Power series for up-down min-max permutations

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Abstract

Calculus and combinatorics overlap in that power series can be used to study combinatorially defined sequences. In this note we use exponential generating functions to study a curious refinement of the Euler numbers, which count the number of "up-down" permutations of length n. Students of calculus are familiar with the *Maclaurin series* of a function of a real variable:

$$f(x) = \sum_{n \ge 0} a_n \frac{x^n}{n!} = a_0 + a_1 x + a_2 \frac{x^2}{2} + a_3 \frac{x^3}{6} + \cdots,$$

where a_n is the *n*th derivative of f evaluated at x = 0.

What students of calculus may be less familiar with is the idea of a generating function for a sequence of numbers, whereby the sequence of coefficients is used to define the function. That is, a generating function for a sequence a_0, a_1, a_2, \ldots is a way of encoding the sequence algebraically, as a formal power series. This approach has been used extensively in enumerative combinatorics; Herb Wilf's wonderful book [6] provides a good introduction to the theory. In this article, we will study a new variant of a classical problem in enumerative combinatorics, highlighting the power of the generating function approach.

1 Generating functions

There are two basic types of generating functions, *ordinary* and *exponential*, corresponding to

$$\sum_{n\geq 0} a_n x^n$$
 and $\sum_{n\geq 0} a_n \frac{x^n}{n!}$,

respectively.

To take a simple example, consider the sequence 1, 1, 1, ... Define its ordinary generating function to be

$$F(x) = \sum_{n \ge 0} x^n.$$

We notice

$$F(x) = 1 + x + x^{2} + x^{3} + \dots = 1 + x(1 + x + x^{2} + \dots) = 1 + xF(x).$$

Thus, (1-x)F(x) = 1, and so it must be that F(x) = 1/(1-x), as expected.

Note that this approach is very different from how the geometric series is introduced in a calculus class. By defining the generating function as a formal series, we neatly sidestep any question about its existence. We also don't have to worry about convergence questions ("Is |x| < 1?") because for us, x is a symbol, not a real number. As Wilf [6, p. 7] puts it, "the *analytic* nature of the generating function doesn't interest us; we love it only as a clothesline on which our sequence is hanging out to dry."

2 Alternating permutations

The first sequence we'd like to "hang out to dry" is the sequence E_0, E_1, E_2, \ldots formed by interleaving the terms in the power series expansions for secant and tangent. Expanded

individually, we have

$$\sec x = \sum_{n \ge 0} E_{2n} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + 5\frac{x^4}{24} + 61\frac{x^6}{720} + \cdots$$

and

$$\tan x = \sum_{n \ge 0} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} = x + 2\frac{x^3}{6} + 16\frac{x^5}{120} + \cdots$$

Thus, $E(x) = \sec x + \tan x$ is the exponential generating function for the sequence

$$1, 1, 1, 2, 5, 16, 61, \ldots$$

which is sometimes known as the sequence of *Euler numbers* (hence the notation E_n), though in number theory the term "Euler numbers" sometimes just refers to the even terms. The sequence is number A000111 in the On-Line Encyclopedia of Integer Sequences [4].¹

For a combinatorialist, the first question we'd like to answer when given a sequence of positive integers is: "What do they count?" Well, in the case of the Euler numbers, the answer to this question has been known since at least 1881, when the Belgian mathematician Désiré André interpreted them as counting *alternating permutations* [1, 2]. Alternating permutations have re-emerged in various parts of algebraic and topological combinatorics in recent decades, as discussed in a survey article by Richard Stanley [5].

Nowadays, when people say "alternating permutation" they usually mean either *up-down alternating* or *down-up alternating*. The terms refer to how the permutation appears in one-line notation, e.g., 3517264 is an up-down permutation because the numbers alternately increase and decrease, whereas 5371624 is a down-up permutation because the numbers alternately decrease and increase.

