# DIVISIBILITY PROPERTIES FOR INTEGER SEQUENCES

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**Abstract**

For a sequence $f = (f_1, f_2, \dots)$ of nonzero integers, define $\Delta(f)$ to be the numerical triangle that lists all the generalized binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_f = \frac{f_n f_{n-1} \cdots f_{n-k+1}}{f_k\, f_{k-1}\ \cdots\ f_1}.$$

Sequence $f$ is called *binomid* if all entries of $\Delta(f)$ are integers. For $I = (1, 2, 3, \dots)$, $\Delta(I)$ is Pascal's Triangle and $I$ is binomid. Surprisingly, every row and column of Pascal's Triangle is also binomid. For any $f$, the rows and columns of $\Delta(f)$ generate their own triangles and all those triangles fit together to form the "Binomid Pyramid" $\mathbb{BP}(f)$. Sequence $f$ is *binomid at every level* if all entries of $\mathbb{BP}(f)$ are integers. We prove that several familiar sequences are binomid at every level. For instance, every sequence $L$ satisfying a linear recurrence of order 2 has that property provided $L(0) = 0$. The sequences $I$, the Fibonacci numbers, and $(2^n - 1)_{n\geq 1}$ provide examples.

## 1. Introduction

In this paper, we consider sequences $f = (f_n)_{n\geq 1} = (f_1, f_2, \dots)$ where every $f_n$ is a nonzero integer. We sometimes write $f(n)$ in place of $f_n$.

**Definition 1.** For integers $n, k$ with $0 \leq k \leq n$, the *f-nomid* coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_f = \frac{f_n f_{n-1} \cdots f_{n-k+1}}{f_k\, f_{k-1}\ \cdots\ f_1}.$$

Defining *f-factorials* as $\langle n\rangle!_f = f_n f_{n-1} \cdots f_2 f_1$ we find that

$$\begin{bmatrix} n \\ k \end{bmatrix}_f = \frac{\langle n\rangle!_f}{\langle k\rangle!_f \cdot \langle n-k\rangle!_f}.$$

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Then $\left[ {n \atop k} \right]_f$ is defined only for $n, k \in \mathbb{Z}$ with $0 \le k \le n$. Set $\langle 0 \rangle\mathbf{!} = 1$ so that $\left[ {n \atop 0} \right]_f = \left[ {n \atop n} \right]_f = 1$. The factorial formula helps explain the symmetry

$$\left[ {n \atop k} \right]_f = \left[ {n \atop n-k} \right]_f, \text{ whenever } 0 \le k \le n.$$

These generalized binomial coefficients have been considered by several authors in various contexts. Ward [25] mentioned them in 1936 and wrote several subsequent papers discussing their properties and applications. For instance, in [26] he developed a theory of calculus that includes analogues of power series. Gould [12] describes the early history of $f$-nomid coefficients (that he calls Fontené-Ward coefficients), and mentions some of their properties. Knuth and Wilf [19] provide a few more early references, and Ballot [2] investigates related ideas.

**Definition 2.** The *binomid triangle* $\Delta(f)$ is the array of all the $f$-nomid coefficients $\left[ {n \atop k} \right]_f$ for $0 \le k \le n$. Sequence $f$ is *binomid* if every entry of $\Delta(f)$ is an integer.

Then $f$ is binomid if and only if $f_1 f_2 \cdots f_k$ divides $f_{m+1} f_{m+2} \cdots f_{m+k}$, for every $m$ and $k$ in $\mathbb{Z}^+$. If $f_1 = 1$ then $\left[ {n \atop 1} \right]_f = f_n$ and $f$ appears as Column 1 of its triangle $\Delta(f)$. The classic "Pascal's Triangle" of binomial coefficients is $\Delta(I)$ where $I = (n) = (1, 2, 3, 4, \dots)$.

It is convenient to allow finite sequences. Suppose there is $N \ge 1$ such that

$$f_n \ne 0 \text{ for } 1 \le n \le N \quad \text{and} \quad f_n \text{ is not defined for } n > N.$$

When $0 \le k \le n \le N$, define $\left[ {n \atop k} \right]_f$ as before, leaving $\left[ {n \atop k} \right]_f$ undefined when $n > N$. Then $\Delta(f)$ is a finite triangular array with $N + 1$ entries along each edge.

