of $M_1 = f^{-1}(y_0)$ and $M_2 = g^{-1}(y_0)$. This N is called a *framed cobordism* between M_1 and M_2, and when it exists the two closed manifolds are said to be *framed cobordant*. An example is shown in Figure 1.

Let $\Omega_{k,n}^{\mathrm{fr}}$ denote the cobordism group of framed k-manifolds in $\mathbf{R}^{n+k}$. The above construction leads to Pontryagin's isomorphism

$$\Omega_{k,n}^{\mathrm{fr}} \xrightarrow{\approx} \pi_{n+k}(S^n).$$

First consider the case $k = 0$. Here the 0-dimensional manifold M is a finite set of points in $\mathbf{R}^n$. Each comes with a framing which can be obtained from a standard one by an element in the orthogonal group $O(n)$. We attach a sign to each point corresponding to the sign of the associated determinant. With these signs we can count the points algebraically and get an integer called the *degree of* f. Two framed 0-manifolds are cobordant iff they

Sidebar 1 The Hopf-Whitehead J-homomorphism

Suppose our framed manifold is S^k with a framing that extends to a D^{k+1}. This will lead to the trivial element in $\pi_{n+k}(S^n)$, but twisting the framing can lead to nontrivial elements. The twist is determined up to homotopy by an element in $\pi_k(SO(n))$. Pontryagin's construction thus leads to the homomorphism

$$\pi_k(SO(n)) \xrightarrow{J} \pi_{n+k}(S^n)$$

introduced by Hopf [Hop35] and Whitehead [Whi42]. Both source and target known to be independent of n for $n > k+1$. In this case the source group for each k (denoted simply by $\pi_k(SO)$ since n is irrelevant) was determined by Bott [Bot59] in his remarkable periodicity theorem. He showed

$$\pi_k(SO) = \begin{cases} \mathbf{Z} & \text{for } k \equiv 3 \text{ or } 7 \bmod 8 \\ \mathbf{Z}/2 & \text{for } k \equiv 0 \text{ or } 1 \bmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Here is a table showing these groups for $k \leq 10$.

k	1	2	3	4	5	6	7	8	9	10
$\pi_k(SO)$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0

In each case where the group is nontrivial, its the image under J of its generator is known to generate a direct summand. In the jth case we denote this image by β_j and its dimension by $\phi(j)$, which is roughly $2j$. The first three of these are the Hopf maps $\eta \in \pi_1^S$, $\nu \in \pi_3^S$ and $\sigma \in \pi_7^S$. After that we have $\beta_4 \in \pi_8^S$, $\beta_5 \in \pi_9^S$, $\beta_6 \in \pi_{11}^S$ and so on.

For the case $\pi_{4m-1}(SO) = \mathbf{Z}$, the image under J is known to be a cyclic group whose order a_m is the denominator of $B_m/4m$, where B_m is the mth Bernoulli number. Details can be found in [Ada66] and [MS74]. Here is a table showing these values for $m \leq 10$.

m	1	2	3	4	5	6	7	8	9	10
a_m	24	240	504	480	264	65,520	24	16,320	28,728	13,200

have the same degree. Figure 2 shows a cobordism between the empty set and a pair of points with opposite signs.

Now consider the case $k = 1$. M is a closed 1-manifold, i.e., a disjoint union of circles. Two framings on a single circle differ by a map from S^1 to the group $SO(n)$, and it is known that

$$\pi_1(SO(n)) = \begin{cases} 0 & \text{for } n = 1 \\ \mathbf{Z} & \text{for } n = 2 \\ \mathbf{Z}/2 & \text{for } n > 2. \end{cases}$$

Figure 2 illustrates the two different framings on S^1 for $n = 2$. It turns about that any disjoint union of framed circles is cobordant to a single framed circle. This can be used to show that

$$\pi_{n+1}(S^n) = \begin{cases} 0 & \text{for } n = 1 \\ \mathbf{Z} & \text{for } n = 2 \\ \mathbf{Z}/2 & \text{for } n > 2. \end{cases}$$

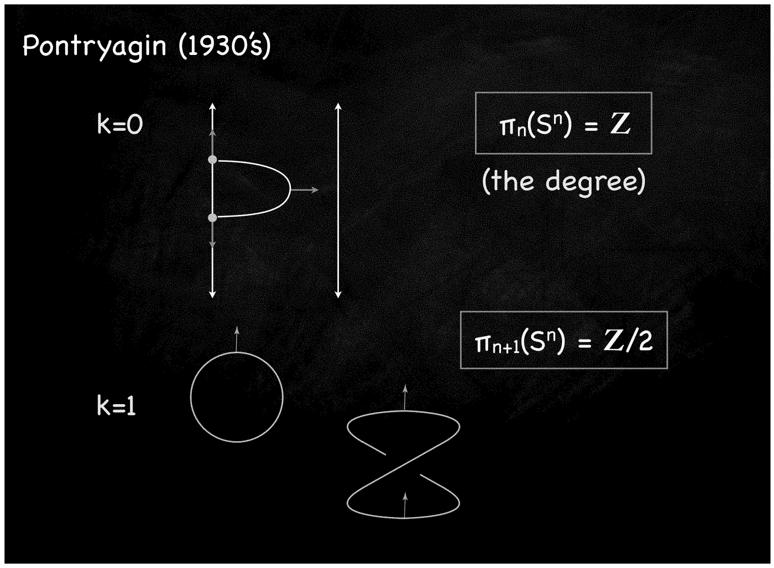


FIGURE 2. The cases $k = 0$ and $k = 1$. The indicated 0-manifold (two points in $\mathbf{R}$ with opposite signs) is framed cobordant (via the yellow line) to the empty set. For $k = 1$, the two circles are framed in $\mathbf{R}^3$. One normal field on each is in the plane of the picture as indicated, and the second (not shown) is pointing out of the plane of the picture toward the reader. Which of these two framings extends to a disk in $\mathbf{R}^3$?

The case $k = 2$ is more subtle. As in the 1-dimensional case we have a complete classification of closed 2-manifolds, and it is only necessary to consider path connected ones. The existence of a framing implies that the surface is orientable, so it is characterized by its genus.

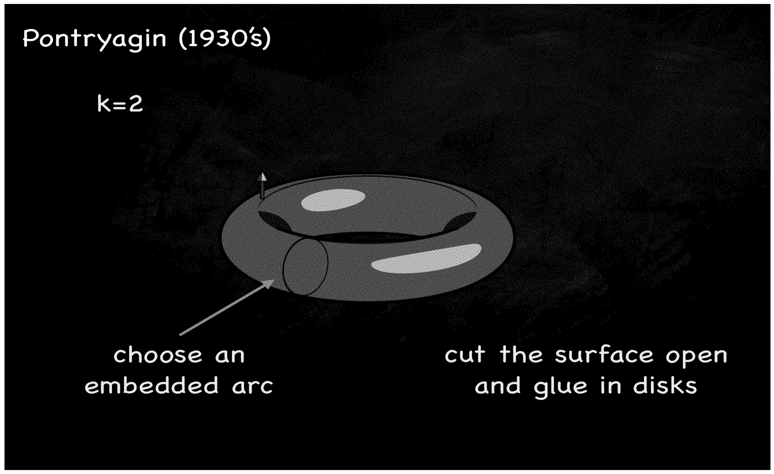
If the genus is zero, namely if $M = S^2$, then there is a framing which extends to a 3-dimensional ball. This makes M cobordant to the empty set, which means that the map is *null homotopic* (or, more briefly, *null*), meaning that it is homotopic to a constant map. Any two framings on S^2 differ by an element in $\pi_2(SO(n))$. This group is known to vanish, so any two framings on S^2 are equivalent, and the map $f : S^{n+2} \to S^n$ is null.

Now suppose the genus is one, as shown in Figure 3. Suppose we can find an embedded arc as shown on which the framing extends to a disk. Then there is a cobordism which effectively cuts along the arc and attaches two disks as shown. This process is called *framed surgery*. If we can do this, then we have converted the torus to a 2-sphere and shown that the map $f : S^{n+2} \to S^n$ is null.

When can we find such a closed curve in M? It must represent a generator of $H_1(M)$ and carry a trivial framing. This leads to a map

$$\varphi : H_1(M; \mathbf{Z}/2) \to \mathbf{Z}/2 \tag{1}$$

defined as follows. Each class in H_1 can be represented by a closed curve which is framed either trivially or nontrivially. It can be shown that homologous curves have the same framing invariant, so φ is well defined. At this point Pontryagin made a famous mistake



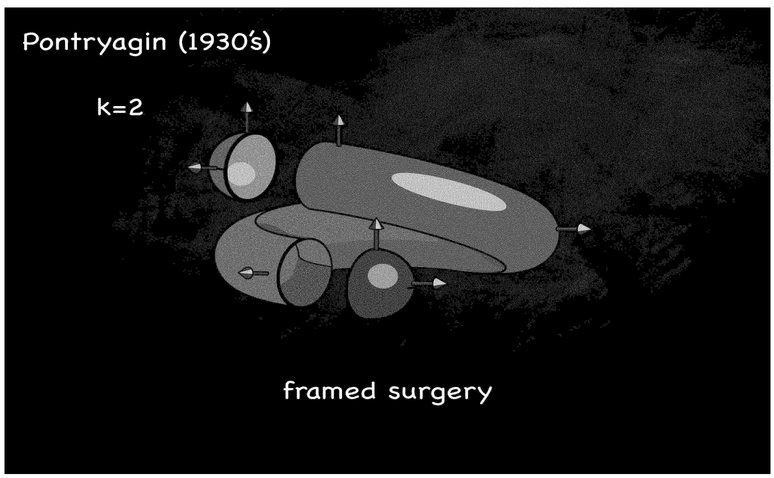


FIGURE 3. The case $k = 2$ and genus 1. If the framing on the embedded arc extends to a disk, then there is a cobordism (called a framed surgery) that converts the torus to a 2-sphere as shown.

which went undedected for over a decade: HE ASSUMED THAT φ WAS A HOMOMORPHISM. We now know this is not the case, and we will say more about it below in §1.3. This nonlinearity is illustrated in Figure 4.

