April 22, 1:00 pm

1 Theta Functions

We've previously seen connections between modular forms and Ramanujan's work by verifying that Eisenstein series are indeed modular forms, and showing that the Discriminant function Δ is a weight 12 cusp form with a product expansion in q. As we said then, the extent to which we can express modular forms in terms of products and quotients of the η function begin to explain many of the q-series identities given in Berndt. Using facts like the additivity of weights of modular forms under multiplication, we can even begin to predict where to look for such impressive identities.

Many of the identities we used for solving the problem of representations by sums of squares could be boiled down to identities about the theta function, a q-series supported on powers n^2. In this section, we'll begin a study of theta functions and their connection to quadratic forms.

2 Poisson Summation for Lattices

The theory of the Fourier transform is often stated for functions of a real variable, but is really no different for a real vector space. Given a function f on $\mathbb{R}$ that is sufficiently well-behaved (For example f is piece-wise continuous with only finitely many discontinuities, of bounded total variation, satisfying

$$f(a) = \frac{1}{2}\left[\lim_{x\to a^-} f(x) + \lim_{x\to a^+} f(x)\right] \quad \text{for all } a,$$

and

$$|f(x)| < c_1 \min(1, x^{-c_2}) \quad \text{for some } c_1 > 0, c_2 > 1.$$

One can fuss with these conditions some, but these will be more than sufficient for us.)

Now for such a function, define the Fourier transform

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{2\pi i x y}\, dy.$$

Proposition 1 (Poisson Summation for $\mathbb{R}$) *Given a function f as above with Fourier transform $\hat{f}$, then*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

Remember that we're working with functions on $\mathbb{R}$ here, which is non-compact. The theory is somewhat different from, say $\mathbb{R}/\mathbb{Z}$, a compact domain for which we can express sufficiently nice functions as a Fourier series. We don't have such a series here, but we can regard the above formula as a partial analogue of this property.

Proof Consider the function

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n),$$

which is sufficiently nice (in the above sense) if f is sufficiently nice, and is periodic of period 1. That is, we can consider F as a function on the compact domain $\mathbb{R}/\mathbb{Z}$, so it has a Fourier expansion such that

$$F(x) = \sum_{n=-\infty}^{\infty} a_m e^{2\pi i m x},$$

where, as usual,

$$a_m = \int_0^1 F(x) e^{-2\pi i m x} dx = \int_0^1 \sum_{n=-\infty}^{\infty} f(x+n) e^{-2\pi i m x} dx$$

Since $e^{2\pi i m x} = e^{2\pi i m(x+n)}$, we may interchange the order of summation and integration, giving

$$a_m = \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n) e^{-2\pi i m(x+n)} dx = \int_{-\infty}^{\infty} f(x) e^{-2\pi i m x} dx = \hat{f}(-m).$$

Thus taking $x = 0$, we have

$$\sum_{n=-\infty}^{\infty} f(n) = F(0) = \sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

as required. $\square$

More generally, to a real vector space V of dimension n with a translation invariant measure μ, we may define (again for a rapidly decreasing smooth function f) the Fourier transform

$$\hat{f}(x) = \int_V e^{-2\pi i\langle y,x\rangle} f(y)\mu(y).$$

(Note that comparing with $V = \mathbb{R}$ indeed reduces to our earlier definition.) Then $\hat{f}$ is a sufficiently nice function on V^*, the dual vector space of V consisting of linear functions $V \to \mathbb{R}$. If Γ is a lattice in V, then the dual lattice Γ' in V^* is defined as the set of $x \in V^*$ such that $\langle y, x\rangle \in \mathbb{Z}$ for all $y \in \Gamma$.