For later references to columns of Pascal's Triangle $\Delta(I)$, we display those values in Table 1, with index $n \ge 0$ listed vertically on the left and index $k \ge 0$ across the top.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | |
| 8 | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

Table 1: Classic Pascal triangle $\Delta(I)$ for $I(n) = n$

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | |
| 1 | 1 | 1 | | | | | | | |
| 2 | 1 | 3 | 1 | | | | | | |
| 3 | 1 | 6 | 6 | 1 | | | | | |
| 4 | 1 | 10 | 20 | 10 | 1 | | | | |
| 5 | 1 | 15 | 50 | 50 | 15 | 1 | | | |
| 6 | 1 | 21 | 105 | 175 | 105 | 21 | 1 | | |
| 7 | 1 | 28 | 196 | 490 | 490 | 196 | 28 | 1 | |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Table 2: Triangle for $T(n) = C_2(n) = n(n+1)/2$.

The following lemma can be used to generate examples of binomid sequences. Proofs of those statements are left to the reader.

**Lemma 3.** *Suppose $f$ and $g$ are integer sequences.*

*(a) Define sequence $fg$ by $(fg)_n = f_n g_n$. Then $\begin{bmatrix} n \\ k \end{bmatrix}_{fg} = \begin{bmatrix} n \\ k \end{bmatrix}_f \cdot \begin{bmatrix} n \\ k \end{bmatrix}_g$. If $f$ and $g$ are binomid, then $fg$ is binomid. If $c$ is nonzero in $\mathbb{Z}$ then $cf$ is binomid if and only if $f$ is binomid.*

*(b) Sequence $f$ is called a* divisor-chain *if $f_n \mid f_{n+1}$ for every $n$. Equivalently, $f_n = \langle n \rangle !_a$ for some integer sequence $a$. Every divisor-chain is binomid. In particular, $(c^n) = (c, c^2, c^3, \dots)$ and $n! = (1, 2, 6, 24, 120, \dots)$ are binomid.*

*(c) Define an integer sequence $\psi$ to be* homomorphic[1] *if $\psi(mn) = \psi(m)\psi(n)$ for every $m, n$. If $\psi$ is homomorphic and $f$ is binomid then $\psi \circ f$ is also binomid.*

*(d) If $f = (f_1, f_2, f_3, \dots)$ is binomid then so are the sequences $(1, f_1, f_2, f_3, \dots)$, $(1, f_1, 1, f_2, 1, f_3, \dots)$ and $(f_1, f_1, f_2, f_2, f_3, f_3, \dots)$.*

The "triangular numbers" $T(n) = \frac{n(n+1)}{2} = (1, 3, 6, 10. \dots)$ are the entries in Column 2 of Pascal's triangle. The first few rows of the triangle $\Delta(T)$ are displayed in Table 2 . Similarly, the third Pascal column $C_3 = (1, 4, 10, 20, 35, \dots)$ generates the triangle in Table 3. Those integer values lead us to suspect that every Pascal column $C_m$ is binomid. That result has been proved by several authors. It also follows from Theorem 15 and Lemma 13(c) below.

The OEIS [23] webpage for sequence A342889 provides many references related to the triangles $\Delta(C_m)$. For instance, $\Delta(T) = \Delta(C_2)$ is the triangle of Narayana numbers. The "generalized binomial coefficients"[2] $(n, k)_m$ mentioned on those OEIS

[1] Other names include "totally multiplicative" and "strongly multiplicative."

[2] Some authors use this term for different numerical triangles. For instance, the numbers $\binom{n}{m}_s$ in [3, p. 15] are not the same as our binomid coefficients. Similarly the "Pascal pyramid" built from trinomial coefficients (as in [3, p. 45] is not one of the Binomid Pyramids mentioned below.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | |
| 1 | 1 | 1 | | | | | | | |
| 2 | 1 | 4 | 1 | | | | | | |
| 3 | 1 | 10 | 10 | 1 | | | | | |
| 4 | 1 | 20 | 50 | 20 | 1 | | | | |
| 5 | 1 | 35 | 175 | 175 | 35 | 1 | | | |
| 6 | 1 | 56 | 490 | 980 | 980 | 56 | 1 | | |
| 7 | 1 | 84 | 1176 | 4116 | 4116 | 1176 | 84 | 1 | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

Table 3: Triangle for $C_3(n) = n(n+1)(n+2)/6$.

pages are our $\left[{n \atop k}\right]_{C_{m+1}}$ built from Pascal columns.