On that basis he argued that φ must have a nontrivial kernel, since the source group is $(\mathbf{Z}/2)^2$. Therefore there is a closed curve along which we can do the surgery shown in Figure 3. It follows that M can be surgered into a 2-sphere, leading to the erroneous conclusion that $\pi_{n+2}(S^n) = 0$ for all n. Freudenthal [Fre38] and later George Whitehead [Whi50] both proved that it is $\mathbf{Z}/2$ for $n \geq 2$. Pontryagin corrected his mistake in [Pon50], and in [Pon55] he gave a complete account of the relation between framed cobordism and homotopy groups of spheres.

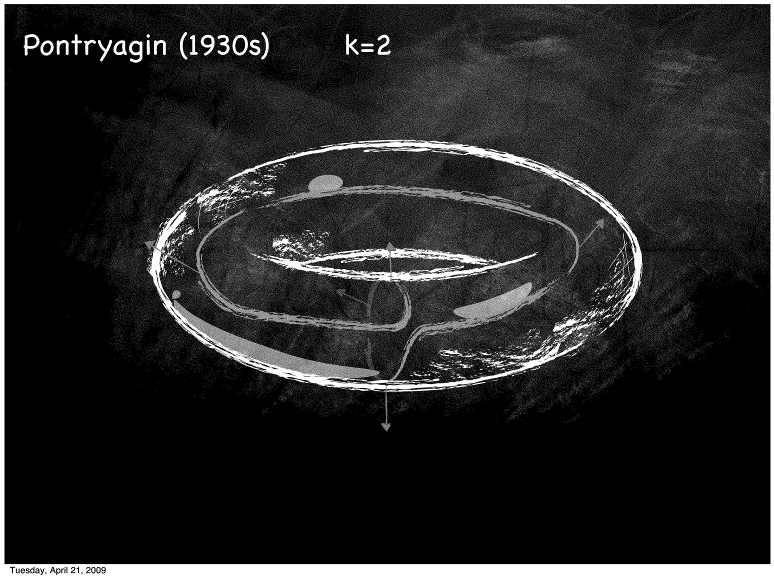


FIGURE 4. The nonlinearity of φ. Even if the framing on the torus is such that its restrictions to the longitudinal and latitudinal circles each extends to a disk, the resulting framing on their sum does not.

1.2. **Our main result.** Our main theorem can be stated in three different but equivalent ways:

- *Manifold formulation:* It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds M is always zero.
- *Stable homotopy theoretic formulation:* It says that certain long sought hypothetical maps between high dimensional spheres do not exist.
- *Unstable homotopy theoretic formulation:* It says something about the EHP sequence (to be defined below), which has to do with unstable homotopy groups of spheres.

The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved.

Here again is the stable homotopy theoretic formulation.

Main Theorem. *The Arf-Kervaire elements $\theta_j \in \pi^S_{2^{j+1}-2}$ do not exist for $j \geq 7$.*

1.3. **The manifold formulation.** Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank $2n$ with mod 2 reduction $\overline{H}$. It is known that $\overline{H}$ has a basis of the form $\{a_i, b_i \colon 1 \leq i \leq n\}$ with

$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

A *quadratic refinement of* λ is a map $q : \overline{H} \to \mathbf{Z}/2$ satisfying

$$q(x+y) = q(x) + q(y) + \lambda(x, y)$$

Its *Arf invariant* is

$$\mathrm{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbf{Z}/2.$$

In 1941 Arf [Arf41] proved that this invariant (along with the number n) determines the isomorphism type of q.

An equivalent definition is the "democratic invariant" of Browder. The elements of $\overline{H}$ "vote" for either 0 or 1 by the function q. The winner of the election (which is never a tie) is $\mathrm{Arf}(q)$. Here is a table illustrating this for three possible refinements q, q' and q'' when $\overline{H}$ has rank 2.

x	0	a	b	$a+b$	Arf invariant
$q(x)$	0	0	0	1	0
$q'(x)$	0	1	1	1	1
$q''(x)$	0	1	0	0	0

The value each refinement on $a+b$ is determined by those on a and b, and q'' is isomorphic to q . Thus the vote is three to one in each case. When $\overline{H}$ has rank 4, it is 10 to 6.

Let M be a $2m$-connected smooth closed manifold of dimension $4m+2$ with a framed embedding in $\mathbf{R}^{4m+2+n}$. We saw above that this leads to a map $f : S^{n+4m+2} \to S^n$ and hence an element in $\pi_{n+4m+2}(S^n)$.

Let $H = H_{2m+1}(M; \mathbf{Z})$, the homology group in the middle dimension. Each $x \in H$ is represented by an immersion $i_x : S^{2m+1} \looparrowright M$ with a stably trivialized normal bundle. H has an antisymmetric bilinear form λ defined in terms of intersection numbers.

In 1960 Kervaire [Ker60] defined a quadratic refinement q on its mod 2 reduction in terms of the trivialization of each sphere's normal bundle. The *Kervaire invariant* $\Phi(M)$ is defined to be the Arf invariant of q. In the case $m = 0$, when the dimension of the manifold is 2, Kervaire's q is Pontryagin's map φ of (1).

What can we say about $\Phi(M)$?

- Kervaire [Ker60] showed it must vanish when $k = 2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure. This is illustrated in Figure 5. N is a smooth 10-manifold with boundary given as the union of two copies of the tangnent disk bundle of S^5. The boundary is homeomorphic to S^9. Thus we can get a closed topological manifold X by gluing on a 10-ball along its common boundary with n, or equivalently collapsing ∂N to a point. X then has nontrivial Kervaire invariant. On the other hand, Kervaire proved that any smooth framed manifold must have trivial

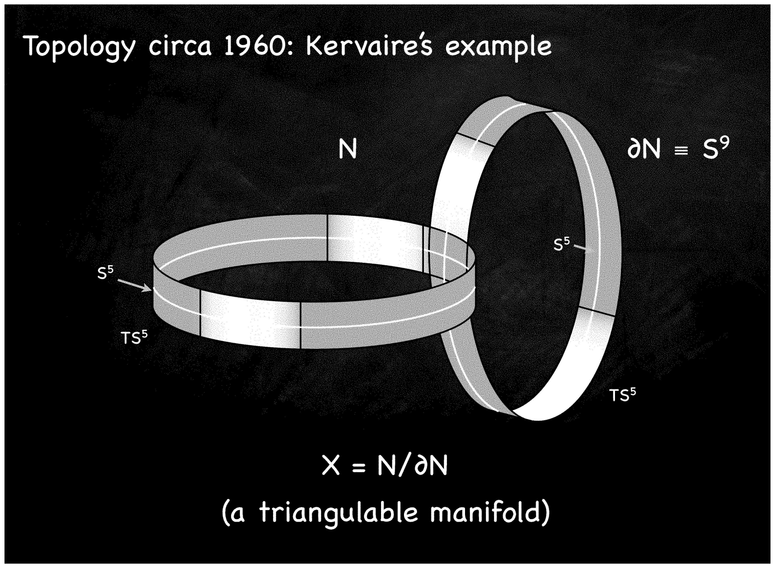


FIGURE 5. Kervaire's example of a nonsmoothable 10-manifold. The manifold N is a smooth 10-manifold with boundary homeomorphic to S^9. The manifold X obtained by collapsing the boundary to a point would have $\Phi(X) = 1$ and is therefore not smooth.

Kervaire invariant. Therefore the topological framed manifold X cannot have a smooth structure. Equivalently, the boundary ∂N cannot be diffeomorphic to S^9. It must be an exotic 9-sphere.

- For $k = 0$ there is a framing on the torus $S^1 \times S^1 \subset \mathbf{R}^4$ with nontrivial Kervaire invariant. Pontryagin used it in [Pon50] (after some false starts in the 30s) to show $\pi_{n+2}(S^n) = \mathbf{Z}/2$ for all $n \geq 2$. It is illustrated in Figure 6. That picture shows a torus immersed in $\mathbf{R}^3$. This immersion is the linear image of an embedding in $\mathbf{R}^4$.
- There are similar constructions for $k = 1$ and $k = 3$, where the framed manifolds are $S^3 \times S^3$ and $S^7 \times S^7$ respectively. Like S^1, S^3 and S^7 are both parallelizable, meaning that their trivial tangent bundles are trivial. The framings can be twisted in such a way as to yield a nontrivial Kervaire invariant.
- Brown-Peterson [BP66] showed that it vanishes for all positive even k. This means that apart from the 2-dimensioanl case, any smooth framed manifold with non-trivail Kervaire invariant must a dimension congruent to 6 modulo 8.
- Browder [Bro69] showed that it can be nontrivial only if $k = 2^{j-1} - 1$ for some positive integer j. This happens iff the element h_j^2 is a permanent cycle in the Adams spectral sequence, which was originally introduced in [Ada58]. (More information about it can be found below in §3.7) in [Rav86] and[Rav04].) The corresponding element in $\pi^S_{n+2^{j+1}-2}$ is θ_j, the subject of our theorem. *This is the stable homotopy theoretic formulation of the problem.*

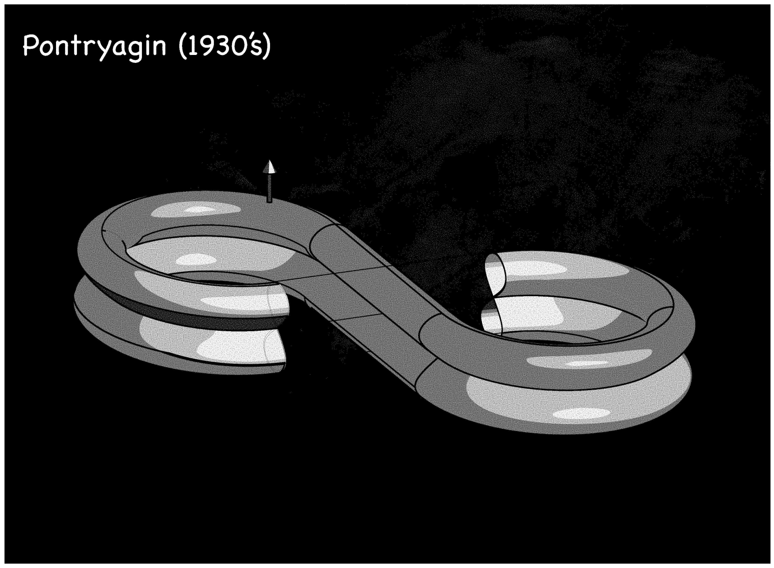


FIGURE 6. A framing on the torus with nontrivial Kervaire invariant. The immersion shown in $\mathbf{R}^3$ is the linear image of an embedding in $\mathbf{R}^4$. This framing on the torus does not exted to any manifold bounded by it.

