1 Introduction

In this unit on elliptic functions, we'll see how two very natural lines of questions interact. The first, as we have met several times in Berndt's book, involves elliptic integrals. In particular, we seek complex functions which are solutions to the indefinite integral

$$\int \frac{dz}{\sqrt{Az^3 + Bz + C}}$$

called an elliptic integral of the first kind. The second concerns periodic functions. For functions of a real variable, the trigonometric functions $\cos(2\pi nx)$ and $\sin(2\pi nx)$ are basic examples of periodic functions with period $1/n$, that is, functions for which

$$f(x + 1/n) = f(x) \quad \forall x \in \mathbb{R}.$$

By Fourier theory, any suitably nice function of period $1/n$ (e.g. piece-wise differentiable functions with finitely many discontinuities over any period) can be expressed in terms of these elementary functions. For functions of a complex variable, we can again ask about periodic functions with period $\omega \in \mathbb{C}$, where again we mean:

$$f(z + \omega) = f(z) \quad \forall z \in \mathbb{C}.$$

Once again, there is a Fourier theory for complex functions, described in terms of the elementary periodic functions $e^{2\pi iz/\omega}$ with period ω.

Let's try to do this somewhat rigorously. Suppose we are given a (non-empty) connected, open domain D with the property that $z \in D$ implies $z + \omega$ and $z - \omega$ are in D. Then let D' be the image of D under $z \mapsto \zeta = e^{2\pi iz/\omega}$. (Convince yourself that D' is still open and connected under this map.) For example, if D is the complex plane (our most common example) then D' is the punctured plane $\mathbb{C} - \{0\}$. If $f(z)$ is a meromorphic function in D with period ω, then let F be the unique function on D' defined by

$$F(\zeta) = F(e^{2\pi iz/\omega}) = f(z).$$

Note F is well-defined as f has period ω, and is meromorphic since f was assumed meromorphic. Then if D' contains an annulus $r_1 < |\zeta| < r_2$ for which F has no poles, then in this region F has a Laurent expansion

$$F(\zeta) = \sum_{n=-\infty}^{\infty} c_n \zeta^n$$

or equivalently

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inz/\omega},$$

valid in the horizontal strip in $\mathbb{C}$ which is the preimage of our annulus under the exponential. The coefficients c_n are given by

$$c_n = \frac{1}{2\pi i} \int_{|\zeta|=r} F(\zeta)\zeta^{-(n+1)} d\zeta$$

where the region of integration may be any circle with radius r with $r_1 < r < r_2$. (If you've seen complex integration and Cauchy's theorem before, then you know that the integral is independent of our choice of r owing to the assumption that there are no poles in the annulus.) Translating back via change of variables, we may write

$$c_n = \frac{1}{\omega} \int_a^{a+\omega} f(z) e^{-2\pi i n z/\omega} dz$$

where we may take a to be any point in the horizontal strip and the integration over any path from a to $a+\omega$ within the strip. In particular, if the initial function $f(z)$ was analytic in the whole complex plane, then this Fourier series description is valid for all points $z \in \mathbb{C}$.

But why settle for the simple periodicity property? Let's demand that our functions be periodic with respect to an arbitrary set of periods S. This set has structure, since if $\omega_1, \omega_2 \in S$ then by definition of a period, $m\omega_1 + n\omega_2 \in S$ for any $m, n \in \mathbb{Z}$ (for algebra lovers, this means S is a $\mathbb{Z}$-module).

Exercise 1: Determine what happens when you allow arbitrary sets of periods for functions of a real variable. (Hint: You might want to consider cases according to the difference of any two periods in your set S, and you'll probably want to assume your functions are "sufficiently nice" as in the assumptions for real variable Fourier theory above.)

For functions of a complex variable, the extra dimension gives a new collection of periodic functions we haven't yet mentioned. Meromorphic functions of a complex variable are quite restricted, and in particular, any such function which is constant on a sequence of complex numbers converging to a limit point must be the constant function. Thus, the set of periods S must not contain such an accumulation point, or else the only such periodic function will be constant. We call such a module without an accumulation point "discrete."