Certain determinants of binomial coefficients are related to sequence $C_m$:

$$\det\left[\binom{n+i}{m+j}\right]_{i,j=0}^{k-1} = \frac{\binom{n+k-1}{m}\cdots\binom{n}{m}}{\binom{m+k-1}{m}\cdots\binom{m}{m}} = \left[{n-m+k \atop k}\right]_{C_m}.$$

Consequently, every Pascal column $C_m$ is a binomid sequence. That determinant formula is mentioned in [3, p. 164], referring to [24, p. 257].[3] Gessel and Viennot [11] found combinatorial interpretations for such Pascal determinants. In a recent exposition, Cigler [5] derives determinant expressions for entries of $\Delta(C_m)$, arrays that he calls "Hoggatt Triangles" following [9].

What about the Pascal rows? Tables 4 - 9 display the triangles for several rows $R_m(n) = \binom{m}{n-1}$.

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | | | |
| 1 | 1 | 1 | | |
| 2 | 1 | 2 | 1 | |
| 3 | 1 | 1 | 1 | 1 |

Table 4: Triangle for $R_2(n) = \binom{2}{n-1}$.

Define the *Binomial Pyramid* to be the numerical array constructed by stacking the triangles built from Pascal Rows. That infinite pyramid has three faces with all outer entries equal to 1. The horizontal slice at depth $m$ is $\Delta(R_m)$, the triangle made from Pascal Row $m$. Our numerical examples indicate that all entries are integers and the pyramid has three-fold rotational symmetry.

Removing one face (of all ones) from that pyramid exposes an infinite triangular face that is the original Pascal triangle. This is seen in the triangles $\Delta(R_m)$ displayed

[3] Netto states that this determinant formula appeared in an 1865 work of v. Zeipel.

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | | | | |
| 1 | 1 | 1 | | | |
| 2 | 1 | 3 | 1 | | |
| 3 | 1 | 3 | 3 | 1 | |
| 4 | 1 | 1 | 1 | 1 | 1 |

Table 5: Triangle for $R_3(n) = \binom{3}{n-1}$.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | | | | | |
| 1 | 1 | 1 | | | | |
| 2 | 1 | 4 | 1 | | | |
| 3 | 1 | 6 | 6 | 1 | | |
| 4 | 1 | 4 | 6 | 4 | 1 | |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 6: Triangle for $R_4(n) = \binom{4}{n-1}$.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 5 | 1 | | | | |
| 3 | 1 | 10 | 10 | 1 | | | |
| 4 | 1 | 10 | 20 | 10 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 7: Triangle for $R_5(n) = \binom{5}{n-1}$.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 1 | | | | | | |
| 2 | 1 | 6 | 1 | | | | | |
| 3 | 1 | 15 | 15 | 1 | | | | |
| 4 | 1 | 20 | 50 | 20 | 1 | | | |
| 5 | 1 | 15 | 50 | 50 | 15 | 1 | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 8: Triangle for $R_6(n) = \binom{6}{n-1}$.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | |
| 1 | 1 | 1 | | | | | | | |
| 2 | 1 | 7 | 1 | | | | | | |
| 3 | 1 | 21 | 21 | 1 | | | | | |
| 4 | 1 | 35 | 105 | 35 | 1 | | | | |
| 5 | 1 | 35 | 175 | 175 | 35 | 1 | | | |
| 6 | 1 | 21 | 105 | 175 | 105 | 21 | 1 | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 9: Triangle for $R_7(n) = \binom{7}{n-1}$.

above each Column 1 is a row of Pascal's triangle (by construction).

Listing Column 2 for those triangles displayed above yields

$$(1,\ 1),\quad (1,\ 3,\ 1),\quad (1,\ 6,\ 6,\ 1),\quad (1,\ 10,\ 20,\ 10,\ 1),\quad \text{etc.}$$

Those are exactly the rows of the binomid triangle for $T = C_2 = (1, 3, 6, 10, \dots)$ displayed earlier! That is, removing two face-layers of the Binomial Pyramid exposes a triangular face that equals $\Delta(C_2)$, built from Pascal Column 2. This pattern continues: Each triangle $\Delta(C_m)$ appears as a slice of our Binomial Pyramid. We generalize those assertions here, and outline proofs in the next section.

**Definition 4.** For a sequence $f$, the *Binomid Pyramid* $\mathbb{BP}(f)$ is made by stacking the binomid triangles constructed from the rows of triangle $\Delta(f)$. Sequence $f$ is *binomid at level* $c$ if column $c$ of $\Delta(f)$ is a binomid sequence.

