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FOUNDATIONS OF UNLIMITED CATEGORY THEORY: WHAT REMAINS TO BE DONE

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Abstract. Following a discussion of various forms of set-theoretical foundations of category theory and the controversial question of whether category theory does or can provide an autonomous foundation of mathematics, this article concentrates on the question whether there is a foundation for "unlimited" or "naive" category theory. The author proposed four criteria for such some years ago. The article describes how much had previously been accomplished on one approach to meeting those criteria, then takes care of one important obstacle that had been met in that approach, and finally explains what remains to be done if one is to have a fully satisfactory solution.

From the very beginnings of the subject of category theory as introduced by Eilenberg & Mac Lane (1945) it was recognized that the notion of category lends itself naturally to instances of self-application. Not only does it appear reasonable to speak of the category ***Grp*** of all groups, the category ***Top*** of all topological spaces, etc., we are also led to consider the category ***Cat*** of all categories, whose morphisms are just all the functors $F : A \to B$, where A, B are arbitrary categories. Even more, given any two categories A and B, one can form the category B^A of all functors from A to B, whose morphisms are all the natural transformations $\eta : F \to G$. Thus one may contemplate as apparently reasonable mathematical objects such categories as $\boldsymbol{Grp}^{\boldsymbol{Top}}$ and $\boldsymbol{Cat}^{\boldsymbol{Cat}}$, with each counting as an object of ***Cat***.

From the beginnings, too, of the subject in Eilenberg & Mac Lane (1945) it was suggested that some sort of set-theoretical foundation is needed for category theory since such naive or unrestricted readings having to do with "large" and "super-large" categories appeared to border on the familiar paradoxes. Subsequently, Mac Lane (1961, 1969, and elsewhere) pushed for finding a suitable set-theoretical framework to deal with these problems. The "one universe" solution that he settled on as presented in the text (Mac Lane, 1971) is one of the approaches that is widely accepted. A universe U in a set-theoretical framework is a nonempty transitive set that contains ω, is closed under the operations of pairing, union and power set, and is closed under strong replacement, that is, $(f : a \to U) \Rightarrow f[a] \in U$ for $a \in U$ and f an arbitrary function. The existence of such a U is equivalent to the existence of a strongly inaccessible cardinal.[1] Relative to any such universe U, a set

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[1] A related approach is that ascribed to Grothendieck, which invokes the assumption of arbitrarily large universes, and thus whose existence is equivalent to the existence of infinitely many strongly inaccessible cardinals. On the other hand, in Feferman (1969, 2004) I introduced refinements of the Mac Lane and Grothendieck approaches in which the condition on a transitive set s to be a universe is weakened to the requirement that s with the membership relation restricted to it forms an elementary substructure of the class of all sets; the resulting theories are conservative over ZFC. Shulman (2008) provides a very useful survey of the various approaches to set-theoretical

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is called *small* if it belongs to U and *large* if it is a subset of U but not a member of U; similarly for categories. Special attention is given to *locally small categories*, that is, those whose *Hom* sets are all small. Thus, for example, in place of ***Grp***, ***Top***, and ***Cat*** one deals in such a set-theoretical reduction with the categories of all small groups, small topological spaces, and small categories, respectively; each is a large, locally small category. In these terms what takes the place of $\boldsymbol{Grp}^{\boldsymbol{Top}}$ is now a "super-large" or "meta" category, that is, lying beyond U though a perfectly reasonable one existing in the class of all sets. It seems to be universally accepted in practice that such distinctions as those of being small, large, and locally small are essential to many of the fundamental theorems of category theory, the prime example being Freyd's Adjoint Functor Theorem (AFT).[2] As Freyd (1964, pp. 85–86) said of his theorem, the crucial "solution set" condition for it "is not baroque" since one has counterexamples to AFT when that condition is dropped. Moreover, such set-theoretical conditions can be verified for the various categories that arise in the applications to such areas as combinatorial topology, homological algebra, and algebraic geometry.