Theorem 1 *A discrete $\mathbb{Z}$-module M consists of either 0 alone, integer multiples of a single non-zero complex number ω, or all linear combinations $\{m\omega_1 + n\omega_2\}$ for a pair of complex numbers ω_1, ω_2 with $\omega_2/\omega_1 \notin \mathbb{R}$.*

Proof If $M \neq \{0\}$, then we may choose an element $\omega_1 \in M$ with smallest positive absolute value (there may be several with the same norm, and in fact you can show the only possibilities are 2,4, or 6, but no matter). Then $\{n\omega_1 \mid n \in \mathbb{Z}\}$ is contained in M.

Suppose there exists a complex number $\omega \in M$ which is not an integer multiple of ω_1. Again, take ω_2 to be the complex number of smallest absolute value with this property. If ω_2/ω_1 real, then there exists an integer n such that

$$n < \omega_2/\omega_1 < n + 1 \ \Rightarrow\ 0 < |n\omega_1 - \omega_2| < |\omega_1|$$

which contradicts the fact that ω_1 was chosen to have smallest positive absolute value. So it just remains to show that, assuming such an ω_2 exists, $M = \{n\omega_1 + m\omega_2 \mid m, n \in \mathbb{Z}\}$.

First, we show any complex number ω can be written in the form $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ with λ_1, λ_2 real. Equivalently, we seek a solution (λ_1, λ_2) to the system of equations

$$\omega = \lambda_1\omega_1 + \lambda_2\omega_2\bar{\omega} = \lambda_1\bar{\omega}_1 + \lambda_2\bar{\omega}_2$$

Because ω_2/ω_1 is not real, this system is non-singular (as functions of the two complex variables λ_1, λ_2). That is, the determinant $\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1 \neq 0$. So there is a unique solution in complex numbers λ_1, λ_2. On the other hand, both pairs (λ_1, λ_2) and $(\bar{\lambda}_1, \bar{\lambda}_2)$ are visibly solutions, so λ_1 and λ_2 must be real.

Given any $\omega \in M$, we finally show that we can find $k, n \in \mathbb{Z}$ with $\omega - k\omega_1 - n\omega_2 = 0$. Given any $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$, with λ_i real, we can find m, n with

$$|\lambda_1 - m| \leq 1/2, \quad |\lambda_2 - n| \leq 1/2$$

Consider

$$\omega' = \omega - m\omega_1 - n\omega_2.$$

Then $|\omega'| < \frac{1}{2}|\omega_1| + \frac{1}{2}|\omega_2| \leq |\omega_2|$, where importantly the first inequality is strict, since ω_2 is not a real multiple of ω_1 (giving a degenerate version of the triangle inequality, as the quotient being real means they lie on a line in the complex plane). This inequality implies ω' is an integral multiple of ω_1 since we chose ω_2 as above. $\square$

We now begin our study of doubly periodic complex functions – functions whose period module is a two-dimensional integer lattice.

2 Relations among Period Lattices

We're assuming that our period module is generated by a pair of complex numbers $\{\omega_1, \omega_2\}$, so that any period $\omega \in M$ can be uniquely written in the form $\omega = m\omega_1 + n\omega_2$ for some $m, n \in \mathbb{Z}$, i.e. ω_1, ω_2 form a basis for our $\mathbb{Z}$-module.

Given another basis (ω_1', ω_2'), then since (ω_1, ω_2) is a basis, we may write:

$$\begin{aligned}\omega_2' &= a\omega_2 + b\omega_1 \\ \omega_1' &= c\omega_2 + d\omega_1\end{aligned}$$

for some integers a, b, c, d. Note the same relations apply upon taking complex conjugates of both sides, and this allows us to formulate an identity entirely in terms of 2×2 matrices:

$$\begin{pmatrix}\omega_2' & \bar{\omega}_2' \\ \omega_1' & \bar{\omega}_1'\end{pmatrix} = \begin{pmatrix}a & b \\ c & d\end{pmatrix}\begin{pmatrix}\omega_2 & \bar{\omega}_2 \\ \omega_1 & \bar{\omega}_1\end{pmatrix}$$