By definition, $f$ is *binomid at every level* if each column of $\Delta(f)$ is binomid. Equivalently, by Corollary 10 below, all entries of the pyramid $\mathbb{BP}(f)$ are integers.

Sequence $T = (1, 3, 6, 10, 15, \dots)$ is binomid at level 1. It is not binomid at level 2 since Column 2 of $\Delta(T)$ is $g = (1, 6, 20, 50, 105, \dots)$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_g$ is not an integer.

Some sequences are binomid at level 2 but not at level 1. For instance, let $f$ be the eventually constant sequence $f = (2^{a_n})$ where $a = (0, 2, 4, 1, 3, 1, 4, 4, 4, \dots)$. It is not binomid because $\begin{bmatrix} 6 \\ 3 \end{bmatrix}_f = \frac{1}{2}$. We can see that Column 2 of $\Delta(f)$ is $(2^{b_n})$ where $b = (0, 4, 3, 2, 2, 3, 6, 6, 6, \dots)$. Checking several cases shows that Column 2 is binomid.

Here are some examples that are fairly easy to verify.

**Proposition 5.** *Continue with the notation used in Lemma 3.*

*(a) Every divisor-chain is binomid at every level.*

*(b) Suppose the integer sequence $\psi$ is homomorphic. If $f$ is binomid at level $c$, then so is $\psi \circ f$.*

*Proof.* Statement (b) follows from Lemma 3(c). To prove (a), suppose $f$ is a divisor-chain. Column $j$ of $\Delta(f)$ is defined as $C_j(n) = \left[{n+j-1 \atop j}\right]_f$. Since $\left[{d \atop j}\right]_f = \frac{f_d}{f_{d-j}}\left[{d-1 \atop j}\right]_f$ and $\frac{f_d}{f_{d-j}}$ is an integer whenever $0 \le j < d$, we conclude that $C_j$ is a divisor-chain. Therefore $C_j$ is binomid by Lemma 3(b). □

As another motivating example, let $G_2(n) = 2^n - 1 = (1, 3, 7, 15, \dots)$. The first few rows of its triangle are displayed in Table 10 .

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 1 | | | | | | |
| 2 | 1 | 3 | 1 | | | | | |
| 3 | 1 | 7 | 7 | 1 | | | | |
| 4 | 1 | 15 | 35 | 15 | 1 | | | |
| 5 | 1 | 31 | 155 | 155 | 31 | 1 | | |
| 6 | 1 | 63 | 651 | 1395 | 651 | 63 | 1 | |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Table 10: Triangle for $G_2(n) = 2^n - 1$.

Let $G_q(n) = \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}$. The entries $\left[{n \atop k}\right]_{G_q}$ in triangle $\Delta(G_q)$ are often called "$q$-nomial" (or Gaussian) coefficients and have appeared in many articles since Gauss [10] introduced them in 1808. For example see [19]. "Fibonomial" coefficients are the entries of $\Delta(F)$ where $F$ is the Fibonacci sequence. As Lucas [22] and Carmichael [4] pointed out long ago, $G_q$ and $F$ are examples of *Lucas sequences*.[4] Those are integer sequences $L$ that satisfy a linear recurrence of order 2 and have $L(0) = 0$. Then $L$ is a constant multiple of a sequence $U$ in Definition 16. In Theorem VII of [4], Carmichael proved that every Lucas sequence is binomid.

Our Theorem 15 and Lemma 16 below show that every Lucas sequence is binomid at every level.

**Remark 6.** The definition of binomid sequences can be restated in additive form. Let $v_p(n)$ be the exponent of the prime $p$ in $n$. That is, $n = \prod_p p^{v_p(n)}$ .

(i) Suppose $(a) = (a_1, a_2, \dots)$ is a sequence of nonzero integers. Then $(a)$ is binomid if and only if $(p^{v_p(a_n)})$ is binomid for every prime $p$.

(ii) For an integer sequence $b = (b_1, b_2, \dots)$, define $s_b(n) = b_1 + \cdots + b_n$. Suppose $c > 1$ is an integer. Then $(c^{b_n})$ is binomid if and only if $s_b(m) + s_b(n) \le s_b(m + n)$ for every $m, n \in \mathbb{Z}^+$.

Each property defined below has an additive version. But those reformulations do not seem to provide significantly better proofs of the theorems.

[4] Ballot [2] refers to entries of $\Delta(L)$ as "Lucanomial" coefficients.