Despite this mass of detailed *prima facie* evidence to the contrary, the case for autonomous categorical foundations is still being made on ideological grounds by workers in the field. Mac Lane himself (Mac Lane, 1986, p. 402)[3] said that "it is now possible to develop almost all of ordinary mathematics in a well-pointed topos with choice and natural number object," that is, within the system called the Elementary Theory of the Category of all Sets (ETCS). The system ETCS, introduced by Lawvere (1964), is formulated in the language of category theory where the variables for objects are interpreted as ranging over sets and the variables for morphisms over set-theoretic mappings between sets with domain and codomain specified. Using the notions of topos theory, ETCS isolates significant and to some extent characteristic properties of the category of all (small) sets.[4] One means by which this is accomplished is that the membership relation is eliminated in favor of arrows from the terminal object 1. The idea that ETCS could count as a foundation of mathematics seems to be that one has thus axiomatized the essential mathematical properties of sets as given in categorical terms, so if set theory is accepted as a foundation of mathematics, so too should one accept the theory of the category of all sets. However, as Osius (1974) showed, ETCS is only equivalent in strength to Zermelo set theory, which is relatively weak; on the other hand, he produced an extension of ETCS which is equivalent in strength to ZFC. Thus, to the extent that the latter is claimed to be a foundation of "all of mathematics," one may argue the same for the Osius system. But any claims for ETCS and such extensions as an autonomous foundation of mathematics must obviously be put into question.

In a recent interesting article, Linnebo & Pettigrew (2011) distinguish three kinds of autonomy for such proposed foundational systems: *logical autonomy*, *conceptual autonomy*,

foundations for category theory, with useful attention to examples of how they work out in practice.

2 As formulated in Mac Lane (1971, p. 117) that gives a necessary and sufficient condition for a functor $G : A \to B$ on a small complete locally small category A to have a left adjoint; namely it is that G should preserve all small limits and that the solution set condition holds, that is, for each b in B there is a small set I of morphisms $f_i : b \to Ga_i$ such that every morphism $h : b \to Ga$ factors through one of the f_i.

3 See also the appendix to the 1998 second edition of Mac Lane (1971).

4 A recent convenient exposition of ETCS may be found in Linnebo & Pettigrew (2011).

and *justificatory autonomy*. They argue that ETCS enjoys logical and conceptual autonomy, but that justificatory autonomy may be disputed:

> We argue that the debate turns crucially on whether the objects of a foundation of mathematics can or indeed should be specified only up to isomorphism, as is customary in other branches of contemporary mathematics. In particular, if sets should be characterized only up to isomorphism, then a category-theoretic approach will be highly appropriate; whereas if sets have a richer 'nature' than is preserved under isomorphism, then such an approach will be inadequate. (Ibid., p. 228)

The idea presumed here that the objects of contemporary mathematics are simply structures specified up to isomorphism is belied by mathematical practice where various representations of such structures are ubiquitous.[5]

But the claim too for logical autonomy of ETCS is also disputable. In Feferman (1977) I raised several criticisms of the general case for category theory as an autonomous foundation of mathematics. I shall not repeat those arguments here except for what Linnebo and Pettigrew call the Logical Dependence Objection; that part of my critique has been continued by Hellman (2003) and elsewhere. The basic argument is very simple: in order to explain what a category is, we need the prior notion of a structure as given by certain collections (the objects and morphisms), relations (composition), and operations (domain, codomain, and identity morphisms). Thus one needs some sort of prior theory of collections and operations, if one regards relations simply to be collections of n-tuples for various n. I am not persuaded by Linnebo and Pettigrew that that objection is met by the fact that ETCS is itself a theory of sets and functions (ibid., p. 233) and so does not need a prior such theory for its logical status. However, as we have seen, one does not have the membership relation in that system, while the notion of set (or collection more generally) is essentially tied to the notion of membership. For just one example among many, consider the operation D that takes one from a first-order structure $M = (A, R)$, to the set $D(M)$ of all subsets of A definable in M from parameters in A; this depends *prima facie* on what exactly are the members of A. (The operation D is what one uses to define the step from L_α to $L_{\alpha+1}$ in the constructible hierarchy.)[6] In general, the substitution for membership of arrows from the terminal object 1 only provides a kind of shadow of that relation; in my view, the use of that procedure to get rid of sets as ordinarily conceived simply comes down to a kind of ideological shell game.

Linnebo and Pettigrew (op. cit.) also consider the system CCAF introduced by Lawvere (1966) (and improved in McLarty, 1991, which axiomatizes significant properties of the category of all small categories.[7] In that case, they argue (p. 232) that it is vulnerable to the Logical Dependence Objection. One should also note that it fails as an actual foundation of working category theory since we cannot speak *within it* of such categories as that of all small groups, all small topological spaces, and even the category of all small categories

[5] Witness permutation and matrix representation of groups, coordinate system representation of geometric objects, Fourier representation of functions on the reals, representation of Lie groups and algebras, etc.