We can similarly express ω_1 and ω_2 in terms of the new basis (ω_1', ω_2') to obtain:

$$\begin{pmatrix}\omega_2 & \bar{\omega}_2 \\ \omega_1 & \bar{\omega}_1\end{pmatrix} = \begin{pmatrix}a' & b' \\ c' & d'\end{pmatrix}\begin{pmatrix}\omega_2' & \bar{\omega}_2' \\ \omega_1' & \bar{\omega}_1'\end{pmatrix}$$

for some integers a', b', c', d'. Thus, substituting,

$$\begin{pmatrix}\omega_2 & \bar{\omega}_2 \\ \omega_1 & \bar{\omega}_1\end{pmatrix} = \begin{pmatrix}a' & b' \\ c' & d'\end{pmatrix}\begin{pmatrix}a & b \\ c & d\end{pmatrix}\begin{pmatrix}\omega_2 & \bar{\omega}_2 \\ \omega_1 & \bar{\omega}_1\end{pmatrix}.$$

Just as in the proof of the above theorem, knowing that the ratio ω_2/ω_1 is not real implies that the determinant $\omega_2\bar{\omega}_1 - \bar{\omega}_2\omega_1$ is non-zero, and thus the matrix on the left-hand side above is invertible. Multiplying on the right by the inverse yields:

$$\begin{pmatrix}a' & b' \\ c' & d'\end{pmatrix}\begin{pmatrix}a & b \\ c & d\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$$

Because determinants are multiplicative, our matrices above must have determinant ± 1. We call such matrices "unimodular" since, as you may recall from linear algebra, the determinant's absolute value measures the factor by which volume increases under the associated linear transformation. In terms of the period lattice, it means that the parallelogram formed by $\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$ has the same area for any basis ω_1, ω_2.

3 Canonical Bases

We can use the action of unimodular integer matrices to try to obtain canonical representatives for our period lattice. The next theorem says that we can almost do this uniquely.

Theorem 2 *There exists a basis (ω_1, ω_2) with ratio $\tau = \omega_2/\omega_1$ satisfying the following conditions:*

1. *Im*$(\tau) > 0$
2. $\frac{-1}{2} < Re(\tau) \leq \frac{1}{2}$
3. $|\tau| \geq 1$
4. *Re*$(\tau) \geq 0$ *if* $|\tau| = 1$

This determines τ *uniquely, which in turn, determines either 2, 4, or 6 possibilities for* (ω_1, ω_2).

Note this domain for τ is essentially the one we defined earlier as a fundamental domain for the action of $SL(2,\mathbb{Z})$ on the upper half plane, and we've made a choice in condition (4) according to equivalent points in this domain. However, strictly speaking, the group we're working with here is $GL(2,\mathbb{Z})$, remembering that the determinant of these integer matrices must be ± 1 in order that the inverse matrix remains integral.

Proof Choosing ω_1 and ω_2 as in the proof of Theorem 1, we have

$$|\omega_1| \leq |\omega_2|, \quad |\omega_2| \leq |\omega_1 + \omega_2|, \quad |\omega_2| \leq |\omega_1 - \omega_2|$$

which in turn implies that $|\tau| \geq 1$ and $|\mathrm{Re}(\tau)| \leq \frac{1}{2}$. By choice of $(-\omega_1, \omega_2)$ or (ω_1, ω_2), we may guarantee that $\mathrm{Im}(\tau) > 0$. Finally, if $\mathrm{Re}(\tau) = -\frac{1}{2}$, replace (ω_1, ω_2) by $(\omega_1, \omega_1 + \omega_2)$ and if $|\tau| = 1$ with $\mathrm{Re}(\tau) < 0$ then replace (ω_1, ω_2) by $(-\omega_2, \omega_1)$. Thus τ will have all required properties of the theorem. (Note, in particular, that none of these subsequent changes to the basis mess up earlier desired properties of τ.)