- θ_j is known to exist for $1 \leq j \leq 3$, i.e., in dimensions 2, 6, and 14. In these cases the relevant framed manifold is $S^{2^j-1} \times S^{2^j-1}$ with a twisted framing as discussed above. The framings on S^{2^j-1} represent the elements h_j in the Adams spectral sequence. The Hopf invariant one theorem of Adams [Ada60] says that for $j > 3$, h_j is not a permanent cycle in the Adams spectral sequence because it supports a nontrivial differential. (His original proof was not written in this language, but had to do with secondary cohomlogy operations.) This means that for $j > 3$, a smooth framed manifold representing θ_j (i.e., having a nontrivial Kervaire invariant) cannot have the form $S^{2^j-1} \times S^{2^j-1}$.
- θ_j is also known to exist for $j = 4$ and $j = 5$, i.e., in dimensions 30 and 62. In both cases the existence was first established by purely homotopy theoretic means, without constructing a suitable framed manifold. For $j = 4$ this was done by Barratt, Mahowald and Tangora in [MT67] and [BMT70]. A framed 30-manifold with nontrivial Kervaire invariant was later constructed by Jones [Jon78]. For $j = 5$ the homotopy theory was done in 1985 by Barratt-Jones-Mahowald in [BJM84].
- Our theorem says θ_j does *not* exist for $j \geq 7$. The case $j = 6$ is still open.

Figure 7 illustrates Kervaire's construction of a framed $(4k+2)$-manifold with nontrivial Kervaire invariant. In all cases except $k = 0$, 1 or 3, any framing of this manifold will do because the tangent bundle of S^{2k+1} is nontrivial and leads to a nontrivial invariant. What the picture does not tell us is whether the bounding sphere S^{4k+1} is diffeomorphic to the standard sphere. If it is, then attaching a $(4k+2)$-disk to it will produce a smooth framed manifold with nontrival Kervaire invariant. If it is not, then we have an exotic $(4k+1)$-sphere bounding a framed manifold and hence not detected by framed cobordism.

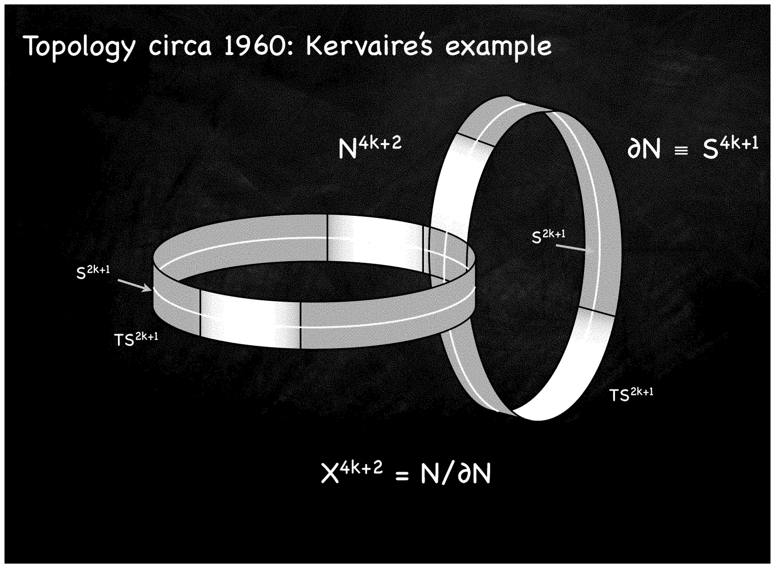


FIGURE 7. Kervaire's example for general k. N is a smooth framed $(4k+2)$-manifold whose boundary is homeomorphic to S^{4k+1}. If ∂N is diffeomorphic to S^{4k+1}, then X is a closed smooth framed $(4k+2)$-manifold with nontrivial Kervaire invariant. We now know this is the case only when $k = 0$, 1, 3, 7, 15 and possibly 31. Otherwise ∂N is an exotic $(4k+1)$-sphere that is a framed boundary.

1.4. **The unstable formulation.** Assume all spaces in sight are localized and the prime 2. For each $n > 0$ there is a fiber sequence (see §3.1) due to James, [Jam55], [Jam56a], [Jam56b] and [Jam57]

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}. \tag{2}$$

Here $\Omega X = \Omega^1 X$ where $\Omega^k X$ denotes the space of continuous base point preserving maps to X from the k-sphere S^k, known as the kth loop space of X. This leads to a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_{m+n}(S^n) \xrightarrow{E} \pi_{m+n+1}(S^{n+1}) \xrightarrow{H} \pi_{m+n+1}(S^{2n+1}) \xrightarrow{P} \pi_{m+n-1}(S^n) \longrightarrow \cdots$$

Here

- E stands for *E*inhängung, the German word for suspension.
- H stands for *H*opf invariant.
- P stands forWhitehead *p*roduct.

Assembling these for fixed m and various n leads to a diagram

$$\begin{tikzcd}
 & \pi_{m+n+1}(S^{2n-1}) \arrow[d, "P"] & \pi_{m+n+2}(S^{2n+1}) \arrow[d, "P"] & \pi_{m+n+3}(S^{2n+3}) \arrow[d, "P"] & \\
\cdots \arrow[r, "E"] & \pi_{m+n-1}(S^{n-1}) \arrow[r, "E"] \arrow[d, "H"] & \pi_{m+n}(S^n) \arrow[r, "E"] \arrow[d, "H"] & \pi_{m+n+1}(S^{n+1}) \arrow[r, "E"] \arrow[d, "H"] & \cdots \\
 & \pi_{m+n-1}(S^{2n-3}) & \pi_{m+n}(S^{2n-1}) & \pi_{m+n+1}(S^{2n+1}) &
\end{tikzcd}$$

where

- Sequences of arrows labeled H, P, E, H (or any subset thereof) in that order are exact.
- The groups in the top and bottom rows are inductively known, and we can compute those in the middle row by induction on n.
- The groups in the top and bottom rows vanish for large n, making E an isomorphism.
- An element in the middle row has trivial suspension (is killed by E) iff it is in the image of P.
- It desuspends (is in the image of E) iff its Hopf invariant (image under H) is trivial.

When $m = n - 1$ this diagram is

$$
\begin{tikzcd}
 & & \pi_{2n+1}(S^{n+1}) \arrow[d, "H"] & & \\
 & \pi_{2n}(S^{2n-1}) \arrow[d, "P"] & \mathbf{Z} \arrow[d, "P"] & 0 \arrow[d, "P"] & \\
\cdots \arrow[r, "E"] & \pi_{2n-2}(S^{n-1}) \arrow[r, "E"] \arrow[d, "H"] & \pi_{2n-1}(S^n) \arrow[r, "E"] \arrow[d, "H"] & \pi_{2n}(S^{n+1}) \arrow[r, "E"] \arrow[d, "H"] & \cdots \\
 & \pi_{2n-2}(S^{2n-3}) & \mathbf{Z} & 0 &
\end{tikzcd}
$$

The image under P of the generator of the upper $\mathbf{Z}$ is denoted by $w_n \in \pi_{2n-1}(S^n)$ and is called the *Whitehead square*.

- When n is even, $H(w_n) = 2$ and w_n has infinite order.
- w_n is trivial for $n = 1$, 3 and 7. In those cases the generator of the upper $\mathbf{Z}$ is the Hopf invariant (image under H) of one of the three Hopf maps in $\pi_{2n+1}(S^{n+1})$,

$$S^3 \xrightarrow{\eta} S^2, \qquad S^7 \xrightarrow{\nu} S^4 \quad \text{and} \qquad S^{15} \xrightarrow{\sigma} S^8.$$

- For other odd values of n, twice the generator of the upper $\mathbf{Z}$ is $H(w_{n+1})$, so w_n has order 2.
- It turns out that w_n is divisible by 2 iff $n = 2^{j+1} - 1$ and θ_j exists, in which case $w_n = 2\theta_j$.
- Each Whitehead square $w_{2n+1} \in \pi_{4n+1}(S^{2n+1})$ (except the cases $n = 0$, 1 and 3) desuspends to a lower sphere until we get an element with a nontrivial Hopf invariant, which is always some β_j (see Sidebar 1). More precisely we have

$$H(w_{(2s+1)2^j-1}) = \beta_j$$

  for each $j > 0$ and $s \geq 0$. This result is essentially Adams' 1962 solution to the vector field problem [Ada62].

Recall the EHP sequence

$$\cdots \longrightarrow \pi_{m+n}(S^n) \xrightarrow{E} \pi_{m+n+1}(S^{n+1}) \xrightarrow{H} \pi_{m+n+1}(S^{2n+1}) \xrightarrow{P} \pi_{m+n-1}(S^n) \longrightarrow \cdots$$

Given some $\beta_j \in \pi_{\phi(j)+2n+1}(S^{2n+1})$ for $\phi(j) < 2n$, one can ask about the Hopf invariant of its image under P, which vanishes when β_j is in the image of H. In most cases the answer is known and is due to Mahowald, [Mah67] and [Mah82]. The remaining cases have to do with θ_j. The answer that he had hoped for is the following, which can be found in [Mah67]. (To our knowledge, Mahowald never referred to this as the World Without End Hypothesis. We chose that term to emphasize its contrast with the Doomsday Hypothesis.)