[6] Conceivably L itself can be defined in categorical terms as the minimal model of set theory containing all the ordinals, but that is another matter.

[7] The designation CCAF is an (almost) acronym for the Category of all Categories as a Foundation for Mathematics.

itself. The Adjoint Functor Theorem, for example, would require us to speak of functors on such categories and we cannot express within the language of CCAF the requisite small completeness and solution set conditions. Perhaps there are ways of doctoring up CCAF to take care of such problems, for example, by introducing an internal idea of a universe, although that would be contrary to the spirit of CCAF. As it stands, both ETCS and CCAF are simply display pieces designed to support the claims for category theory as an autonomous foundation of mathematics without demonstrating that in any significant way beyond their formulation. By the way, the situation is different for another theory considered alongside these two by Linnebo and Pettigrew, namely SDG, a formal system developed by Lawvere and Kock in purely categorical terms for Synthetic Differential Geometry (cf. Kock, 1991). Successful as that is, in this respect one is simply comparing apples and oranges since the scope of application of SDG is limited to smooth spaces and smooth maps, with no pretenses at all to be a foundation of all of mathematics.

I shall not pursue any further here the general foundational claims for category theory; instead I want to return to the issues posed at the opening of this article. Despite the success of working set-theoretical foundations for working mathematicians (categorists among them) there still remains the question why the unrestricted notions with which we started are mathematically reasonable on the face of it and do not obviously lead to paradoxical conclusions. There is no sensible way, for example, to form a category of all categories which do not belong to themselves. So we should be interested in seeing whether the unlimited or "naive" constructions with which we opened can be accounted for in a demonstrably consistent framework that extends usual set theory in some way or other. A call for some such foundation (that was only recently brought to my attention) seems first to have been made by Engeler & Röhrl (1969). More specifically, in my "Categorical foundations and foundations of category theory" (Feferman, 1977, CF-FCT in the following), I suggested the following requirements to meet that purpose within a suitable axiomatic system S:

(R1) Form the category of all structures of a given kind, e.g. the category ***Grp*** of all groups, ***Top*** of all topological space, and ***Cat*** of all categories.

(R2) Form the category B^A of all functors from A to B, where A, B are any two categories.

(R3) Establish the existence of the natural numbers N, and carry out familiar operations on objects $a, b, \ldots$ and collections $A, B, \ldots$, including the formation of $\{a, b\}$, (a, b), $A \cup B$, $A \cap B$, $A - B$, $A \times B$, B^A, $\cup A$, $\cap A$, $\prod B_x[x \in A]$, etc.

In addition to these requirements, since (R1) guarantees forms of self-application, one would expect that the justification for using such S would not be direct but would rather be achieved by reduction to some familiar accepted axiom system. That was taken for granted in the CF-FCT article, but should have been made explicit as follows:

(R4) Establish the consistency of S relative to a currently accepted system of set theory.

All the candidate systems S that I will describe here have been shown to satisfy (R4), though via reductions to systems of set theory of widely varying strength, depending on S.

Returning to (R3), first note that implicit in it is that we have a general notion of operation which can take collections in addition to individuals as both arguments and values. In particular, the notation $\prod B_x[x \in A]$ should have been explained as given by an operation f from A to collections, with $f(x) = B_x$ for each x in A, for which the product consists as usual of all operations g with domain A such that for each x in A, $g(x) \in B_x$. The list should also have included passage from a collection A and equivalence relation E

on A to the canonical projection operation p: $A \to A/E$, where $p(x)$ is the equivalence class $[x]_E$ for each x in A.