It remains to show that these conditions uniquely determine τ (and hence finitely many choices of basis as indicated). Indeed, given any other basis (ω_1', ω_2'), then one readily checks $\tau' = \omega_2'/\omega_1'$ satisfies

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = \pm 1.$$

This gives

$$\mathrm{Im}(\tau') = \frac{\pm \mathrm{Im}(\tau)}{|c\tau + d|^2} \quad \text{with sign in the numerator matching } ad - bc.$$

It suffices to show that if τ and τ' satisfy the four conditions, then they must be equal. Condition (1) implies that the sign in the numerator is positive, so $ad - bc = 1$.

Without loss of generality, we may assume $\mathrm{Im}(\tau') \geq \mathrm{Im}(\tau)$ since τ and τ' play symmetric roles in the above equation. This implies $|c\tau + d| \leq 1$ and this greatly reduces the possible integer c, d.

Exercise: Work out the remaining possible choices of c, d for which τ and τ' are in the fundamental domain and show that each implies $\tau = \tau'$. You should be able to reduce to the two cases $c = 0$ and $|c| = 1$ first, and then treat these cases separately.

To finish, note we can always replace (ω_1, ω_2) by $(-\omega_1, -\omega_2)$ without changing τ. This gives at least two choices of basis for any value of τ.

Exercise: Using your work in the previous exercise, determine the values of τ for which there are 2,4, or 6 possible choices of basis and explain your answer.

These two exercises complete the proof of the theorem.

$\square$

4 Properties of Complex Elliptic Functions

In this section, we use basic results from a first course in complex analysis to elucidate properties of elliptic functions. As complex analysis is not a prerequisite for the course, the reader may happily take results in this section on faith, or consult any standard reference on the subject.

Let $f(z)$ be a meromorphic function with period lattice M generated by (ω_1, ω_2). We will write $z_1 \sim z_2$ for two complex numbers z_1, z_2 if $z_1 - z_2 \in M$. In particular, the value of f is constant on equivalence classes defined by this relation. Note that any such elliptic $f(z)$ is completely determined by its values in any parallelogram $P(z_0)$ with vertices $\{z_0, z_0 + \omega_1, z_0 + \omega_2, z_0 + \omega_1 + \omega_2\}$ for any complex number z_0.

Proposition 1 *An elliptic function without poles is constant.*

Proof If $f(z)$ has no poles, it is bounded on the closed, compact domain $P(z_0)$ for any z_0, and hence bounded on the entire complex plane. Then we may apply Liouville's theorem (cf. Ahlfors, Ch. 4, sect. 2.3), a consequence of complex integration theory, which states that any analytic function bounded on the entire plane is constant. $\square$

Proposition 2 *The sum of residues of an elliptic function is 0.*

Here, we mean the sum of residues at a complete set of representatives for inequivalent poles of $f(z)$. Note that to any fundamental parallelogram $P(z_0)$, there can be at most finitely many poles (otherwise the poles would necessarily have an accumulation point in this compact region, hence f would not be meromorphic), so this set is finite.

Proof Since the number of poles in $P(z_0)$ is finite for any z_0, we may choose z_0 so that $P(z_0)$ has no poles lying on the boundary. Let ∂P denote this boundary. Then Cauchy's integral theorem states that the sum of the residues at poles is given by

$$\frac{1}{2\pi i}\int_{\partial P} f(z)dz$$

where we travel along the boundary with respect to a choice of positive orientation. But then we travel in opposite directions on parallel edges of the boundary, while the value of the function is identical on parallel edges, so their integrals along each pair of edges cancel, giving 0. □

Corollary 1 *There are no elliptic functions with a single simple pole.*

Proposition 3 *A non-constant elliptic functions has equally many inequivalent poles as inequivalent zeros.*

Proof This is a common trick in complex function theory – to consider zeros and poles of a function, with multiplicity, consider the same integral as in the proof above, but replace the integrand $f(z)$ by the logarithmic derivative $\frac{d}{dz}\log(f(z)) = \frac{f'(z)}{f(z)}$. The same argument as above shows this integral is 0 over ∂P as well. □

Interestingly, since $f(z)$ and $f(z) - c$ have the same poles for any value of c (and yet the zero sets change), the above result implies that the number of incongruent solutions to $f(z) = c$ is a constant independent of c. This is sometimes called the "order" of the elliptic function. (Is this true for periodic functions of a real variable?)