There is a simple correspondence between up-down alternating and down-up alternating permutations: for any up-down permutation of $\{1, 2, ..., n\}$ replacing the letter k with n+1-k will yield a corresponding down-up permutation (and vice-versa). Notice the permutations 3517264 and 5371624 can be obtained from each other in this way. Thus the set of alternating permutations splits evenly into these two types and André's result says:

 $E_n = |\{\text{up-down permutations of length } n\}|,$ $= |\{\text{down-up permutations of length } n\}|.$

That is, $E(x) = \sec x + \tan x$ is the exponential generating function for the number of up-down (or down-up) permutations.

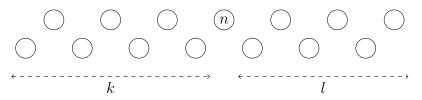
¹This database, founded by Neil Sloane, is an indispensible resource for the study of enumerative combinatorics.

As an exercise, let's see how to derive this result combinatorially.

Our starting point is to define E_n as the number of up-down alternating permutations of length n (with $E_0 = 1$ for convenience), and to define the exponential generating function for this sequence:

$$E(x) = \sum_{n \ge 0} E_n \frac{x^n}{n!}.$$

Now consider how to create one of the $2E_n$ alternating permutations (up-down or down-up) of length n. First suppose that we put n in postion k + 1. Then to the left of n we will have an alternating permutation of length k that ends in a down step, and to the right of n we will have an up-down permutation of length l, where k + l = n - 1. We can sketch the situation like this:



Note that while we have drawn an up-down alternating permutation, it could also be down-up. It depends on whether k is even or, as in this case, k is odd.

To fill in the rest of the picture, we need to do two things:

- 1. choose the k elements that go to the left of n and arrange them as an alternating permutation, and
- 2. arrange the remaining l elements to the right of n as an alternating permutation.

Step 1 can be done in $\binom{n-1}{k}E_k$ ways. Indeed, we are choosing k of n-1 elements, and these can be ordered in E_k ways, according to the reversal of any up-down permutation of length k. For step 2, we also want to form an up-down permutation, this time written left to right. Since there are l elements to the right of n, this can be done in E_l ways.

Now we can conclude that there are $\binom{n-1}{k}E_kE_l$ alternating permutations with n in position k+1. Summing over all k, we find the total number of alternating permutations of length n (down-up or up-down) is:

$$2E_{n} = \sum_{\substack{k,l \ge 0 \\ k+l=n-1}} {\binom{n-1}{k}} E_{k}E_{l}$$

= $\sum_{\substack{k,l \ge 0 \\ k+l=n-1}} \frac{(n-1)!}{k!l!} E_{k}E_{l}$
= $(n-1)! \sum_{\substack{k,l \ge 0 \\ k+l=n-1}} \frac{E_{k}}{k!} \frac{E_{l}}{l!}$ (for $n \ge 2$). (1)

Next, we will multiply both sides of (1) by $x^{n-1}/(n-1)!$ and sum this recurrence over all $n \ge 2$. We get:

$$\sum_{n\geq 2} 2E_n \frac{x^{n-1}}{(n-1)!} = \sum_{n\geq 2} \left(\sum_{k+l=n-1} \frac{E_k}{k!} \frac{E_l}{l!} \right) x^{n-1}$$
$$= \sum_{n\geq 2} \sum_{k+l=n-1} \left(E_k \frac{x^k}{k!} \right) \left(E_l \frac{x^l}{l!} \right)$$
$$= \left(\sum_{k\geq 0} E_k \frac{x^k}{k!} \right) \left(\sum_{l\geq 0} E_l \frac{x^l}{l!} \right) - 1$$
$$= E(x)^2 - 1,$$

where the second to last identity comes from thinking of the index n as simply a way to keep track of the total degree in the expansion of the product. The only term missing from the product was $E_0^2 = 1$. So we have

$$E(x)^{2} - 1 = 2 \sum_{n \ge 2} E_{n} \frac{x^{n-1}}{(n-1)!}.$$
(2)

But notice

$$E'(x) = \sum_{n \ge 1} E_n \frac{x^{n-1}}{(n-1)!},$$

so the right-hand side of (2) is 2(E'(x) - 1). Therefore we can write:

$$E(x)^2 + 1 = 2E'(x).$$

It easy to check that taking $E(x) = \sec x + \tan x$ satisfies this differential equation with initial condition E(0) = 1. This completes the derivation of André's generating function for up-down permutations.