In the CF-FCT article I suggested for the choice of S to meet (R1)–(R3) some sort of theory—possibly nonextensional—of operations and classes (aka classifications). And in the second part of that paper I proposed a specific system of partial operations and partial classes of that character. But, as it worked out, it only met restricted forms of (R1)–(R3),[8] and so I did not continue to pursue that approach afterward. Actually, I had earlier explored a much more successful system in an unpublished manuscript, Feferman (1974), that met (R1) and (R2) in full, and all of (R3) except for the products $\prod B_x[x \in A]$ and the quotients A/E. Because those and related constructions are ubiquitous in category theory, I did not publish the manuscript.[9] However, I returned to that approach a few years ago because I thought it was worthwhile for people to see that one could go quite far to meet the requirements while highlighting the problems in meeting them in full. It was thus that I published in Feferman (2006) a summary of the notions and results of the 1974 manuscript, together with an outline of the proof; in addition to that, I scanned the 1974 MS and had it placed on my home page in order to make it directly available (see the references). More recently, I realized that a natural extension of the 1974 system could be used to take care of the problem of passage to equivalence classes; that is the main new result dealt with here.

The earlier system in question is denoted S* in Feferman (1974, 2006). S* is an extension of the MKC (Morse–Kelly with Choice) theory of sets and classes by an enriched stratified theory of classes, NFU(P). The subsystem NFU(P) already satisfies (R1) and (R2) *in full* and (R3) for all the operations there except for the passage to equivalence classes and closure under general cartesian product As to (R4), NFU(P) is consistent relative to Zermelo set theory Z, while the full system S* is consistent relative to ZFC + "there exist two strongly inaccessible cardinals." The consistency proofs are given in full in Feferman (1974). Here are quick descriptions of the systems involved, first of all of NFU . Recall the system NF introduced by Quine (1937); its language has variables $A, B, C, \ldots, X, Y, Z$, and the only relation symbols are $=$ and $\in$. A formula is called *stratified* if it results from a formula of the simple theory of types by suppression of type superscripts. The axioms of NF are Extensionality (Ext) and the Stratified Comprehension Axiom (SCA), that is, $(\exists A)(\forall X)[X \in A \leftrightarrow \varphi]$, for all stratified formulas φ which do not contain 'A' as a free variable. Despite many, many efforts, no proof of the consistency of NF is known .[10] NFU is the system obtained from NF by weakening Extensionality to allow Urelements, that is,

$$\text{(Ext)}' \qquad (\exists X)(X \in A) \wedge (\forall X)[X \in A \leftrightarrow X \in B] \to A = B.$$

The consistency of NFU was proved by Jensen (1969) relative to PA, and for NFU plus the axiom of infinity, relative to Z. In order to obtain these proofs of consistency, Jensen applied the results of Specker (1962) and Ehrenfeucht & Mostowski (1965) in a novel way.

8 For example, with respect to (R1), one could only form in it the *partial* category of all *total* groups, etc.

9 Moreover, I subsequently became aware of the work of Oberschelp (1973) which seemed to have considerable overlap with the work of my 1974 manuscript; I have recently studied the Oberschelp paper again and see that my system had significant advantages over his.

10 In the final section of Engeler & Röhrl (1969) the authors suggested use of some sort of system for the foundations of unlimited category theory whose consistency could be reduced to that of NF. Since the consistency of NF is still a major open problem, I moved to related systems whose consistency was known (e.g., NFU) or could be established by known methods, and which would suffice just as well for the specific purposes given by the criteria (R1)–(R3).

In NFU, take Λ to be some fixed urelement and then write $\{X \mid \varphi(X)\}$ for the unique A satisfying (SCA) relative to given parameters Y when φ is stratified and $(\exists X)\varphi(X)$, otherwise Λ. Then closure under the following operations on classes is provable: $\{Y\}$, $\{Y_1, Y_2\}$, $Y_1 \cup Y_2$ and $Y_1 \cap Y_2$; more generally, $\cup Y = \{X \mid \exists Z(X \in Z \wedge Z \in Y)\}$ and $\cap Y = \{X \mid \forall Z(Z \in Y \rightarrow X \in Z)\}$. We also have $\wp(Y) = \{X \mid X \subseteq Y\}$. Finally we have $-Y = \{X \mid X \notin Y\}$ and $V = \{X \mid X = X\}$; note $-\Lambda = V$ and $V \in V$. But in addition to these we need for (R3) a pairing operation in order to define relations, functions, and finite Cartesian products. The usual one can be defined in NFU by $\{\{X\}, \{X, Y\}\}$ but this pair is of type level 2 over the levels of X, Y. In order to have a pairing operation that preserves type levels, we add a binary operation symbol P to the language of NFU with axiom

(Pair) $$P(X_1, Y_1) = P(X_2, Y_2) \rightarrow X_1 = X_2 \wedge Y_1 = Y_2.$$

Now a formula φ is called stratified in the language of NFU plus P if it comes from assigning the same type level to (t_1, t_2) (i.e., $P(t_1, t_2)$) as is assigned to both t_1 and t_2, for any terms t_1 and t_2. For example, $(X, Y) \in Z$ is now stratified, but not $[(X, Y) \in Z \wedge X \in Y]$. Write NFU(P) for the system obtained by adding the Pair axiom and extending the SCA axiom using this stratification condition. The consistency of NFU(P) is proved by a simple modification of Jensen's for NFU.