Proposition 4 *Let* $\{p_1, \ldots, p_n\}$ *be the poles of an elliptic function* $f(z)$ *and let* $\{z_1, \ldots, z_n\}$ *be the zeros of* $f(z)$*. Then*

$$p_1 + \cdots + p_n \sim z_1 + \cdots + z_n$$

Proof We consider the same contour integral as in the previous proofs, choosing P so that it doesn't contain poles on the boundary. But now we consider a third option for the integrand:

$$\frac{1}{2\pi i}\int_{\partial P} \frac{zf'(z)}{f(z)}dz$$

Again, remembering that the Cauchy integral theorem says the value of this integral is the sum of the poles of this integrand, then this integral will be

$$z_1 + \cdots + z_n - p_1 - \cdots - p_n$$

provided we choose the representatives to lie inside P. It remains to show this integral evaluates to a member of M. Consider the integral over a pair of parallel edges:

$$\frac{1}{2\pi i}\left[\int_{z_0}^{z_0+\omega_1} \frac{zf'(z)}{f(z)}dz - \int_{z_0+\omega_2}^{z_0+\omega_1+\omega_2} \frac{zf'(z)}{f(z)}dz\right].$$

Setting $z \mapsto z + \omega_2$ in the second integral (remembering $f(z) = f(z + \omega_2)$) this simplifies to

$$-\frac{\omega_2}{2\pi i}\int_{z_0}^{z_0+\omega_1} \frac{zf'(z)}{f(z)}dz.$$

But upon factoring out the ω_2, the remaining integral is just the same integral we examined before, which counts the numbers of zeros and poles in P, and so is in particular an integer. Hence, this pair of parallel edge integrals evaluated to $m\omega_2$ for some $m \in \mathbb{Z}$. A similar calculation for the other two parallel edge integrals yields $n\omega_1$ for some $n \in \mathbb{Z}$, thereby completing the proof. $\square$

5 The Weierstrass $\wp$ Function

In Latex, the $\wp$ in the Weierstrass function can be typeset by $\backslash wp$. It's the name we give to the simplest non-trivial elliptic function. As we saw in the previous section, there are no non-trivial elliptic functions of order 0 (no poles, so constant) or order 1 (single simple pole, contradicting Corollary 1 of the previous section). So we seek an elliptic function of order 2, which can have either a double pole inside any parallelogram $P(z_0)$ with residue 0, or two simple poles with residues cancelling. (Again, we know the residues sum to 0 by Proposition 2.)

Suppose our elliptic function has a double pole, which we may take to be at the origin. Further normalize so that the Laurent series at the origin begins $\frac{1}{z^2} + c_0 + c_1 z + \cdots$ by dividing by a constant if necessary. As we noted in the last section, if $f(z)$ is elliptic, so is $f(z) + c$, so we may further assume that $c_0 = 0$. Lastly, supposing that our function $\wp(z)$ has a double pole at the origin and is otherwise analytic, then $\wp(z) - \wp(-z)$ is a holomorphic function with the same period module M, and is therefore a constant by Proposition 1. To determine this constant, pick for example $z_0 = \omega_1/2$, then $z_0 \sim -z_0$ and $\wp(z_0) - \wp(-z_0) = 0$, so $\wp(z) = \wp(-z)$ for all $z \in \mathbb{C}$. That is, $\wp(z)$ is an even function.

So far, we know that if such an order 2 function exists, with double pole at the origin, then its Laurent expansion about the origin has form

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + a_6 z^6 + \cdots$$

We will show the following formula:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in M, \omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

In other words, given any period module M, we may associate an elliptic function $\wp(z)$ of order 2 to M as above (though we suppress the dependence on M in the notation). Note this is not a Laurent expansion, but just an alternate representation for the function. It's also the simplest one we can imagine, given that each point in the lattice of periods M must be a pole of order 2. The subtraction of each $\frac{1}{\omega^2}$ plays two roles: it will give convergence of the series (convince yourself the series diverges otherwise) and gives the right limiting behavior as z approaches 0 according to our Laurent series above.