## 2. Binomid Pyramids

For a sequence $f$, the Binomid Pyramid $\mathbb{BP}(f)$ is formed by stacking the binomid triangles of the row-sequences of the triangle $\Delta(f)$. In this section we verify that the binomid triangles for the column-sequences of $\Delta(f)$ appear by slicing that pyramid along planes parallel to a face.

The notation is chosen so that sequences begin with index 1. We often restrict attention to sequences $f$ with $f_1 = 1$.

**Definition 7.** If $f$ is a sequence pf nonzero integers, define the *Row* and *Column sequences* of its triangle $\Delta(f)$ by

$$R_m(N) = \begin{bmatrix} m \\ N-1 \end{bmatrix}_f \qquad \text{and} \qquad C_j(N) = \begin{bmatrix} N+j-1 \\ j \end{bmatrix}_f.$$

The row sequence $R_m$ has only $m+1$ entries

$$R_m = \left( \begin{bmatrix} m \\ 0 \end{bmatrix}_f, \begin{bmatrix} m \\ 1 \end{bmatrix}_f, \begin{bmatrix} m \\ 2 \end{bmatrix}_f, \ldots, \begin{bmatrix} m \\ m \end{bmatrix}_f \right),$$

with $R_m(N)$ undefined for $N > m+1$. Assuming $f_1 = 1$, we find

$$R_0 = (1),\ R_1 = (1,1), \quad R_2 = (1, f_2, 1), \quad \ldots \quad , R_m = (1, f_m, \tfrac{f_m f_{m-1}}{f_2}, \ldots, f_m, 1).$$

Column sequences are infinite, with entries

$$C_j = \left( \begin{bmatrix} j \\ j \end{bmatrix}_f, \begin{bmatrix} j+1 \\ j \end{bmatrix}_f, \begin{bmatrix} j+2 \\ j \end{bmatrix}_f, \ldots \right) = \left(1, f_{j+1}, \frac{f_{j+2} f_{j+1}}{f_2}, \ldots \right).$$

Note that $C_0 = (1,1,1,\ldots)$ and if $f_1 = 1$ then $C_1 = (1, f_2, f_3, \ldots) = f$.

For a finite sequence $f$, Lemma 8 shows that the triangle $\Delta(f)$ has rotational symnmetry whenever $f$ has left/right symmetry.

**Lemma 8.** *Suppose $f = (f_1, f_2, \ldots, f_n)$ is a symmetric list of $n$ terms; that is, $f_k = f_{n+1-k}$. Then the triangle $\Delta(f)$ has 3-fold rotational symmetry.*

*Proof.* We need to show that Column $c$ = Row $n-c$, whenever $0 \le c \le n$. The $(k+1)^{\text{st}}$ entries of $C_c$ and $R_{n-c}$ are

$$\begin{bmatrix} c+k \\ k \end{bmatrix}_f = \frac{f_{c+k} f_{c+k-1} \cdots f_{c+1}}{\langle k \rangle !_f} \qquad \text{and} \qquad \begin{bmatrix} n-c \\ k \end{bmatrix}_f = \frac{f_{n-c} f_{n-c-1} \cdots f_{n-c-k+1}}{\langle k \rangle !_f}.$$

By symmetry of $f$, those numerators are equal term by term

$$f_{c+k} = f_{n-c-k+1}, \quad f_{c+k-1} = f_{n-c-k+2}, \quad \ldots \ , \quad f_{c+1} = f_{n-c}.$$ □

For the finite sequence above, the symmetry shows that $R_n = (1, 1, \ldots, 1)$, and every later row is undefined.

The Binomid Pyramid $\mathbb{BP}(f)$ in Definition 4 is built by stacking the triangles $\Delta(R_m)$. To show that the triangle $\Delta(C_j)$ appears as a slice of this pyramid, we must check that each row of $\Delta(C_j)$ equals a corresponding row and column of the horizontal slice $\Delta(R_m)$. Numerical observations indicate that

$$\text{Row } n \text{ of } \Delta(C_2) = \text{ Column 2 of } \Delta(R_{n+1}) = \text{Row } n \text{ of } \Delta(R_{n+1}),$$
$$\text{Row } n \text{ of } \Delta(C_3) = \text{ Column 3 of } \Delta(R_{n+2}) = \text{Row } n \text{ of } \Delta(R_{n+2}).$$

The observed pattern provides the next result

**Proposition 9.** *For a sequence $f$ and every $n$ and $m$*
$$\textit{Row } n \textit{ of } \Delta(C_m) = \textit{Column } m \textit{ of } \Delta(R_{n+m-1}) = \textit{Row } n \textit{ of } \Delta(R_{n+m-1}).$$