World Without End Hypothesis (Mahowald 1967).

- *The Arf-Kervaire element* $\theta_j \in \pi^S_{2^{j+1}-2}$ *exists for all* $j > 0$.
- *It desuspends to* $S^{2^{j+1}-1-\phi(j)}$ *and its Hopf invariant is* β_j.
- *Let* $j, s > 0$ *and suppose that* $m = 2^{j+2}(s+1) - 4 - \phi(j)$ *and* $n = 2^{j+1}(s+1) - 2 - \phi(j)$. *Then* $P(\beta_j)$ *has Hopf invariant* θ_j.

This describes the systematic behavior in the EHP sequence of elements related to the image of J, and the θ_j are an essential part of the picture. Because of our theorem, *we now know that this hypothesis is incorrect*.

1.5. **Questions raised by our theorem.**

EHP sequence formulation. The World Without End Hypothesis was the nicest possible statement of its kind given all that was known prior to our theorem. Now we know it cannot be true since θ_j does not exist for $j \geq 7$. *This means the behavior of the indicated elements* $P(\beta_j)$ *for* $j \geq 7$ *is a mystery.*

Adams spectral sequence formulation. (See §3.7.) We now know that the h_j^2 for $j \geq 7$ are not permanent cycles, so they have to support nontrivial differentials. *We have no idea what their targets are.*

Manifold formulation. Here our result does *not* lead to any obvious new questions. It appears rather to be the final page in the story.

Our method of proof offers a new tool for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future.

2. OUR STRATEGY

2.1. **Ingredients of the proof.** Our proof has several ingredients.

- It uses methods of stable homotopy theory, which means it uses spectra instead of topological spaces. For more information about this see §4. Recall that a space X has a homotopy group $\pi_k(X)$ for each positive integer k. A spectrum X has an abelian homotopy group $\pi_k(X)$ *defined for every integer* k. For the sphere spectrum S^0, $\pi_k(S^0)$ is the stable k-stem homotopy group π_k^S. The hypothetical θ_j is an element of this group for $k = 2^{j+1} - 2$.
- It uses complex cobordism theory. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory. It includes some highly developed computational techniques that began with work by Milnor [Mil60], Novikov ([Nov60], [Nov62] and [Nov67]) and Quillen [Qui69] in the 60s. A pivotal tool in the subject is the theory of formal group laws. On this subject the definitive reference is Hazewinkel's book [Haz78]. A much briefer account covering the most relevant aspects of the subject can be found in [Rav86, Appendix 2].
- It also makes use of newer less familiar methods from equivariant stable homotopy theory. A helpful introduction to this subject is the paper of Greenlees-May [GM95]. This means there is a finite group G (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers $\mathbf{Z}$, *but by* $RO(G)$*, the orthogonal representation ring of* G. Our calculations make use of this richer structure.

2.2. **The spectrum Ω.** We will produce a map $S^0 \to \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) *Detection Theorem.* It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial. *This means that if θ_j exists, we will see its image in $\pi_*(\Omega)$.*
(ii) *Periodicity Theorem.* It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.
(iii) *Gap Theorem.* $\pi_k(\Omega) = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.

If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in $\pi_{254}(\Omega)$. On the other hand, (ii) and (iii) imply that $\pi_{254}(\Omega) = 0$, so θ_7 cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.

2.3. **How we construct Ω.** Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.

To construct it we start with the complex cobordism spectrum MU. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of C_2 defined by complex conjugation. The fixed point set of this action is the set of real points, known to topologists as MO, the unoriented cobordism spectrum. In this notation, U and O stand for the unitary and orthogonal groups.

To get a C_8-spectrum, we use the following general construction for getting from a space or spectrum X acted on by a group H to one acted on by a larger group G containing H as a subgroup. Let

$$Y = \mathrm{Map}_H(G, X),$$

the space (or spectrum) of H-equivariant maps from G to X. Here the action of H on G is by right multiplication, and the resulting object has an action of G by left multiplication. As a set, $Y = X^{|G/H|}$, the $|G/H|$-fold Cartesian power of X. A general element of G permutes these factors, each of which is left invariant by the subgroup H.

In particular we get a C_8-spectrum

$$MU^{(4)} = \mathrm{Map}_{C_2}(C_8, MU).$$

This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.

3. Some classical algebraic topology.

3.1. **Fibrations.** A map $p : E \to B$ is a *fibration* (sometimes called a Hurewicz fibration [Hur35][Hur36]) if the following commutuative diagram can always be completed:

(3)
$$\begin{tikzcd} x \arrow[d, mapsto] & X \arrow[r, "\tilde{f}_0"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ (x,0) & X \times I \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B, \end{tikzcd}$$

where I denotes the closed unit interval $[0, 1]$. In other words, given maps f and $\tilde{f}_0$ as shown, one can always find a map $\tilde{f}$ such that $\tilde{f}i = \tilde{f}_0$ and $p\tilde{f} = f$. This is called the *homotopy lifting property* because it says that if one end of the homotopy f can be lifted from B to E via $\tilde{f}_0$, then we can find a lifting $\tilde{f}$ of the entire homotopy.

Experience has shown that it often suffices to assume (3) holds only in the case where X is an n-dimensional disk D^n for any n. In this case p is called a *Serre fibration* [Ser51]. It is enough to establish the long exact sequence of homotopy groups (4).

One example of a fibration is a *fiber bundle* [Ste99], a map for which

(i) the preimage $p^{-1}(b)$ for each $b \in B$ is homeomorphic to a space F called the *fiber of* p and
(ii) each $b \in B$ has a neighborhood U such that $p^{-1}(U)$ is homeomorphic to $U \times F$ in such a way that the composition of p with the homeomorphism is projection onto U.

In this case there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots \tag{4}$$

where $i : F \to E$ is the inclusion of the fiber and ∂ is a certain boundary homomorphism. More details can be found in any textbook on algebraic topology, such as Hatcher [Hat02], May [May99] or Gray [Gra75].

There is a way to replace any map $p : E \to B$ by a fibration between homotopy equivalent spaces and construct a homotopy theoretic fiber. First we replace B by the *mapping cylinder* $\tilde{B} = M_p$, which is the quotient of the disjoint union

$$(E \times I) \bigcup B$$

obtained by identifying $(e, 1) \in E \times I$ with $p(e) \in B$. For technical reasons it is convenient to assume that E and B have base points e_0 and b_0 with $p(e_0) = b_0$, and to collapse $\{e_0\} \times I$ to a point which becomes the base point in $\tilde{B}$. The inclusion of B into $\tilde{B}$ is a homotopy equivalence.

Next we replace E by the space $\tilde{E}$ of paths (continuous maps from the unit interval I) in $\tilde{B}$ that begin in $E \times \{0\}$. The inclusion of E into $\tilde{E}$ via paths of the form $t \mapsto (e, t)$ is again a homotopy equivalence. Finally we define $\tilde{p} : \tilde{E} \to \tilde{B}$ by sending a path to its endpoint. Then the diagram

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[d, "p"] & \tilde{E} \arrow[d, "\tilde{p}"] \\ B \arrow[r, "\simeq"] & \tilde{B} \end{tikzcd}$$

commutes, the horizontal maps are homotopy equivalences, and $\tilde{p}$ is a fibration.

Now let $\tilde{F} \subset \tilde{E}$ be the space of paths starting in $E \times \{0\}$ and ending at the base point; we denote the inclusion map by $\tilde{i}$. This is the *homotopy theoretic fiber of* p and there is a long exact sequence similar to (4),

$$\cdots \longrightarrow \pi_n(\tilde{F}) \xrightarrow{\tilde{i}_*} \pi_n(\tilde{E}) \xrightarrow{\tilde{p}_*} \pi_n(\tilde{B}) \xrightarrow{\tilde{\partial}} \pi_{n-1}(\tilde{F}) \longrightarrow \cdots$$

A *fiber sequence* (such as (2)) is any composite

$$W \xrightarrow{f} X \xrightarrow{g} Y$$

where f is equivalent to the inclusion of the homotopy theoretic fiber of g.

When the map p is inclusion of the base point b_0 of B, we find that

Sidebar 2 CW complexes

A *CW-complex* (first introduced by Henry Whitehead in [Whi49]) X is a space constructed as a union of skeleta X^n defined as follows. X^0 is a discrete set. X^n is obtained from X^{n-1} by attaching a collection of n-disks $\{D_\alpha^n\}$ by identifying points on their boundaries with points in X^{n-1} using maps $f_\alpha : \partial D_\alpha^n \to X^{n-1}$ called *attaching maps*. X^n is the cofiber (see §3.2) of the map to X^{n-1} from the disjoint union of the $(n-1)$-spheres ∂D_α^n. The n-disks are called n*-cells*. A CW-complex is said to be *finite* if its has only a finite number of cells altogether, and to have *finite type* if it has only finitely many in each dimension.

This collection of spaces is general enough to include all manifolds and all real and complex algebraic varieties. Indeed most homotopy theorists never have to deal with spaces that are not at least homotopy equivalent to CW-complexes. They have the following convenient properties:

- The product of two CW-complexes is a CW-complex.
- A map $f : X \to Y$ of CW-complexes is a homotopy equivalence (which means there is a map $g : Y \to X$ such that fg and gf are homotopic to the identities maps on Y and X) iff it is a weak homotopy equivalence, meaning that it induces an isomorphism in all homotopy groups.
- Any such map is homotopic to one that sends X^n to Y^n for all n.
- The space of maps $X \to Y$, while not a CW-complex itself in general, always has the homotopy type of one by a theorem of Milnor [Mil59]. The same holds if one requires that certain subcomplexes of X map to certain subcomplexes of Y. For example, ΩS^{n+1} is equivalent to a CW-complex with a single cell in each dimension divisible by n, and no others.
- $H_*(X)$ can be computed in terms of a *cellular chain complex* $C_*(X)$ in which the nth chain group $C_n(X)$ is free abelian on the n-cells of X.