3 A new refinement of Euler numbers

In a short note [3], David Callan suggested partitioning the alternating permutations in a new way. Rather than consider whether an alternating permutation is down-up or up-down, he considered the relative positions of 1 and n. We call a permutation *min-max* if the number 1 appears to the left of n; if it appears to the right of n, we say it is *max-min*.

There is a correspondence between min-max alternating permutations and max-min alternating permutations through reversal of permutations. For example, the min-max permutation 153624 becomes the max-min permutation 426351, while 3517264 becomes 4627153. (Notice that this reversal does not preserve up-downness in the even length

case, but does in the odd.) Thus, there are the same number of min-max alternating permutations as there are max-min alternating permutations. Since the total number of alternating permutations is $2E_n$, this implies that E_n counts the number of min-max alternating permutations.

				max-min		min-max	
n = 4	max-min	min-max					
up-down	$2413 \\ 3412$	1324 1423 2314	up-down	$\begin{array}{c} 35241 \\ 35142 \\ 34251 \\ 35241 \end{array}$	45231 45132 24351 25341	$14253 \\ 24153 \\ 15243 \\ 14253$	$13254 \\ 23154 \\ 15342 \\ 14352$
down-up	4231 4132 3241	$3142 \\ 2143$	down-up	$\begin{array}{r} 42513 \\ 43512 \\ 52314 \\ 32514 \end{array}$	51423 53412 51324 52413	$31524 \\ 21534 \\ 41325 \\ 41523$	$32415 \\ 21435 \\ 42315 \\ 31425$

Table 1: Alternating permutations for n = 4 and n = 5 organized by the properties max-min/min-max and up-down/down-up.

But what if we combine Callan's idea with the classical one? This partitions alternating permutations into four subsets according to whether a permutation is up-down or down-up and according to whether it is min-max or max-min. We see these subsets shown in Table 1 for n = 4 and n = 5.

Let E_n^{\times} denote the the number of up-down min-max permutations of n (the arrow \nearrow points from 1 to n), and let E_n^{\times} denote the number of up-down max-min permutations of n. Notice that

$$E_n = E_n^{\checkmark} + E_n^{\backsim}$$

We include the first few values of these sequences and some related quantities in Table 2. The sequences E_n^{\times} and E_n^{\times} do not appear in the On-Line Encyclopedia of Integer Sequences [4] and thus are unlikely to have been studied before.

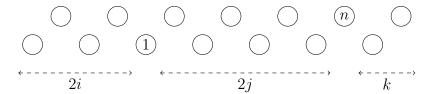
We can see from Table 2 that the numbers E_n^{\checkmark} and E_n^{\checkmark} are roughly equal. In fact, for odd n, they are exactly equal. (This is because reversal gives a bijection between updown min-max permutations and up-down max-min permutations.) In the even case, we find it slightly more likely that an up-down permutation is min-max than max-min. But notice something: the difference between the number of min-max up-down permutations and max-min up-down permutations is an Euler number! This will turn out to imply that, as $n \to \infty$, the ratio of min-max up-down permutations and max-min up-down permutations tends to 1.

n	2	3	4	5	6	7	8	9	10
E_n	1	2	5	16	61	272	1385	7936	50521
E_n^{\checkmark}	1	1	3	8	33	136	723	3968	25953
$E_n^{``}$	0	1	2	8	28	136	662	3968	24568
$E_n^{\checkmark} - E_n^{\backsim}$	1	0	1	0	5	0	61	0	1385
E_n E_n^{\checkmark} E_n^{\backsim} $E_n^{\checkmark} - E_n^{\backsim}$ $\frac{E_n^{\backsim}}{E_n^{\checkmark}}$	0	1	$\frac{2}{3}$	1	.737	1	.916	1	.947