*Proof.* Since $R_k$ has $k+1$ terms, Lemma 8 shows
$$\text{Column } m \text{ of } \Delta(R_k) = \text{Row } k+1-m \text{ of } \Delta(R_k).$$
This proves the second equality in the statement of the Proposition. To complete the proof we will show $\begin{bmatrix} n \\ k \end{bmatrix}_{C_m} = \begin{bmatrix} n \\ k \end{bmatrix}_{R_{n+m-1}}$ for every $n, k, m$. Those two sequences are
$$C_m(N) = \begin{bmatrix} N+m-1 \\ m \end{bmatrix}_f \text{ and } R_{n+m-1}(N) = \begin{bmatrix} n+m-1 \\ N-1 \end{bmatrix}_f.$$
Then
$$\begin{bmatrix} n \\ k \end{bmatrix}_{C_m} = \frac{\begin{bmatrix} n+m-1 \\ m \end{bmatrix}_f \begin{bmatrix} n+m-2 \\ m \end{bmatrix}_f \cdots \begin{bmatrix} n+m-k \\ m \end{bmatrix}_f}{\begin{bmatrix} k+m-1 \\ m \end{bmatrix}_f \begin{bmatrix} k+m-2 \\ m \end{bmatrix}_f \cdots \begin{bmatrix} m \\ m \end{bmatrix}_f} \quad \text{and}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{R_{n+m-1}} = \frac{\begin{bmatrix} n+m-1 \\ n-1 \end{bmatrix}_f \begin{bmatrix} n+m-1 \\ n-2 \end{bmatrix}_f \cdots \begin{bmatrix} n+m-1 \\ n-k \end{bmatrix}_f}{\begin{bmatrix} n+m-1 \\ k-1 \end{bmatrix}_f \begin{bmatrix} n+m-1 \\ k-2 \end{bmatrix}_f \cdots \begin{bmatrix} n+m-1 \\ 0 \end{bmatrix}_f}.$$

The proof strategy is to substitute the $f$-factorial definitions for all those binomid coefficients and then simplify the fractions. We omit the details of this straightforward, but very long, calculation. □

Proposition 9 verifies our earlier statement that slices parallel to a face in the pyramid $\mathbb{BP}(f)$ yield the binomid triangles for the columns of $\Delta(f)$.

**Corollary 10.** *If $f$ is a sequence on nonzero integers, then all columns of $\Delta(f)$ are binomid if and only if all rows of $\Delta(f)$ are binomid. Those conditions hold when $f$ is binomid at every level, as in Definition 4.*

Note. The 3-fold symmetry of $\Delta(R_m)$ (in Proposition 8) implies that the formula in Proposition 9 is equivalent to
$$\begin{bmatrix} n \\ k \end{bmatrix}_{C_m} = \begin{bmatrix} k+m \\ m \end{bmatrix}_{R_{n+m-1}}, \text{ for every } n, k, \text{ and } m.$$

## 3. Divisor-Product Sequences

**Definition 11.** For a sequence $g$, define sequence $\mathcal{P}(g)$ by $\mathcal{P}(g)(n) = \prod_{d|n} g(d)$. Sequence $f$ is a *divisor-product* if $f = \mathcal{P}(g)$ for an integer sequence $g$.

That notation indicates that $d$ runs over all the positive integer divisors of $n$. Cyclotomic polynomials provide motivation. Define the homogeneous polynomials $\Phi_n(x, y)$ in $\mathbb{Z}[x, y]$ by requiring

$$x^n - y^n = \prod_{d|n} \Phi_d(x, y).$$

For example,

$$\begin{aligned} \Phi_1(x,y) &= x - y, & \Phi_4(x,y) &= x^2 + y^2, \\ \Phi_2(x,y) &= x + y, & \Phi_5(x,y) &= x^4 + x^3y + x^2y^2 + xy^3 + y^4, \\ \Phi_3(x,y) &= x^2 + xy + y^2, & \Phi_6(x,y) &= x^2 - xy + y^2. \end{aligned}$$

Note that $\Phi_n(y, x) = \Phi_n(x, y)$ for every $n > 1$. Each (inhomogeneous) cyclotomic polynomial $\Phi_n(x) = \Phi_n(x, 1)$ is monic of degree $\varphi(n)$ with integer coefficients.[5]

For example, the sequence $G_2 = (2^n - 1) = (1, 3, 7, 15, 31, 63, 127, 255, \dots)$ is a divisor-product since it factors as

$$2^n - 1 = \prod_{d|n} \Phi_d(2),$$

and $\big(\Phi_n(2)\big) = (1, 3, 7, 5, 31, 3, 127, 17, 73, 11, \dots)$ has integer entries. These sequences are "divisible" in the following sense.