Tuples are obtained by iterating pairing in NFU(P), and then relations and functions of any number of arguments are defined as usual. We can now prove existence of $X \times Y$ and Y^X for any X, Y. We can also deal with single sorted and many sorted structures

$$A = (A_1, \ldots A_k, R_1, \ldots, R_m, F_1, \ldots, F_n)$$

as usual, where the domains, relations, and functions all have the same type level. In particular, we can define what it means for A to be a group, a topological space,[11] a category, etc. and form the classes *Grp*, *Top*, *Cat* of all such. In particular, we can form a category ***Cat*** = (*Cat*, *Funct*,...) whose objects are all the categories, and whose morphisms are all the functors between categories ("the category of *all* categories"), and prove ***Cat*** $\in$ *Cat*. Similarly, we can prove that $A, B \in Cat$ implies $B^A \in Cat$, where $B^A = (Funct(A, B), Nat\ldots)$ has as objects all functors from A to B and as morphisms all natural transformations between such functors. This is an indication of how (R1) and (R2) are satisfied in full in NFU(P); cf. Feferman (2006) for the requisite details.

So now NFU(P) meets all the requirements of (R3) except for the following two problems already indicated above:

(Prob. 1) To go from an equivalence relation on a class to the class of its equivalence classes.

(Prob. 2) To form unrestricted Cartesian products.

As to (Prob. 1), let (A, E) be a structure with E an equivalence relation on A. Then the equivalence classes $[X]_E = \{Y \mid (X, Y) \in E\}$ may be defined as usual in NFU; for X in A, each such $[X]_E$ is on the same type level as A. But then the class A/E of all such $[X]_E$ is one type level higher than A and we can't form the canonical map from A to A_E. The main new step here is to provide an alternative way to deal with (Prob. 1) by forming a

[11] The usual explanation of topological spaces is second order, but we can get around this by taking a topological space to be a collection O of classes satisfying the usual conditions for the open sets on a space X, and then retrieve X as the union of O.

further extension NFU(P, C) of our system by means of the addition of a Universal Choice Operator C, that is, a unary operation symbol with axiom:

(UC) $$(\exists X)[X \in A] \rightarrow C(A) \in A,$$

where the expanded stratification condition is that $C(t)$ has (implicit) type level one lower than that of t. In particular, we may now define $X/E = C([X]_E)$, which is at the same type level as X, and then $A/E = \{X/E \mid X \in A\}$. That is, instead of working with equivalence classes we use C to work in a uniform way with representatives of the equivalence classes. Of course we have $X/E = Y/E$ if and only if $(X, Y) \in E$.

The consistency of NFU(P, C) is obtained by a modification of the consistency proof of the much stronger system S*, mentioned above, that had been introduced in Feferman (1974, 2006) and which is an extension of both NFU(P) and MKC. The language L* of S* adds set variables $a, b, c, \ldots, x, y, z$, and a class constant V_0.[12] The stratification conditions for the Comprehension scheme in S* is liberalized to allow set variables and the constant V_0 to take any type whatever. For example, the formulas $[V_0 \in X \wedge \neg(X \in V_0)]$, $(X \in x)$, and $[(x, X) \in x]$ are all stratified, but $[(x, X) \in X]$ is not. The ontological axioms for sets and classes in S* state that each set is a class and that V_0 is the class of all sets. The axioms for sets are the usual ones for ZF, that is, extensionality, empty set, unordered pair, union, power set, and infinity, but with the Replacement scheme strengthened by the admission of all formulas in the language of L*; Foundation is also strengthened to a scheme whose instances are given by all formulas in the language of L*. Finally, S* has a kind of universal choice axiom stating that there exists a function F such that for each nonempty class X, $F(X) = \{Y\}$ for some Y in X. Instead of this we now take S† to be the system obtained from S* by replacing that universal choice axiom by the (UC) axiom formulated above. S† is formally weaker than S* in that the relation which holds between X and Y just in case $C(X) = Y$ is not a function since the type of Y is lower than that of X; but its advantage is that C is now an explicitly given stratified operation on the class of all classes. The consistency of S† is proved by a simple modification of the proof I gave in 1974 of the consistency of S. A model of it is built as an end extension of L_κ, the constructible sets up to the first strongly inaccessible cardinal κ; some details of the modified proof are given in the Appendix below.