- $\tilde{B} = B$,
- $\tilde{E} = PB$, the space of paths in B starting at b_0, which is contractible, and
- $\tilde{F} = \Omega B$, the space of closed path starting and ending at b_0, which is called the *loop space of* B.

This is called the *path fibration of* B. The long exact sequence in this case gives an isomorphism

$$\pi_n(\Omega B) \approx \pi_{n+1}(B).$$

When we apply this constuction to the map $\tilde{d}$, we find that its homotopy theoretic fiber is equivalent to ΩB. The homotopy theoretic fiber of the map $\Omega B \to \tilde{F}$ is equivalent to ΩE, and so on.

3.2. **Cofibrations.** A *cofibration* is a map $i : A \to X$ satisfying a diagram dual to (3), namely

$$\begin{tikzcd}
\omega(0) & Y & X \arrow[l, "\tilde{f}_0"'] \arrow[dl, dashed, "\tilde{f}"'] \\
\omega \arrow[u, mapsto] & Y^I \arrow[u, "p_0"] & A \arrow[l, "f"] \arrow[u, "i"']
\end{tikzcd} \tag{5}$$

Here Y^I denotes the space of maps $\omega : I \to Y$, i.e., all paths in Y. The map $f : A \to Y^I$ is formally equivalent to a homotopy $A \times I \to Y$. $\tilde{f}_0$ is an extension of one end of it to X and the hypothetical $\tilde{f}$ is an extension of all of it to X, so this is called the *homotopy extension property*.

The inclusion i of a subcomplex A into a CW-complex X (see Sidebar 2) is always a cofibration. It has a *cofiber* C_i (also know as the *mapping cone*), which, like the mapping cylinder above, is the quotient of the disjoint union

$$(A \times I) \bigcup X$$

obtained by identifying $(a, 1) \in A \times I$ with $i(a) \in X$ (as in the mapping cylinder) and by collapsing all of $A \times \{0\}$ to a point. As before, if a base point is desired, it can be obtained by collapsing the path along the base point of A to a point. We denote the inclusion map $X \to C_i$ by j .

Then the reduced homology $\overline{H}_*(C_i)$ of the cofiber is the same as the relative homology $H_*(X, A)$. There is a long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(C_i) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots \tag{6}$$

The procedure for replacing an arbitrary map $i : A \to X$ by a homotopy equivalent cofibration is easier here than it was for fibrations above. All we have to do is replace X by the mapping cylinder $\tilde{X} = M_i$. This makes the evident inclusion $\tilde{i} : A \to \tilde{X}$ a cofibration, and the mapping cone C_i is its cofiber. The simplicity of this construction renders the term "homotopy theoretic cofiber" unnecessary. C_i is called the simply the *cofiber of the map* i. The cofiber of the map $j : M_i \to C_i$ is easily seen to be the suspension ΣA.

A *cofiber sequence* is any composite

$$W \xrightarrow{f} X \xrightarrow{g} Y$$

where g is equivalent to the map j to C_f. One has similar notions in the category of spectra, to be defined in §4. In that world fiber sequence and cofiber sequences are the same: W is equivalent to the fiber of g iff Y is equivalent to the cofiber of f. This means we have long exact sequences in both homotopy and homology.

3.3. **Eilenberg-Mac Lane spaces and cohomology operations.**

Theorem 1. Eilenberg-Mac Lane spaces and cohomology. *For any CW-complex X, positive integer n and discrete abelian group A, there is a natural isomorphism between the cohomology group $H^n(X; A)$ and the group of homotopy classes of maps $[X, K(A, n)]$,*

$$H^n(X; A) \approx [X, K(A, n)] \tag{7}$$

where $K(A, n)$ is the Eilenberg-Mac Lane space constructed in Sidebar 3.

The group structure on the set of homotopy classes arises from a map $K(A,n) \times K(A,n) \to K(A,n)$ with suitable properties. This isomorphism holds if X is also an Eilenberg-Mac Lane space. When it is $K(A,n)$, we get the identity map on the right corresponds to a canonical element ι_n on the left called the *fundamental class*.

Let $\theta \in H^{n+k}(K(A,n);A')$ and $x \in H^n(X;A)$ for $k \geq 0$. (The case $k < 0$ is uninteresting because the cohomology group is trivial.) Using (7), these correspond to maps

$$X \xrightarrow{x} K(A,n) \xrightarrow{\theta} K(A',n+k),$$

so the composite θx corresponds to an element in $H^{n+k}(X;A')$. Hence θ gives us a natural transformation from the functor $H^n(\cdot;A)$ to $H^{n+k}(\cdot;A')$ called a *cohomology operation*. It may or may not be a group homomorphism. When it is, we say it is *additive*.

Now assume that all cohomology groups have coefficients in $\mathbf{Z}/2$ and let $K_n = K(\mathbf{Z}/2,n)$. Let $s_n : \Sigma K_n \to K_{n+1}$ be adjoint to the equivalence $K_n \to \Omega K_{n+1}$. It induces a homomorphism

$$H^{n+k+1}(K_{n+1}) \xrightarrow{s_n^*} H^{n+k+1}(\Sigma K_n) = H^{n+k}(K_n) \ni \theta$$

It is known that θ is additive iff it is in the image of s_n^*. In that case it is also in the image of a similar map from $H^{n+k+t}(K_{n+t})$ for any $t > 0$, and there is a way to choose these preimages canonically. Such an operation θ is said to be *stable*.

3.4. **The Steenrod algebra.** Here is an important example of a mod 2 stabel operation. Let $\theta = \iota_n^2 \in H^{2n}(K_n)$. The corresponding cohomology operation sends x to x^2, which is additive since we are working mod 2. Since it is additive, it is stable and we have similar operations

$$H^{n+t}(X) \xrightarrow{Sq^n} H^{2n+t}(X)$$

for all $t \geq 0$. This is called the *nth Steenrod squaring operation* [Ste62]. These operations have been studied extensively and it is known that any mod 2 stable operation can be expressed in terms of them. They have the following properties:

Theorem 2. Properties of mod 2 Steenrod operations

(i) Sq^0 *is the identity map.*

(ii) CARTAN FORMULA.

$$Sq^n(xy) = \sum_{0 \leq i \leq n} Sq^i(x) Sq^{n-i}(y).$$

(iii) ADEM RELATION. *For* $0 < a < 2b$,

$$Sq^a Sq^b = \prod_{j=0}^{j=[a/2]} \binom{b-i-2j}{a-2j} Sq^{a+b-j} Sq^j.$$

(iv) UNSTABLE CONDITION. $Sq^n(x) = x^2$ *for* $x \in H^n$ *and* $Sq^n(x) = 0$ *for* $x \in H^i$ *with* $i < n$.

For $n > 0$, the Adem relation gives $Sq^{2n+1} = Sq^1 Sq^{2n}$ and

$$Sq^{(2n+1)2^{k+1}} = Sq^{2^{k+1}} Sq^{2^{k+2}n} + Sq^{2^{k+2}n+2^k} Sq^{2^k} \qquad \text{for } k \geq 0,$$

so Sq^i is decomposable unless i is a power of 2.

A monomial $Sq^I = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_m}$ is *admissible* if

$$i_1 \geq 2i_2,\ i_2 \geq 2i_3, \ldots,\ i_{m-1} \geq 2i_m. \tag{8}$$

Repeated use of the Adem relation will convert any monomial to a sum of admissible ones.

The *mod 2 Steenrod algebra* $\mathcal{A}$ is the algebra of all mod 2 stable cohomology operations. It is the associative algebra generated by the Sq^n for $n > 0$ and subject to the Adem relation. The admissible monomials form a basis for it (the *Adem basis*) and it is generated as an algebra by the elements Sq^{2^j} for $j \geq 0$.

For any space X, $H^*(X)$ is an $\mathcal{A}$-module subject to the Cartan formula and the unstable condition. For a spectrum X, these restrictions are vacuous, so we just get an $\mathcal{A}$-module.

We will now describe $H^*(K_n)$ for all n. The *excess* of an admissible monomial Sq^I as in (8) is

$$e(I) = (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + (i_{m-1} - 2i_m) = i_1 - i_2 - i_3 - \cdots - i_m.$$

Theorem 3. Mod 2 cohomology of mod 2 Eilenberg-Mac Lane spaces.

$$H^*(K_n) = \mathbf{Z}/2[Sq^I\iota_n \colon e(I) < n,\ I \textit{ admissible}].$$

The only admissible monomial with excess 0 is 1, so the theorem says

$$H^*(K_1) = \mathbf{Z}/2[\iota_1],$$

which is consistent with the fact that $K_1 = \mathbf{R}P^\infty$.

The set of admissible monomials with excess 1 is

$$\left\{ Sq^1,\ Sq^2Sq^1,\ Sq^4Sq^2Sq^1,\ Sq^8Sq^4Sq^2Sq^1,\ \ldots \right\},$$

which leads to

$$H^*(K_2) = \mathbf{Z}/2[x_2,\ x_3,\ \ldots, x_{1+2^k}, \ldots]$$

where $x_2 = \iota_2$ and $x_{1+2^{k+1}} = Sq^{2^k} x_{1+2^k}$.

This entire discussion has an odd primary analog, but we do not need it here.

3.5. Milnor's formulation.

The results of this section are due to Milnor [Mil58].

The Cartan formula (Theorem 2 (ii)) leads to a coproduct

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \otimes \mathcal{A} \\ Sq^n & \longmapsto & \sum_{0 \leq i \leq n} Sq^i \otimes Sq^{n-i} \end{array}$$

which is an algebra map and is cocommutative. This leads to a commutative algebra structure on the dual

$$\mathcal{A}_* = \mathrm{Hom}_{\mathbf{Z}/2}(\mathcal{A}, \mathbf{Z}/2).$$

The noncommutative product on $\mathcal{A}$, which is a map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, translates to a noncommutative coproduct on $\mathcal{A}_*$ which we denote by Δ.