Table 2: Number of up-down min-max/max-min permutations up to n = 10

4 New generating functions

All of the facts just mentioned will be quickly deduced once we have found generating functions for E_n^{\times} and E_n^{\times} . Just as we did for the case of all alternating permutations, we will do this by first considering how to build a min-max up-down permutation. Since it must be min-max, we will write a 1 somewhere to the left of n, and we will put a bunch of blank spaces in for the rest of the numbers. We will also draw it in a way that reminds us that the whole thing better be up-down alternating. We end up with a picture like this:



Now to fill in the rest of the picture, we need to do three things:

- 1. choose the elements that go to the left of 1 and arrange them as an up-down permutation of even length,
- 2. choose the elements that go between 1 and n and arrange them as a down-up permutation of even length, and
- 3. arrange the remaining elements as an up-down permutation to the right of n.

Step 1 can be done in $\binom{n-2}{2i}E_{2i}$ ways, step 2 can be done in $\binom{n-2-2i}{2j}E_{2j}$ ways, and, letting k = n - 2 - 2i - 2j, step 3 can be done in E_k ways. Thus, for a fixed *n*, the number of

min-max up-down permutations is:

$$E_{n}^{\prime} = \sum_{\substack{i,j,k\geq 0\\2i+2j+k=n-2}} {\binom{n-2}{2i}} E_{2i} {\binom{n-2-2i}{2j}} E_{2j} E_{k}$$
$$= (n-2)! \sum_{\substack{i,j,k\geq 0\\2i+2j+k=n-2}} \frac{E_{2i}}{(2i)!} \frac{E_{2j}}{(2j)!} \frac{E_{k}}{k!}$$
(3)

Now let's formally write down the generating function we want. Because of the (n-2)! that cropped up above, and the fact that the notion of "min-max" doesn't make sense if n < 2, it makes sense to shift the coefficients of our generating function two steps and define:

$$E^{\not(x)} = \sum_{n \ge 0} E_{n+2}^{\not(x)} \frac{x^n}{n!} = 1 + x + 3\frac{x^2}{2} + 8\frac{x^3}{6} + 33\frac{x^4}{24} + \cdots.$$

Now we can use (3) to manipulate the series as follows:

$$E_n^{\nearrow}(x) = \sum_{n \ge 0} E_{n+2}^{\nearrow} \frac{x^n}{n!}$$

= $\sum_{n \ge 0} \left(\sum_{\substack{0 \le i, j, k \\ 2i+2j+k=n}} \frac{E_{2j}}{(2j)!} \frac{E_{2k}}{(2k)!} \frac{E_l}{l!} \right) x^n$
= $\sum_{i \ge 0} E_{2i} \frac{x^{2i}}{(2i)!} \cdot \sum_{j \ge 0} E_{2j} \frac{x^{2j}}{(2j)!} \cdot \sum_{k \ge 0} E_k \frac{x^k}{k!}.$

But we know these power series! The first two are the same: the generating function for the even index Euler numbers, and the third is the generating function for all Euler numbers. That is,

$$E_n^{\prime}(x) = \sec x \cdot \sec x \cdot (\sec x + \tan x) = \sec^3 x + \sec^2 \tan x. \tag{4}$$

This makes perfect sense if we think back to our picture for an up-down min-max permutation. We have an arbitrary up-down permutation of even length (encoded by $\sec x$), followed by a unique minimum, followed by an arbitrary down-up permutation of even length (also encoded by $\sec x$), followed by a unique maximum, followed by an arbitrary up-down permutation (encoded by $\sec x + \tan x$). Easy!