**Definition 12.** An integer sequence is *divisible* if for $k, n \in \mathbb{Z}^+$

$$k \mid n \;\text{ implies }\; f(k) \mid f(n).$$

Such an $f$ is called a *divisibility sequence* or a *division sequence.*

**Lemma 13.**

*(i) Every divisor-product is divisible.*

*(ii) Let* $G_{a,b}(n) = \dfrac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^n$*. If $a, b$ are integers, then $G_{a,b}$ is a divisor-product.*

*(iii) For $c \in \mathbb{Z}$, the sequences*

$$I = (n) = (1, 2, 3, \dots), \quad (c) = (c, c, c, \dots), \;\text{ and }\; (c^n) = (c, c^2, c^3, \dots)$$

*are divisor-products.*

*(iv) If $(a_n)$ and $(b_n)$ are divisor-products, then so is $(a_n b_n)$.*

[5] Further information about $\Phi_n(x)$ appears in many number theory texts.

(v) *Suppose the integer sequence* $\psi$ *is homomorphic (as in Lemma 3). If* $f$ *is a divisor-product then so is* $\psi \circ f$*.*

*Proof.* Statement (i) follows from Definition 11. Furthermore, if $f$ is a divisor-product and $d = \gcd(m, n)$, then $f(m)f(n) \mid f(mn)f(d)$. For (ii), set $g(1) = 1$ and $g(n) = \Phi_n(a, b)$ when $n > 1$. Then $G_{a,b}(n) = \prod_{d|n} g(d)$. This remains valid when $a = b$ provided that we set $G_{a,a}(n) = na^{n-1}$. To prove (iii), note that $I = G_{1,1}$ and apply (2). Explicitly, $I = \mathcal{P}(j)$ where

$$j(n) = \begin{cases} p & \text{if } n = p^k > 1 \text{ is a prime power,} \\ 1 & \text{otherwise.} \end{cases}$$

Sequence $(c)$ equals $\mathcal{P}(\delta_c)$ where $\delta_c = (c, 1, 1, 1, \dots)$. Also $(c^n) = \mathcal{P}(c^{\varphi(n)})$, where $\varphi$ is the Euler function. To see why (iv) and (v) are true, check that $\mathcal{P}(gh) = \mathcal{P}(g)\mathcal{P}(h)$ and $\psi \circ \mathcal{P}(g) = \mathcal{P}(\psi \circ g)$. □

For any sequence $f$ with entries in $\mathbb{Q}^\times$, there exists a unique sequence $g$ in $\mathbb{Q}$ with $f = \mathcal{P}(g)$. The multiplicative version of the Möbius inversion formula provides an explicit formula for $g = \mathcal{P}^{-1}(f)$.

**Lemma 14** (Möbius inversion)**.** *If* $f$ *is a sequence of nonzero numbers then* $f = \mathcal{P}(g)$ *where*

$$g(n) = \prod_{d|n} f(d)^{\mu(n/d)}.$$

The Möbius function $\mu(k)$ has values in $\{0, 1, -1\}$. By definition, $f$ is a divisor-product exactly when every $g(n)$ is an integer. The additive form of Möbius inversion is discussed in many number theory texts. This multiplicative form is a variant mentioned explcitily in some textbooks[6] and webpages.

In 1939, Ward [27] pointed out that divisor-product sequences are binomid. The following generalization is the major result of this article.

**Theorem 15.** *A divisor-product sequence is binomid at every level.*

*Proof.* If $g$ is a sequence of nonzero integers, the theorem asserts that $\mathcal{P}(g)$ is binomid at every level. Equivalently, every entry of the pyramid $\mathbb{BP}\big(\mathcal{P}(g)\big)$ is an integer. To prove this, consider a "generic" situation.

For a sequence $X = (x_1, x_2, x_3, \dots)$ of independent indeterminates, let $\mathbb{Z}[X]$ be the polynomial ring in those variables using integer coefficients. The theorem is reduced to the following statement:

Claim. Each entry of $\mathbb{BP}\big(\mathcal{P}(X)\big)$ is a polynomial in $\mathbb{Z}[X]$.