Proposition 2 (Poisson Summation for Lattices) *The volume of the lattice Γ in V is $\mu(V/\Gamma)$. For f sufficiently nice as above, we have*

$$\sum_{y\in\Gamma} f(y) = \frac{1}{\mu(V/\Gamma)} \sum_{x\in\Gamma'} \hat{f}(x)$$

Proof We just want to reduce to the usual Poisson summation over $\mathbb{R}^n$, whose proof is essentially identical to the one we gave above for $\mathbb{R}$. We can rescale the measure by the volume, so we may assume $\mu(V/\Gamma) = 1$. Let $e_1, \ldots, e_n$ be a basis of Γ. Then we may identify V with $\mathbb{R}^n$ according to the coefficients of these basis vectors, Γ may be identified with $\mathbb{Z}^n$, and μ with $dx_1 \ldots dx_n$. The space of linear maps may also be identified with $\mathbb{R}^n$ and Γ' with $\mathbb{Z}^n$, so we reduce to Poisson summation for $\mathbb{R}^n$. $\square$

3 Functional Equations for Theta Functions

Suppose that V has a symmetric, bilinear form $B(x, y) = x \cdot y$ which is positive and non-degenerate ($x \cdot x > 0$ if $x \neq 0$). This simplifies the situation above, as V^* can be identified with V using the form, and Γ' is now a lattice in V. To any lattice Γ, define

$$\Theta_\Gamma(t) = \sum_{x\in\Gamma} e^{-\pi t x\cdot x}, \quad \text{where } t > 0,\ t \in \mathbb{R}.$$

Taking $V = \mathbb{R}, \Gamma = \mathbb{Z}, q = e^{2\pi i z}$, and $z = iy$ for $y > 0$ (or actually, somewhat annoyingly, $z = iy/2$) recovers the theta function φ defined in Berndt's book.

Theorem 1 *The function Θ_Γ satisfies the functional equation*

$$\Theta_\Gamma(t) = t^{-n/2} \frac{1}{\mu(V/\Gamma)} \Theta_{\Gamma'}(t^{-1}).$$

In particular, if $\Gamma = \Gamma' = \mathbb{Z} \subset \mathbb{R}$, *we have*

$$\Theta(t) = \frac{1}{\sqrt{t}}\Theta(\frac{1}{t}).$$

Proof We will apply Poisson summation to the function $f(x) = e^{-\pi x \cdot x}$, a rapidly decreasing smooth function on V. To determine the Fourier transform of f, choose an orthonormal basis for V to identify the vector space with $\mathbb{R}^n$ so that the measure becomes $dx = dx_1 \cdots dx_n$ and the inner product simplifies to give $f = e^{-\pi(x_1^2+\cdots+x_n^2)}$. Hence the Fourier coefficient

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-2\pi i(x_1y_1+\cdots+x_ny_n)} e^{-\pi(y_1^2+\cdots+y_n^2)} dy$$

can be realized as an iterated integral which is identical in each coordinate. Choose one such integral, complete the square in the exponent and evaluate. We find the Fourier transform of $e^{-\pi x^2}$ is again $e^{-\pi x^2}$, so f is equal to $\hat{f}$.

Our theta function has summands $e^{-\pi t x \cdot x}$. Again, use the function f defined above, now for the lattice $t^{1/2}\Gamma$, all translates of elements of Γ by $t^{1/2}$. It's volume in V is $t^{n/2}\mu(V/\Gamma)$ where n is the dimension of V, and its dual is $t^{-1/2}\Gamma'$ according to the definition of the dual. Applying Poisson summation for lattices gives the desired result. $\square$

4 Matrix Description of Theta Functions

Note that in our theorem, we need not take $e_1, \ldots, e_n$ to be an orthonormal basis of Γ. More generally, to obtain a symmetric bilinear form which is positive and non-degenerate, then setting $a_{ij} = e_i \cdot e_j$, the matrix $A = (a_{ij})$ must be positive, non-degenerate, and symmetric. With this choice of basis and bilinear form,

$$x \cdot x = \sum a_{ij}x_ix_j \quad \text{for } x = x_1e_1 + \cdots + x_ne_n$$

and the corresponding theta function is

$$\Theta_\Gamma(t) = \sum_{x \in \mathbb{Z}^n} e^{-\pi t \sum a_{ij}x_ix_j}.$$

By comparing the basis $e_1, \ldots, e_n$ chosen above to an orthonormal basis, we see that the volume of our lattice Γ is $\det(A)^{1/2}$ (see Serre, p. 108 for a slick proof using wedge products). Further if $B = (b_{ij})$ is the inverse matrix to A, then B plays the same role as A but now for the dual basis. That is, setting $e'_i = \sum b_{ij}e_i$, then the e'_i are a dual basis whose inner products recover B, and whose volume is $1/\det(A)^{1/2}$.