Lemma 1 *The Weierstrass $\wp$ function defined above converges uniformly on any compact set.*

Proof Indeed

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{z(2\omega - z)}{\omega^2 (z-\omega)^2} \right|$$

and to demonstrate convergence, we want to bound this summand when $|\omega|$ is sufficiently large. A nifty way to do this is to assume that for a given z, $|\omega| > 2|z|$. This implies by reverse triangle inequality that $|z - \omega| > \frac{1}{2}|\omega|$. Hence,

$$\left| \frac{z(2\omega - z)}{\omega^2 (z-\omega)^2} \right| \leq \left| \frac{z\omega}{\omega^2 (z-\omega)^2} \right| + \left| \frac{z(\omega - z)}{\omega^2 (z-\omega)^2} \right| < \frac{6|z|}{|\omega|^3}.$$

This implies that if z is restricted to any compact set, then the series will converge uniformly, provided we can show that

$$\sum_{\omega \in M, \omega \neq 0} \frac{1}{|\omega|^3}$$

converges. To do this, we use the form of the period lattice to compare this sum to special values of the zeta function. That is, since ω_2/ω_1 is not real, so there exists a constant $k > 0$ for which

$$|m\omega_1 + n\omega_2| \geq k(|m| + |n|) \qquad \text{for all real pairs } (m, n).$$

But there are precisely $4N$ pairs (m, n) with $|m| + |n| = N$. Hence,

$$\sum_{\omega \in M, \omega \neq 0} \frac{1}{|\omega|^3} \leq 4k^{-3} \sum_{N=1}^{\infty} \frac{1}{N^2} < 8k^{-3}.$$

$\square$

We still have not shown that $\wp(z)$ is periodic with respect to M. An elegant way to show this is to note that, because $\wp(z)$ converges uniformly, we are permitted to do term-wise differentiation to conclude

$$\wp'(z) = -\frac{2}{z^3} - \sum_{\omega \neq 0} \frac{2}{(z-\omega)^3} = -2 \sum_{\omega \in M} \frac{1}{(z-\omega)^3}.$$

This shows $\wp'(z)$ is doubly periodic, as the sum is clearly invariant under translation by points in M (again, uniform convergence ensures we may rearrange our sum). This implies that the derivative of the functions $F_1(z) = (\wp(z+\omega_1) - \wp(z))$ and $F_2(z) = (\wp(z+\omega_2) - \wp(z))$ are 0 and hence F_1 and F_2 are constant. Taking $z = -\omega_1$ in F_1 and $z = -\omega_2$ in F_2 shows these constants are in fact 0.

Note that in the process of verifying that $\wp(z)$ has period module M, we also proved that $\wp'(z)$ is an odd elliptic function of order 3.

6 The Antiderivative of $\wp(z)$

The anti-derivative of $\frac{1}{z}$ is $\log(z)$, which is a multi-valued function. This basic example illustrates that care must be taken in interpreting anti-derivatives. However, in our case, $\wp(z)$ has residue 0 at each of its double poles, so by general theorem of complex analysis, its antiderivative is a single-valued function.

Following historical tradition, we let $-\zeta(z)$ denote the anti-derivative of $\wp(z)$, where

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in M, \omega \neq 0} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right) \tag{1}$$

which we may regard as obtained (for all but the $\frac{1}{z}$ term) by term-by-term integration from 0 to z along any path not containing poles, and hence converges.