Other than this, I have still not been able to find any way of dealing with full Cartesian Products (and Sums) in such systems. The difficulty is that if A is a class of classes and F is a function which associates with each X in A a class $F(X)$ then we cannot define in a stratified way the class of all functions G such that for each X in A, $G(X) \in F(X)$. However, in the systems S* and S†, since sets are free floating as to type level we can form such things as $\prod_{x \in A} F(x)$, where A is any class of sets, that is, $A \subseteq V_0$, and F is a function from A to classes. That is a more liberal operation than one has in the various standard set-theoretic foundations of category theory. So one thing that remains to be done in continuation of this work is to examine the possible advantages and disadvantages of using the system S† for the actual development of category theory in various leading test cases, such as the Freyd Adjoint Functor Theorem, the Yoneda Lemma, and the Kan Extension Theorem.

Quite another thing is, still, to pursue altogether other possible approaches. In CF-FCT I took a very ecumenical stance about what theory of collections and operations would be best to use as a foundation for category theory meeting (R1)–(R2) (and implicitly (R3)

[12] This should not be confused with the usual V_0 in the cumulative hierarchy.

and (R4)). It seems to me we've reached the limit of what enriched stratified systems can do, and I would now like to encourage pursuit along other directions. I have reasons to believe that working with systems of explicit mathematics (Feferman, 1975) and what I call operational set theory (Feferman, 2009) may provide a useful alternative, and would like to encourage investigating these systems to see what possible advantages they might have. For example, (R2) and (R3) are met in full in these systems, but on the face of it we still need a suitable distinction between small and large classes in order to meet (R1). The big question that remains is whether any workable system that meets (R1)–(R3) in full can be shown to be consistent relative to some accepted system of set theory.

Appendix: Proof of the consistency of S†. This is carried out as follows by a few modifications of the proof of the consistency of S* in Feferman (1974). In place of the ranks R_α we use the constructibles L_α up to level α. In particular, on p. 22 op. cit., replace R_κ and R_δ by L_κ and L_δ, resp.; κ there is the first (strongly) inaccessible cardinal and δ is the first (strongly) inaccessible cardinal greater than κ. Let $y \prec x$ hold if y precedes x in the usual well-ordering of L_δ. Then define $C_1(x)$ for $x \neq 0$ to be the least y in this ordering such that $y \in x$, otherwise 0; this has the property that if $x \in L_\alpha$ then $C_1(x) \in L_\alpha$. On p. 23, add to the conditions on the Ehrenfeucht–Mostowski indiscernibles c_i, (8)(i)–(iii), the following two conditions:

(iv) If $x \in c_i$ and $y \prec x$ then $y \in c_i$.
(v) $\mathfrak{M}_1$ satisfies $\forall x(x \in c_i \wedge x \neq 0 \rightarrow C_1(x) \in x)$.

The rest of p. 23 and pp. 24–25 proceeds as before. Finally, on p. 26, given the automorphism σ established by (14) on p. 24, define $C(x) = C_1(\sigma(x))$. This has the property that if $y \in^* x$ then $C(x) \in^* x$. The proof that all the axioms of S† hold in the model $\mathfrak{M}^*$ given by (17) on p. 26 is then carried out as before on pp. 26–32, with the proof that (UC) holds now established by fiat.

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BIBLIOGRAPHY

Ehrenfeucht, A., & Mostowski, A. (1956). Models of axiomatic theories admitting automorphisms. *Fundamenta Mathematicae*, **43**, 50–68.

Eilenberg, S., & Mac Lane, S. (1945). General theory of natural equivalences. *Transactions of the American Mathematical Society*, **58**, 231–294.

Engeler, E., & Röhrl, H. (1969). On the problem of foundations of category theory. *Dialectica*, **23**, 58–66.