5 Theta Functions as Modular Forms

First we analyze convergence of the theta functions. Given any integer $m \geq 0$, let $r_\Gamma(m)$ denote the number of elements $x \in \Gamma$ such that $x \cdot x = 2m$. Then $r_\Gamma(m)$ is bounded by a polynomial in m. This is clear since we're counting lattice points on the surface of a sphere of radius $2m$ in n dimensions. Serre quotes the bound $O(m^{n/2})$. What's the best you can prove? For our purposes, we just need polynomial growth for convergence since the terms in our theta functions have exponential decay. This implies that the q-series

$$\sum_{m=0}^{\infty} r_\Gamma(m) q^m$$

converges for $|q| < 1$. So letting $q = e^{2\pi i z}$ as usual, for $z \in \mathcal{H}$ defines a holomorphic function

$$\theta_\Gamma(z) = \sum_{m=0}^{\infty} r_\Gamma(m) q^m = \sum_{x \in \Gamma} e^{\pi i z (x \cdot x)} = \sum_{x \in \Gamma} q^{(x \cdot x)/2}.$$

Because we want integral powers of q only in our q expansion, we see now that we need to require an additional condition on our lattice Γ: $x \cdot x \equiv 0 \pmod 2$ for all $x \in \Gamma$. In matrix terms, this implies $A = (e_i \cdot e_j)$ has even entries along the diagonal.

Note further the intentional use of the lower case θ, as this function is slightly different from the functions defined in the previous sections. In those sections, we made careful study of theta functions with real parameter t, which we now see recovered by setting $z = it$. In fact, we'd like to use the symmetry property of the real-valued theta function as $t \mapsto 1/t$ to conclude a similar property for $z \mapsto -1/z$ for $\theta(z)$.

Using our new lower-case notation, we have

$$\Theta_\Gamma(t) = \theta_\Gamma(it), \quad \Theta_\Gamma(t^{-1}) = \theta_\Gamma(-1/it)$$

We'd like to apply Poisson summation for lattices, to obtain a relation for θ. To do this, we require the dual lattice Γ' to be equal to Γ – in terms of the definition of the dual lattice, this means that $x \cdot y \in \mathbb{Z}$ for all $x, y \in \mathbb{Z}$ so that $x \cdot y$ defines an isomorphism between Γ and its dual. In matrix terms, $A = (a_{ij}) = (e_i \cdot e_j)$ has integer coefficients with determinant equal to 1. Given this assumption, we may apply Theorem 1 to give

$$\theta_\Gamma(-1/it) = t^{n/2} \theta_\Gamma(it).$$

Because both $\theta_\Gamma(-1/z)$ and $(iz)^{n/2} \theta_\Gamma(z)$ are analytic functions in z, and we have just proved they are equal for z on the positive imaginary axis, then by analytic continuation, it is true for all $z \in \mathcal{H}$. We record this result as:

Proposition 3 *Given Γ a self-dual lattice with notation as above, for any $z \in \mathcal{H}$,*

$$\theta_\Gamma(-1/z) = (iz)^{n/2}\theta_\Gamma(z).$$

Corollary 1 *If V is a self-dual lattice with $x \cdot x \equiv 0$ (2) for all $x \in \Gamma$ and $\dim V = n$ is a multiple of 8, then $\theta_\Gamma(z)$ is a modular form of weight $n/2$.*

Proof Indeed, the modular group G is generated by two matrices S and T and the q-series expansion (with the $\equiv 0$ mod 2 assumption) shows invariance under T, while the proposition shows invariance under S provided we get rid of the power of i. $\square$

In fact, we can show that any lattice V under these assumptions (self-dual, $x \cdot x \equiv 0$ (2)) must have dimension divisible by 8. We may assume that $n \equiv 4$ (8) (else consider $\Gamma \oplus \Gamma$ or $\Gamma \oplus \Gamma \oplus \Gamma \oplus \Gamma$ and we'll arrive at a similar contradiction). Then, by Proposition 3, the differential form $\omega(z) = \theta_\Gamma(z) dz^{n/4}$ is acted on by S as $S(\omega) = -\omega$ and $T(\omega) = \omega$ hence $(ST)^2(\omega) = \omega$. But the order of ST in G is 3, giving a contradiction.