We'll skip the details, but the reader should now be able to use similar reasoning to come up with the generating function for up-down max-min permutations. It is:

$$E_n^{\kappa}(x) = \tan x \cdot \sec x \cdot (\sec x + \tan x) = \sec^2 x \tan x + \sec x \tan^2 x.$$
(5)

5 Consequences

Now that we have formulas (4) and (5) in hand, we can address the interesting properties hinted at in Table 2. First note that the sum of our functions is the second derivative of $\sec z + \tan z$, as it should be, since

$$\frac{d^2}{dx^2} \left[\sec x + \tan x\right] = \frac{d^2}{dx^2} \sum_{n \ge 0} E_n \frac{x^n}{n!} = \sum_{n \ge 0} E_{n+2} \frac{x^n}{n!}$$

while

$$E^{*}(x) + E^{(x)}(z) = \sum_{n\geq 0} \left(E_{n+2}^{*} + E_{n+2}^{(x)} \right) \frac{x^{n}}{n!}.$$

This is the generating function analogue of the identity $E_n = E_n^{\times} + E_n^{\times}$ (which we already knew). Similarly, we can check that the difference of the two functions is sec x:

$$E^{\prime}(x) - E^{\prime}(x) = \sec^3 x - \sec x \tan^2 x = \sec x \cdot (\sec^2 x - \tan^2 x) = \sec x.$$

Comparing coefficients of the series for the function on the far left with the series for the function on the far right gives the following combinatorial identities for $n \ge 1$:

$$E_{2n}^{\prime} - E_{2n}^{\varsigma} = E_{2n-2}, \tag{6}$$

$$E_{2n+1}^{\prime} - E_{2n+1}^{\kappa} = 0. \tag{7}$$

With these observations we are able to prove that the ratio of min-max up-down permutations to max-min up-down permutations approaches 1. For odd integers, (7) shows the ratio is exactly 1. For even integers, (6) implies that

$$\frac{E_{2n}^{\land}}{E_{2n}^{\checkmark}} = \frac{E_{2n}^{\checkmark} - E_{2n-2}}{E_{2n}^{\checkmark}} = 1 - \frac{E_{2n-2}}{E_{2n}^{\checkmark}}.$$

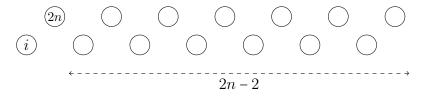
But since $E_{2n}^{\nearrow} > \frac{1}{2}E_{2n}$, we get

$$\frac{E_{2n-2}}{E_{2n}^{\,\prime}} < \frac{E_{2n-2}}{\frac{1}{2}E_{2n}},$$

and it suffices to show

$$\lim_{n \to \infty} \frac{E_{2n-2}}{E_{2n}} = 0$$

To prove this limit, we imagine an up-down permutation of 2n formed as follows. Put 2n in the second position, and put any i = 1, 2, ..., 2n - 1 in position 1:



There are E_{2n-2} ways to fill in the rest of the positions by arranging the remaining numbers in an up-down permutation. This shows us there are at least $(2n-1)E_{2n-2}$ up-down permutations of 2n, i.e., $E_{2n} > (2n-1)E_{2n-2}$.

Returning to our ratio, we therefore have

$$\frac{E_{2n-2}}{E_{2n}} < \frac{E_{2n-2}}{(2n-1)E_{2n-2}} = \frac{1}{2n-1}.$$

Thus $\lim_{n\to\infty} E_{2n-2}/E_{2n} = 0$, and we have

$$\lim_{n \to \infty} \frac{E_n^{\nwarrow}}{E_n^{\checkmark}} = 1,$$

as desired.

6 Conclusions

We hope this article gives the reader some appreciation for the generating function approach to combinatorial enumeration. For example, the method gives identities such as (6) and (7) without much trouble. Reversal of permutations gives a direct combinatorial proof of (7), though the identity in (6), while proved, is still in some sense "unexplained." We are left with a tantalizing combinatorial question: is there a bijective explanation for (6)?

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