Just as we remarked above, the periodicity of $\wp(z)$ implies $\zeta(z+\omega_1)-\zeta(z)=\eta_1$ and $\zeta(z+\omega_2)-\zeta(z)=\eta_2$ for some complex constants η_1, η_2. Note that $\zeta(z)$ is an order 1 function, so

$$\frac{1}{2\pi i}\int_{\partial P}\zeta(z)dz = 1.$$

Evaluating this integral explicitly using pairs of parallel sides of P, we find

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$$

so $\zeta(z)$ is not a periodic function, but nearly so. If we try to take another antiderivative, we see that the $\frac{1}{z}$ in $\zeta(z)$ will lead to a multi-valued function. But we can remove this indeterminacy by composing the result with the exponential function.

Exercise: Show that the result of this process of taking an antiderivative of $\zeta(z)$ and composing with e^z results in the function:

$$\sigma(z) = z\prod_{\omega\neq 0}\left(1-\frac{z}{\omega}\right)e^{z/\omega+\frac{1}{2}(z/\omega)^2}$$

Further show that $\sigma(z)$ is an entire function (i.e. analytic everywhere) satisfying

$$\sigma'(z)/\sigma(z) = \zeta(z).$$

Use this result to prove relationships between $\sigma(z+\omega_1)$ and $\sigma(z)$ and between $\sigma(z+\omega_2)$ and $\sigma(z)$.

7 A differential equation for $\wp(z)$

The expression for $\zeta(z)$ in (1) is useful to us because unlike that for $\wp(z)$, it can be easily translated into a Laurent expansion for $\zeta(z)$ at 0. Each summand

$$\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2} = -\frac{z^2}{\omega^3}-\frac{z^3}{\omega^4}-\cdots$$

after expanding the first term on the right-hand side in a geometric series. After recollecting terms in $\zeta(z)$, we may write:

$$\zeta(z) = \frac{1}{z}-\sum_{k=2}^{\infty}\left(\sum_{\omega\in M,\omega\neq 0}\frac{1}{\omega^{2k}}\right)z^{2k-1}. \tag{2}$$

Note that the terms z^{2k} have cancelled out in the sum over M. Let's define

$$G_k = \sum_{\omega\in M,\omega\neq 0} \frac{1}{\omega^{2k}} = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1+n\omega_2)^{2k}} = \frac{1}{(\omega_1)^{2k}} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}},$$

which realizes (up to a constant factor) an earlier definition we had for Eisenstein series of weight $2k$. Note our change in perspective, though: previously, we were considering these Eisenstein series as functions on the upper half plane in the variable τ. Here, we want to think of the period lattice M as being fixed, so that G_k is just a complex number.

Differentiating term-wise in (2) then gives a Laurent expansion for $\wp(z)$ and $\wp'(z)$ (remembering that $\wp(z) = -\zeta'(z)$) of form:

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty}(2k-1)G_k z^{2k-2} \tag{3}$$

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=2}^{\infty}(2k-1)(2k-2)G_k z^{2k-3} \tag{4}$$

Since both $\wp(z)$ and $\wp'(z)$ are elliptic, so are all polynomial expressions in the two functions. Coupled with the fact that any entire elliptic function is constant, then any polynomial in $\wp(z)$ and $\wp'(z)$ whose Laurent expansion has only positive powers of z must, in fact, be 0.

We begin by cancelling off the largest negative powers in $\wp(z)$ and $\wp'(z)$:

$$\wp'(z)^2 - 4\wp(z)^3 = -60\frac{G_2}{z^2} - 140G_3 + O(z).$$

Hence,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2\wp(z) + 140G_3 = 0.$$

In particular, $w = \wp(z)$ is the solution to the above first-order differential equation

$$(w')^2 = 4w^3 - 60G_2 w - 140G_3.$$

This differential equation can also be solved explicitly by the indefinite integral

$$z = \int \frac{dw}{\sqrt{4w^3 - 60G_2 w - 140G_3}},$$

and so $\wp(z)$ may be realized as the inverse to this elliptic integral.

Moreover, considering $y = \wp'(z)$ and $x = \wp(z)$ as complex variables, we see how our period lattice M may be associated to the elliptic curve:

$$y^2 = 4x^3 - 60G_2 x - 140G_3.$$

References

[1] Lars Ahlfors, *Complex Analysis*, 3rd. ed., McGraw-Hill (1979).