This is all well and good, except that we haven't actually shown that any such vector spaces V exist. Translated into conditions on matrices, we seek $n \times n$ invertible integer matrices with $8|n$ which are symmetric, even along the diagonal, and of determinant 1.

6 Lattices in $n = 8k$ Dimensions

The most natural lattice to consider is $L = \mathbb{Z}^n$ in the vector space $V = \mathbb{Q}^n$ equipped with the usual "dot product" bilinear form, where $\mathbb{Q}$ denotes the rational numbers. It has all the necessary properties, except for the condition that $x \cdot x \equiv 0$ (2) for all x. If we consider the submodule L_2 of L of elements satisfying this condition, this has index 2 in L, so since L has volume 1 as a lattice in $\mathbb{R}^n$, then L_2 has volume 2. To fix this, we add the vector $e = (1/2, \cdots, 1/2) \in V$ and consider $\Gamma_n = \langle L_2, e \rangle$, the submodule of V generated by L_2 and e. Then

$$x \cdot e = \frac{1}{2}\sum x_i \in \mathbb{Z} \text{ for all } x \in L_2, \quad \text{and} \quad e \cdot e = 2k$$

so Γ_n satisfies the mod 2 property. Moreover, L_2 has index 2 in Γ_n since $2e \in L_2$, so Γ_n has finite volume equal to 1 as desired, and defines a lattice in $\mathbb{R}^n$.

Let's analyze $n = 8$, the first example of such a lattice. How many elements $x \in \Gamma_8$ have $x \cdot x = 2$? We can use modular forms to obtain the answer. Indeed, the lattice satisfies all the necessary properties for θ_{Γ_8} to define a modular form of weight 4. Analyzing its q-series, the coefficient of q^0 is $r_\Gamma(0)$, the number of lattice points with inner product 0. Our form is non-degenerate so the only such element of Γ is the origin. That is, $r_\Gamma(0) = 1$.

Previously, we asserted (but have yet to prove) the space of modular forms of weight 4 has dimension 1, so $\theta_{\Gamma_8} = cG_2$ (where G_2 is the Eisenstein series of weight 4), for some constant c which is determined by normalizing the constant term of G_2 to be 1. This implies that the coefficient of q^m: $r_\Gamma(m) = 240\sigma_3(m)$ for all $m \geq 1$. In particular, there are 240 elements x of Γ_8 with $x \cdot x = 2$. They can be listed as follows (with e_i the standard basis in $\mathbb{Q}^n$):

$$\pm e_i \pm e_k \ (i \neq k) \quad \textit{and} \quad \frac{1}{2}\sum_{i=1}^{8} \delta_i e_i \text{ where } \delta_i = \pm 1, \prod_{i=1}^{8} \delta_i = 1.$$

Remark 1 This lattice is extraordinarily important in the theory of Lie groups. The mutual scalar products of these vectors are integers which form a root system of type E_8, where E has nothing to do with Eisenstein series, but rather the Cartan classification of simple Lie groups. Roughly, any simple Lie group is of type A_n, B_n, C_n or D_n for $n \geq 1$ or an exceptional type E_6, E_7, E_8, F_4, or G_2 (again unfortunate notation, as there's no connection to Eisenstein series here). Of these exceptional groups, E_8 is by far the most complex. A discussion of E_8 as providing a so-called "theory of everything" uniting fundamental forces in physics has recently been revived by Garrett Lisi and others, for which our own Prof. Bert Kostant has delivered expository lectures. Check the web for such references.

To finish our discussion, the lattice Γ_8 can be given basis

$$\frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + \cdots + e_7), \quad e_1 + e_2, \quad \text{and } e_i - e_{i-1} \text{ (for } 2 \leq i \leq 7)$$

with corresponding matrix (!)

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

For $k > 1$ (so n at least 16), then the vectors in $x \in \Gamma_{8k}$ with $x \cdot x = 2$ are just $\pm e_i \pm e_k$ with $i \neq k$, which no longer generate Γ_{8k}. In particular, $\Gamma_8 \oplus \Gamma_8$ is not isomorphic to Γ_{16}. Serre deals with the structure theory for lattices satisfying the required properties of this section in Chapter V of his book. See those pages for more detail. To quote just one result, the space of such lattices has dimensions 1, 2, and 24 for k equal to 1, 2, or 3, respectively.

References

[1] J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer-Verlag (1973).