Theorem 4. The structure of the dual Steenrod algebra. *As an algebra,*

$$\mathcal{A}_* = \mathbf{Z}/2[\xi_1, \xi_2, \ldots]$$

where the dimension of ξ_j is $2^j - 1$. The coproduct is

$$\Delta(\xi_n) = \sum_{0 \leq i \leq n} \xi_{n-i}^{2^i} \otimes \xi_i \qquad \textit{where } \xi_0 = 1.$$

This coproduct formula is equivalent to (and much easier to remember than) the Adem relation, but proving this explicitly is difficult.

3.6. **Serre's method of computing homotopy groups.** Let X be a $(n-1)$-connected space for $n > 1$ with $\pi_n(X)$, nontrivial, and let $L_0 = K(\pi_n(X), n)$. The Hurewicz theorem [Hur35] [Hur36] says that $\pi_n(X)$ is isomorphic to $H_n(X; \mathbf{Z})$, and Theorem 1 then implies there is a map $f_0 : X \to L_0$ inducing an isomorphism in π_n. Let X_1 be its fiber as explained in 3.1. The the long exact sequence of homotopy groups (4) implies that

$$\pi_i(X_1) = \begin{cases} 0 & \text{for } i \leq n \\ \pi_i(X) & \text{otherwise} \end{cases}$$

The Serre spectral sequence of [Ser51] is a device for computing the homology of one of the spaces in a fiber sequence in terms of the other two. In theory we can use it to find the homology of X_1 and hence its first nontrivial homotopy group, which lies somewhere above dimension n. Then we can treat X_1 the same way we treated X and find a map $f_1 : X_1 \to L_1$ where L_1 is the appropriate Eilenberg-Mac Lane space. Then we could use the Serre spectral sequence to find the homology of the fiber X_2, and so on. Continuing in this manner we could get a diagram

(9)
$$\begin{tikzcd} X \arrow[d, "f_0"] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"] & X_3 \arrow[l, "g_2"'] \arrow[d, "f_3"] & \cdots \arrow[l, "g_3"'] \\ L_0 & L_1 & L_2 & L_3 & \end{tikzcd}$$

where

(i) each L_i is an Eilenberg-Mac Lane space,
(ii) f_i induces an isomorphism in the first nontrivial homotopy group of X_i and
(ii) X_{i+1} is the fiber of f_i.

This method requires knowing the homology of all spaces in sight, at least through the range of dimensions in which one hopes to compute. It was used with brilliant effect by Serre in [Ser51] to calculate many previously unknown homotopy groups of spheres, but a few years later Adams found a better method [Ada58] for doing this.

Before turning to it, we need to make one observation about the Serre spectral sequence. In each of the fiber sequences in (9), below dimension $2n$ it simplifies to a long exact sequence

(10)
$$\cdots \longrightarrow H_i(X_{s+1}) \xrightarrow{(g_{s+1})_*} H_i(X_s) \xrightarrow{f_{s*}} H_i(L_s) \xrightarrow{\partial} H_{i-1}(X_{s+1}) \longrightarrow \cdots$$

3.7. **The Adams spectral sequence.** In this section we return to our convention that all cohomology groups are reduced and have coefficients in $\mathbf{Z}/2$.

We modify Serre's diagram (9) below dimension $2n$ in the following ways:

(i) Each L_s is a product of mod 2 Eilenberg-Mac Lane spaces K_m, with possibly different values of m (all at least n) for the various factors. Theorem 3 implies that in our range $H^*(K_m) = \Sigma^m \mathcal{A}$, the mth suspension of the Steenrod algebra.
(ii) L_s and f_s are chosen so that the induced map in cohomology is onto. This can always be done, but not uniquely. Choose a set of elements in $H^*(X_s)$ which generate it as an $\mathcal{A}$-module. Each one corresponds to a map to some K_m, so collectively they give a map to the product of K_ms, which is L_s.

A diagram of the form (9) having these two properties is called a *mod 2 Adams resolution for X*. When X is a CW-complex of finite type, it leads to a method for computing $\pi_*(X)$ modulo odd primary torsion below dimension $2n$.

The requirement on $H^*(f_s)$ means that $H^*(g_{s+1})$ is trivial, and the cohomological analog of (10) reduces to a short exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow H^*(\Sigma X_{s+1}) \longrightarrow H^*(L_s) \xrightarrow{f_s^*} H^*(X_s) \longrightarrow 0$$

where $H^*(L_s)$ is free. These can be spliced together for various s, giving us a long exact sequence

$$0 \longleftarrow H^*(X) \longleftarrow H^*(L_0) \longleftarrow H^*(\Sigma L_1) \longleftarrow H^*(\Sigma^2 L_2) \longleftarrow \cdots \tag{11}$$

This is a free $\mathcal{A}$-resolution of $H^*(X)$. We can recover $\pi_*(L_s)$ from its cohomology by applying the functor $\mathrm{Hom}_{\mathcal{A}}(\cdot, \mathbf{Z}/2)$. Doing this to (11) gives us a cochain complex

$$\pi_*(L_0) \longrightarrow \pi_*(\Sigma L_1) \longrightarrow \pi_*(\Sigma^2 L_2) \longrightarrow \cdots \tag{12}$$

Theorem 5. The Adams spectral sequence. *For an $(n-1)$-connected space X, in dimensons less than $2n$ the cohomology of (12) is independent of the choices of the L_s, and is by definition $\mathrm{Ext}_{\mathcal{A}}\left(H^*(X), \mathbf{Z}/2\right)$. This is the E_2-term of the Adams spectral sequence, which converges to $\pi_*(X)$ modulo odd torsion. More specifically,*

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}\left(H^*(X), \mathbf{Z}/2\right),$$

$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$, and $E_\infty^{s,t}$ is a subquotient of $\pi_{t-s}(X)$.

This theorem is crying out to be reformulated in term of spectra (to be defined in the next section), which is how it is usually stated. Then the caveats about n can be eliminated, provided X is connective and of finite type.

The example of interest to us is the sphere spectrum, for which $E_2^{*,*} = \mathrm{Ext}_{\mathcal{A}}^{*,*}\left(\mathbf{Z}/2, \mathbf{Z}/2\right)$. This group is difficult to compute but has been studied extensively, and we will abbreviate it simply by $\mathrm{Ext}^{*,*}$. Here is the first stage of the free resolution of (11) in this case.

$$\begin{array}{ccccccc} 0 \longleftarrow & \mathbf{Z}/2 & \xleftarrow{\epsilon} & \mathcal{A} & \longleftarrow & \mathcal{A}\{x_j : j \geq 0\} & \longleftarrow \cdots \\ & & & Sq^{2^j} & \longleftarrow\!\!\shortmid & x_j & \end{array}$$

We know that $\ker \epsilon$ is generated as an $\mathcal{A}$-module by $\{x_j : j \geq 0\}$ because these elements generate $\mathcal{A}$ as an algebra.

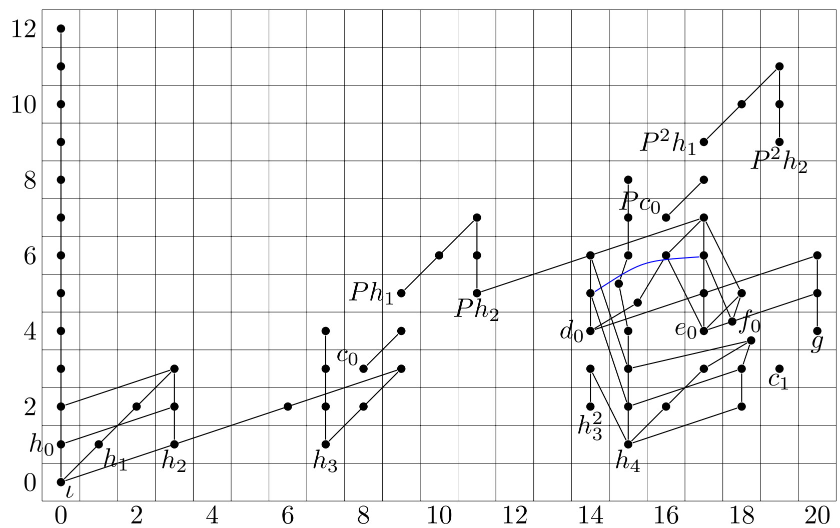
This leads to elements $h_j \in \mathrm{Ext}^{1,2^j}$, and they are known to form a basis of Ext^1. The group Ext has a ring structure, and the following set forms a basis of Ext^2:

$$\{h_j h_k : 0 \leq j \leq k,\, k \neq j+1\}.$$

The following facts are known about these elements.

- The h_j for $0 \leq j \leq 3$ are nontrivial permanent cycles, meaning that they detects elements in π_*^S. h_0 detects the degree 2 map in π_0^S, and the other three detect maps constructed by Hopf in [Hop30] and [Hop35].
- Adams [Ada60] showed that the h_j for $j > 3$ do not detect homotopy elements. Instead they support nontrivial differentials, namely $d_2(h_j) = h_0 h_{j-1}^2$.
- Browder's theorem [Bro69] says there is a framed manifold with non trivial Kervaire invariant in dimension $2^{j+1} - 2$ iff h_j^2 is a permanent cycle, in which case it detects the corresponding homotopy element, θ_j.

Here is a chart showing $\mathrm{Ext}^{s,t}$ for $t-s \leq 20$.



Here lines going up and to the right indicate multiplication by h_0, h_1 and h_2, and lines going up and to the left lines indicate differentials. For more information we refer the reader to [Rav86].

4. Spectra and equivariant spectra

In this section we will define ordinary and equivariant spectra, the objects of study in stable homotopy theory and equivariant stable homotopy theory.

4.1. **An informal definition of spectra.** Informally, a *prespectrum* D is a collection of pointed spaces (spaces equipped with base points that are preserved by all maps in sight) $\{D_n : n \gg 0\}$ with structure maps $\Sigma D_n \to D_{n+1}$. Here ΣX denotes the suspension or double cone on X, wth the cone line through the basepoint $x_0 \in X$ collapsed to be the base point of ΣX. Then we can define

$$\pi_i(D) = \varinjlim \pi_{n+i}(D_n) \qquad \text{and} \qquad H_i(D) = \varinjlim H_{n+i}(D_n).$$

A map $\Sigma X \to Y$ is equivalent (adjoint) to a map $X \to \Omega Y$, where ΩY is the loop space of Y, the space of base point preserving map to Y from the circle S^1.