Feferman, S. (1969). Set-theoretical foundations for category theory (with an appendix by G. Kreisel). In Barr, M., et al., editors. *Reports of the Midwest Category Seminar III, Lecture Notes in Mathematics*, Vol. 106. Berlin, Germany: Springer, pp. 201–247.

Feferman, S. (1974). Some formal systems for the unlimited theory of structures and categories. Unpublished MS, available online at http://math.stanford.edu/~feferman/papers/Unlimited.pdf. Abstract in *Journal of Symbolic Logic*, **39** [1974], 374–375.

Feferman, S. (1975). A language and axioms for explicit mathematics. In *Algebra and Logic, Lecture Notes in Mathematics*, Vol. 450. Berlin, Germany: Springer, pp. 87–139.

Feferman, S. (1977). Categorical foundations and foundations of category theory. In Butts, R. E., and Hintikka, J., editors. *Logic, Foundations of Mathematics and Computability Theory*, Vol. 1. Dordrecht, The Netherlands: Reidel, pp. 149–165.

Feferman, S. (2004). Typical ambiguity: Trying to have your cake and eat it too. In Link, G., editor. *One Hundred Years of Russell's Paradox*. Berlin: De Gruyter, pp. 135–151.

Feferman, S. (2006). Enriched stratified systems for the foundations of category theory. In Sica, G., editor. *What is Category Theory?* Monza, Italy: Polimetrica. Reprinted in (G. Sommaruga, ed.) *Foundational Theories of Classical and Constructive Mathematics* (2011). Berlin: Springer, pp. 127–143.

Feferman, S. (2009). Operational set theory and small large cardinals. *Information and Computation*, **207**, 971–979.

Freyd, P. (1964). *Abelian Categories*. New York: Harper & Row.

Hellman, G. (2003). Does category theory provide a framework for mathematical structuralism?. *Philosophia Mathematica III*, **11**, 129–157.

Jensen, R. (1969). On the consistency of a slight (?) modification of Quine's *New Foundations*. In Davidson, D., and Hintikka, J., editors. *Words and Objections. Essays on the work of W.V.O. Quine*. Dordrecht, The Netherlands: Reidel, pp. 278–291.

Kock, A. (1991). *Synthetic Differential Geometry*. Cambridge, UK: Cambridge University Press.

Lawvere, F. W. (1964). An elementary theory of the category of sets. *Proceedings of the National Academy of Sciences of the United States of America*, **52**, 1506–1511.

Lawvere, F. W. (1966). The category of all categories as a foundation for mathematics. In *Proceedings of the La Jolla Conference on Categorical Algebra*. Berlin: Springer-Verlag, pp. 1–20.

Linnebo, ∅., & Pettigrew, R. (2011). Category theory as an autonomous foundation. *Philosophia Mathematica III*, **19**, 227–254.

Mac Lane, S. (1961). Locally small categories and the foundations of mathematics. In *Infinitistic Methods*. Oxford: Pergamon Press, pp. 25–43.

Mac Lane, S. (1969). One universe as a foundation for category theory. In Barr, M., et al., editors. *Reports of the Midwest Category Seminar III, Lecture Notes in Mathematics*, Vol. 106. Berlin, Germany: Springer, pp. 192–200.

Mac Lane, S. (1971). *Categories for the Working Mathematician*. Berlin: Springer-Verlag. Second edition, 1998.

Mac Lane, S. (1986). *Mathematics: Form and Function*. Berlin: Springer-Verlag.

McLarty, C. (1991). Axiomatizing a category of categories, *Journal of Symbolic Logic*, **56**, 1243–1260.

Oberschelp, A. (1973). Set theory over classes, *Dissertationes Mathematicae* (*Rozprawy Matematyczne*), **106**, 1–62.

Osius, G. (1974). Categorical set theory: A characterization of the category of sets. *Journal of Pure and Applied Algebra*, **4**, 79–119.

Quine, W. V. O. (1937). New foundations for mathematical logic. *American Mathematical Monthly*, **44**, 77–80.

Shulman, M. (2008). Set theory for category theory. *arXiv*:0810.1279v2 [math.CT] 7 Oct 2008.

Specker, E. (1962). Typical ambiguity in logic, In Nagel, E., et al., editors. *Methodology and Philosophy of Science. Proceedings of the 1960 International Congress*. Stanford, CA: Stanford University Press, pp. 116–123.

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