Thus the adjoint of the structure map is a map $D_n \to \Omega D_{n+1}$. A *spectrum* is a prespectrum for which this map is a homeomorphism for each n. We can always get a spectrum $E = \{E_n : n \in \mathbf{Z}\}$ from the prespectrum D by defining

$$E_n = \varinjlim_k \Omega^k D_{n+k}$$

This makes $E_n = \Omega E_{n+1}$ and infinite loop space. The evident map $D \to E$ does not alter homotopy or homology and is thus a weak homotopy equivalence.

A spectrum E can be suspended or desuspended by defining

$$(\Sigma^j E)_n = E_{j+n} \qquad \text{for any integer } j,$$

and we have

$$\pi_k(\Sigma^j E) = \pi_{n-j}(E) \qquad \text{and} \qquad H_k(\Sigma^j E) = H_{n-j}(E).$$

We can make similar statements about looping, defining

$$(\Omega^j E)_n = \Omega^j E_n = E_{n-j},$$

so $\Omega^j E = \Sigma^{-j} E$. Hence a spectrum can be desuspended or delooped any number of times.

Here are two examples:

(i) For a given pointed space X, let $D_n = \Sigma^n X$, the nth iterated suspension of X. Then D is the *suspension prespectrum of* X, denoted by $\Sigma^\infty X$ or (abusively but commonly) simply by X. The homology of $\Sigma^\infty X$ is the reduced homology the space X. Converting to a spectrum as described above does not change its homotopy or homology. When the space X is S^0, we get the sphere spectrum, whose kth homotopy group is π_k^S.

(ii) For an abelian group A, let $K(A, n)$ be the Eilenberg-Mac Lane space (see Sidebar 3) characterized (up to homotopy equivalence) by

$$\pi_k(K(A,n)) = \begin{cases} A & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Then $K(A, n)$ is equivalent to $\Omega K(A, n+1)$. We define a spectrum D by $D_n = K(A, n)$ with structure map $\Sigma K(A, n) \to K(A, n+1)$ being the adjoint of this equivalence. In this case E_n is equivalent to D_n. The resulting spectrum is denoted by HA, the Eilenberg-Mac Lane spectrum for A.

4.2. **Orthogonal spectra.** Let $\mathcal{T}$ denote the category of compactly generated [ML71, VII.8] weak Hausdorff [McC69, §2] pointed topological spaces. Compactly generated means that a subset is closed iff its intersection with every compact subspace is closed. Weak Hausdorff means that the continuous image of every compact space is closed. These are technical conditions designed to keep us out of trouble, as explained in the references cited.

Following [May80, §5] we define an *orthogonal spectrum* to be a functor to this category from an indexing category $\mathscr{I}$. Its objects are are finite dimensional real inner product spaces V. When $V \subset W$, we will denote the orthogonal compliment of V in W by $W - V$.

The category $\mathscr{I}$ is *enriched over* $\mathcal{T}$ [Kel82] , which means that the "set" $\mathscr{I}(V, W)$ of morphisms from V to W is actually a pointed topological space, to be defined shortly. $\mathcal{T}$ is enriched over itself; the set of continuous pointed maps $X \to Y$ can be given the compact-open topology with the constant map as base point. A functor of enriched categories $D : \mathscr{I} \to \mathcal{T}$ is required to induce a continuous pointed map from $\mathscr{I}(V, W) \to \mathcal{T}(D_V, D_W)$ for each V and W in $\mathscr{I}$.

To define $\mathscr{I}(V, W)$, let $O(V, W)$ be the (possibly empty) Stiefel manifold of orthogonal embeddings $f : V \to W$. The space

$$\{(f, w) \in O(V, W) \times W : w \in W - f(V)\}$$

is a vector bundle over $O(V, W)$, and $\mathscr{I}(V, W)$ is defined to be its one point compactification or Thom space. As a set it is

$$\mathscr{I}(V, W) = \bigvee_{f \in O(V,W)} S^{W - f(V)} \tag{13}$$

where S^V denotes the one point compactification of V.

In particular, $\mathscr{I}(V,V) = O(V)_+$, the orthogonal group of V with a disjoint base point. Hence the functor D gives a base point preserving action of $O(V)$ on the pointed space D_V. If V and W have the same dimension, then a choice of isomorphism between them leads to identifications of $\mathscr{I}(V,W)$ and $\mathscr{I}(W,V)$ with $O(V)_+$ and to homeomorphisms between D_V and D_W. Hence the topology of the pointed space D_V depends only on the dimension of V.

If $V \subset W$ is a proper subspace, then $\mathscr{I}(V,W) \supset S^{W-V}$ by (13), and we get a structure map $S^{W-V} \wedge D_V \to D_W$ as in 4.1.

4.3. **Equivariant orthogonal spectra.** Here we define equivariant orthogonal spectra following [MM02].

For a group G, let $\mathcal{T}_G$ denote the category of compactly generated weak Hausdorff pointed topological G-*spaces*, i.e., spaces with an action of G by base point preserving maps. The base point is always fixed by G. When we encounter a G-space X without a fixed point, we adjoin a disjoint base point and denote it by X_+. The groups we will be concerned with in this paper are finite cyclic 2-groups C_{2^n}.

Like $\mathcal{T}$, $\mathcal{T}_G$ is enriched over itself. The space $\mathcal{T}_G(X,Y) = \mathrm{Map}_*(X,Y)$ of *all* pointed maps $X \to Y$ (not just the equivariant ones) is itself a pointed G-space, with the constant map as base point and the action of a group element $\gamma \in G$ on a map $f : X \to Y$ is given by $(\gamma f)(x) = \gamma f(\gamma^{-1}x)$. The fixed point set $\mathcal{T}_G(X,Y)^G$ is the space of equivariant maps.

One could define another category with the same objects as $\mathcal{T}_G$ but enriched instead over $\mathcal{T}$ in which the morphism space is $\mathcal{T}_G(X,Y)^G$. Both enriched categories have underlying ordinary categories in which the spaces of morphisms are replaced by the corresponding sets.

We will use an indexing category $\mathscr{I}^G$ whose objects are finite dimensional orthogonal representations V of G. There is an obvious forgetful functor $\mathscr{I}^G \to \mathscr{I}$. We define $\mathscr{I}^G(V,W)$ to be the space $\mathscr{I}(V,W)$ equipped with the evident G-action. Its fixed point space is

$$\mathscr{I}^G(V,W)^G = \mathscr{I}(V^G, W^G) \wedge O(V^\perp, W^\perp)^G_+,$$

where $V^\perp$ denotes the orthogonal compliment of the invariant subspace V^G of V.

We define an *equivariant orthogonal G-spectrum* to be a functor

$$E : \mathscr{I}^G \to \mathcal{T}_G$$

enriched over G-spaces and equivariant maps. This means that for each representation V we get a pointed G-space E_V, and for each V and W we get a continuous pointed equivariant map

$$\mathscr{I}^G(V,W) \to \mathcal{T}_G(E_V, E_W).$$

As before we have structure maps

$$S^V \wedge E_W \to E_{V+W}$$

which are equivariant. Here S^V is the one point compactification of the representation V, which has its own action of G determined by the representation.

We denote the category of equivariant orthogonal G-spectra by $\mathcal{S}^G$. It is an enriched functor category [ML71, II.4] in which objects are continuous functors $\mathscr{I}^G \to \mathcal{T}_G$ as described above, and morphisms are natural transformations. Thus a morphism of equivariant orthogonal G-spectra $g : D \to E$ consists of continuous equivariant maps $g_V : D_V \to E_V$

Sidebar 3 Classifying spaces

The join $X * Y$ of topological spaces X and Y is the topological quotient of $X \times I \times Y$ obtained by identifying $(x, 0, y)$ with $(x', 0, y)$ and $(x, 1, y)$ with $(x, 1, y')$ for $x, x' \in X$ and $y, y' \in Y$. (meaning that its homotopy groups vanish below dimension m) and Y is n-connected, then $X * Y$ is $(m+n)$-connected, e.g. $S^m * S^n = S^{m+n+1}$. This construction can be iterated, namely it is possible to define the n-fold join

$$X_0 * X_1 * \cdots * X_n$$

as a quotient of the space

$$\left\{((t_0, \ldots, t_n), (x_0, \ldots, x_n)) \in I^{n+1} \times \prod_{0 \le i \le n} X_i \colon \sum_{0 \le i \le n} t_i = 1\right\}.$$

When $t_i = 0$, the coordinate x_i is irrelevant. The set of points with $t_n = 0$ is the $(n-1)$-fold join $X_0 * \cdots * X_{n-1}$.

For a topological group G, let $E_n G$ be the n-fold join of G with itself. It is $(n-1)$-connected, even if G itself is not path connected. It is also a G-space, with G acting by left multiplication on the coordinates in G. This action is free; no point is left fixed by any nontrivial element of G. Using the inclusion maps $E_n G \to E_{n+1} G$ (which commute with the G-action) we can let n go to ∞ and obtain a contractible free G-space EG. Its orbit space is denoted by BG and is called the *classifying space of* G. This construction is due to Milnor [Mil56]. It has the following properties.

(i) BG is path connected and G is homotopy equivalent to ΩBG, which means that $\pi_i(G) = \pi_{i+1}(BG)$. If G is discrete, then

$$\pi_i(BG) = \begin{cases} G & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) BG is functorial in G, meaning the a group homomorphism $\phi : G \to H$ induces a map $B\phi : BG \to BH$. If A is a abelian group, the multiplication map $A \times A \to A$ is a homomorphism leading to a map $BA \times BA \to BA$. It is known that this makes BA itself into a topological abelian group, so the classifying space construction can be iterated. If A is discrete, then

$$\pi_i(B^n A) = \begin{cases} A & \text{for } i = n \\ 0 & \text{otherwise,} \end{cases}$$

so $B^n A$ is the Eilenberg-Mac Lane space $K(A, n)$.

(iii) Let G be the n-dimensional orthogonal group $O(n)$ or the n-dimensional unitary group $U(n)$. There is an n-dimensional real [complex] vector bundle $\xi_{\mathbf{R}}^n$ over $BO(n)$ [$\xi_{\mathbf{C}}^n$ over $BU(n)$] with the following universal property: any such bundle over a paracompact space X is induced by a unique (up to homotopy) from X to the classifying space, and the bundles over X induced by two such maps are isomorphic iff the maps are homotopic. Details can be found in [MS74].

for each $V \in \mathscr{I}^G$ such that for each $V, W \in \mathscr{I}^G$ the following diagram commutes

$$\begin{tikzcd}
\mathscr{I}^G(V, W) \arrow[r, "D"] \arrow[d, "E"'] & \mathcal{T}_G(D_V, D_W) \arrow[d, "\mathcal{T}_G(D_V, g_W)"] \\
\mathcal{T}_G(E_V, E_W) \arrow[r, "\mathcal{T}_G(g_V, E_W)"] & \mathcal{T}_G(D_V, E_W).
\end{tikzcd}$$

Here the vertical and horizontal arrows pointing toward $\mathcal{T}_G(D_V, E_W)$ are respectively composition with g_W and precomposition with g_V.

For each virtual representation $U - W$ there is a spectrum S^{U-W} defined by

$$(S^{U-W})_V = \mathcal{T}(S^W, S^{U+V}).$$

The group of homotopy classes of equivariant maps $\pi^G_{V-W}(X) = [S^{V-W}, X]$ depends only on the isomorphism class of $V - W$, which is an element in the orthogonal representation ring $RO(G)$. *Thus we get homotopy groups graded over $RO(G)$ rather than the integers.* These groups are collectively denoted by $\pi^G_\star(X)$, with the five pointed star $\star$ in place of the usual six pointed asterisk $*$.

For any subgroup $H \subset G$, there is a restriction functor $i^G_H : \mathcal{S}^G \to \mathcal{S}^H$, so we also have homotopy groups $\pi^H_\star(X)$ graded over $RO(H)$. In the case of the trivial subgroup we denote this by $\pi^u_*(X)$, the homotopy of the underlying (nonequivariant) spectrum of X. For each subgroup H, when the subscript is an ordinary integer (meaning a trivial representation of H), we are looking at H-equivariant maps from spheres fixed by H, so we have

$$\pi^H_*(X) = \pi_*(X^H)$$

where X^H is the fixed point spectrum of H.

References

[Ada58] J. F. Adams. On the structure and applications of the Steenrod algebra. *Comment. Math. Helv.*, 32:180–214, 1958.

[Ada60] J. F. Adams. On the non-existence of elements of Hopf invariant one. *Ann. of Math. (2)*, 72:20–104, 1960.

[Ada62] J. F. Adams. Vector fields on spheres. *Ann. of Math. (2)*, 75:603–632, 1962.

[Ada66] J. F. Adams. On the groups $J(X)$. IV. *Topology*, 5:21–71, 1966.

[Arf41] Cahit Arf. Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. I. *J. Reine Angew. Math.*, 183:148–167, 1941.

[BJM84] M. G. Barratt, J. D. S. Jones, and M. E. Mahowald. Relations amongst Toda brackets and the Kervaire invariant in dimension 62. *J. London Math. Soc. (2)*, 30(3):533–550, 1984.

[BMT70] M. G. Barratt, M. E. Mahowald, and M. C. Tangora. Some differentials in the Adams spectral sequence. II. *Topology*, 9:309–316, 1970.

[Bot59] R. Bott. The stable homotopy of the classical groups. *Annals of Mathematics*, 70:313–337, 1959.

[BP66] Edgar H. Brown, Jr. and Franklin P. Peterson. The Kervaire invariant of $(8k+2)$-manifolds. *Amer. J. Math.*, 88:815–826, 1966.

[Bro69] William Browder. The Kervaire invariant of framed manifolds and its generalization. *Ann. of Math. (2)*, 90:157–186, 1969.

[Fre38] H. Freudenthal. Über die Klassen der Sphärenabbildungen I. Grosse Dimnesionionen . *Compositio Math.*, 5:299–314, 1938.

[GM95] J. P. C. Greenlees and J. P. May. Equivariant stable homotopy theory. In *Handbook of algebraic topology*, pages 277–323. North-Holland, Amsterdam, 1995.

[Gra75] Brayton Gray. *Homotopy theory*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. An introduction to algebraic topology, Pure and Applied Mathematics, Vol. 64.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Haz78] Michiel Hazewinkel. *Formal groups and applications*, volume 78 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[Hop30] H. Hopf. Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. *Mathematische Annalen*, 104:639–665, 1930.

[Hop35] H. Hopf. Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension. *Fundamenta Mathematica*, 25:427–440, 1935.

[Hur35] W. Hurewicz. Beiträge zur Topologie Deformationen I–II. *Nederl. Akad. Wetensch. Proc., Series A*, 38:112–119, 521–528, 1935.

[Hur36] W. Hurewicz. Beiträge zur Topologie Deformationen III–IV. *Nederl. Akad. Wetensch. Proc., Series A*, 39:117–126, 215–224, 1936.

[Jam55] I. M. James. Reduced product spaces. *Ann. of Math. (2)*, 62:170–197, 1955.
[Jam56a] I. M. James. On the suspension triad. *Ann. of Math. (2)*, 63:191–247, 1956.
[Jam56b] I. M. James. The suspension triad of a sphere. *Ann. of Math. (2)*, 63:407–429, 1956.
[Jam57] I. M. James. On the suspension sequence. *Ann. of Math. (2)*, 65:74–107, 1957.
[Jon78] John D. S. Jones. The Kervaire invariant of extended power manifolds. *Topology*, 17(3):249–266, 1978.
[Kel82] Gregory Maxwell Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982.
[Ker60] Michel A. Kervaire. A manifold which does not admit any differentiable structure. *Comment. Math. Helv.*, 34:257–270, 1960.
[Mah67] Mark Mahowald. *The metastable homotopy of S^n*. Memoirs of the American Mathematical Society, No. 72. American Mathematical Society, Providence, R.I., 1967.
[Mah82] Mark Mahowald. The image of J in the EHP sequence. *Ann. of Math. (2)*, 116(1):65–112, 1982.
[May80] J. P. May. Pairings of categories and spectra. *J. Pure Appl. Algebra*, 19:299–346, 1980.
[May99] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.
[McC69] M. C. McCord. Classifying spaces and infinite symmetric products. *Trans. Amer. Math. Soc.*, 146:273–298, 1969.
[Mil56] John Milnor. Construction of universal bundles. II. *Ann. of Math. (2)*, 63:430–436, 1956.
[Mil58] John Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958.
[Mil59] John Milnor. On spaces having the homotopy type of a CW-complex. *Trans. Amer. Math. Soc.*, 90:272–280, 1959.
[Mil60] J. Milnor. On the cobordism ring Ω^* and a complex analogue. I. *Amer. J. Math.*, 82:505–521, 1960.
[ML71] Saunders Mac Lane. *Categories for the working mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.
[MM02] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002.
[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.
[MT67] Mark Mahowald and Martin Tangora. Some differentials in the Adams spectral sequence. *Topology*, 6:349–369, 1967.
[Nov60] S. P. Novikov. Some problems in the topology of manifolds connected with the theory of Thom spaces. *Soviet Math. Dokl.*, 1:717–720, 1960.
[Nov62] S. P. Novikov. Homotopy properties of Thom complexes. *Mat. Sb. (N.S.)*, 57 (99):407–442, 1962.
[Nov67] S. P. Novikov. Methods of algebraic topology from the point of view of cobordism theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 31:855–951, 1967.
[Pon38] L. S. Pontryagin. A classification of continuous transformations of a complex into a sphere, 1 and 2. *Doklady Akad. Nauk SSSR (N.S.)*, 19:147–149 and 361–363, 1938.
[Pon50] L. S. Pontryagin. Homotopy classification of the mappings of an $(n + 2)$-dimensional sphere on an n-dimensional one. *Doklady Akad. Nauk SSSR (N.S.)*, 70:957–959, 1950.
[Pon55] L. S. Pontryagin. *Gladkie mnogoobraziya i ih primeneniya v teorii gomotopiĭ[Smooth manifolds and their applications in homotopy theory]*. Trudy Mat. Inst. im. Steklov. no. 45. Izdat. Akad. Nauk SSSR, Moscow, 1955.
[Qui69] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.*, 75:1293–1298, 1969.
[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986. Errata and second edition available online at author's home page.
[Rav04] Douglas C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres, Second Edition*. American Mathematical Society, Providence, 2004. Available online at www.math.rochester.edu/u/faculty/doug/mu.html.
[Ser51] Jean-Pierre Serre. Homologie singulière des espaces fibrés. Applications. *Ann. of Math. (2)*, 54:425–505, 1951.
[Sna09] Victor P. Snaith. *Stable homotopy around the Arf-Kervaire invariant*, volume 273 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2009.
[Ste62] N. E. Steenrod. *Cohomology operations*. Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50. Princeton University Press, Princeton, N.J., 1962.
[Ste99] Norman Steenrod. *The topology of fibre bundles*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. Reprint of the 1957 edition, Princeton Paperbacks.

[Sto68] Robert E. Stong. *Notes on Cobordism Theory*. Princeton University Press, Princeton, 1968.
[Tho54] René Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.
[Whi42] G. W. Whitehead. On the homotopy groups of spheres and rotation groups. *Annals of Mathematics*, 43:634–640, 1942.
[Whi49] J. H. C. Whitehead. Combinatorial homotopy. I. *Bull. Amer. Math. Soc.*, 55:213–245, 1949.
[Whi50] George W. Whitehead. The $(n+2)^{\mathrm{nd}}$ homotopy group of the n-sphere. *Ann. of Math. (2)*, 52:245–247, 1950.

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