For if this Claim is true, we may substitute $g$ for $X$ to conclude that each term in $\mathbb{BP}\big(\mathcal{P}(g)\big)$ is in $\mathbb{Z}$, proving the theorem.

[6] One reference is section 9.2.4 of [8]. Also see the Wikipedia article on Möbius Inversion.

To begin the proving the Claim, note that each entry in the pyramid $\mathbb{BP}\big(\mathcal{P}(X)\big)$ is a fraction involving products of terms $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\mathcal{P}(X)}$ for various $n$ and $k$. Then it is a "rational monomial," a quotient of monomials involving the variables $x_1, x_2, \ldots$ .

For real $\alpha$, we write $\lfloor \alpha \rfloor$ for the "floor function," the greatest integer less than or equal to $\alpha$. Note that

$$\begin{aligned} \langle n \rangle !_{\mathcal{P}(X)} &= \prod_{j=1}^{n} \mathcal{P}(X)(j) = (x_1)(x_1x_2)(x_1x_3)(x_1x_2x_4)(x_1x_5)(x_1x_2x_3x_6)\cdots \\ &= x_1^n x_2^{\lfloor n/2 \rfloor} x_3^{\lfloor n/3 \rfloor} \cdots x_n^{\lfloor n/n \rfloor} \;=\; \prod_{r>0} x_r^{\lfloor n/r \rfloor}. \end{aligned} \tag{1}$$

For each $n$, that infinite product (with $r = 1, 2, 3, 4, \ldots$) is actually finite because $\lfloor n/r \rfloor = 0$ when $r > n$. For rational monomial $M$ and index $r$, write $v_r(M)$ for the exponent of $x_r$ in the factorization of $M$. That is,

$$M = \prod_{r>0} x_r^{v_r(M)}.$$

Then $v_r(M) \in \mathbb{Z}$, and $v_r(M) = 0$ for all but finitely many $r$. Moreover, $M$ is a polynomial exactly when $v_r(M) \geq 0$ for every index $r$.

Formula (1) shows that $v_r\big(\langle n \rangle !_{\mathcal{P}(X)}\big) = \lfloor n/r \rfloor$. Therefore the exponent

$$\delta_{m,r}(j) = v_r\Bigg( \left[ \begin{matrix} m+j \\ m \end{matrix} \right]_{\mathcal{P}(X)} \Bigg)$$

has the simpler formula

$$\delta_{m,r}(j) \;=\; \left\lfloor \frac{m+j}{r} \right\rfloor - \left\lfloor \frac{m}{r} \right\rfloor - \left\lfloor \frac{j}{r} \right\rfloor .$$

We will see below that this quantity is either 0 or 1.

To prove that $\mathcal{P}(X)$ is binomid at every level, we need to show that for every $m$, the column sequence $C_m(n) = \left[ \begin{smallmatrix} m+n-1 \\ m \end{smallmatrix} \right]_{\mathcal{P}(X)}$ is binomid in $\mathbb{Z}[X]$. In other words, for every $n, a \in \mathbb{Z}^+$,

$$\left[ \begin{matrix} m \\ m \end{matrix} \right]_{\mathcal{P}(X)} \cdot \left[ \begin{matrix} m+1 \\ m \end{matrix} \right]_{\mathcal{P}(X)} \cdots \left[ \begin{matrix} m+n-1 \\ m \end{matrix} \right]_{\mathcal{P}(X)}$$

divides

$$\left[ \begin{matrix} a+m \\ m \end{matrix} \right]_{\mathcal{P}(X)} \cdot \left[ \begin{matrix} a+m+1 \\ m \end{matrix} \right]_{\mathcal{P}(X)} \cdots \left[ \begin{matrix} a+m+n-1 \\ m \end{matrix} \right]_{\mathcal{P}(X)}$$

in $\mathbb{Z}[X]$. The definition of $\delta_{m,r}(j)$ then shows that $\mathcal{P}(X)$ is binomid at level $m$ if and only if for every $n, a, r \in \mathbb{Z}^+$,

$$\sum_{j=0}^{n-1} \delta_{m,r}(j) \;\leq\; \sum_{j=a}^{a+n-1} \delta_{m,r}(j). \tag{2}$$

The formula $\delta_{m,r}(j) = \left\lfloor \frac{m+j}{r} \right\rfloor - \left\lfloor \frac{m}{r} \right\rfloor - \left\lfloor \frac{j}{r} \right\rfloor$, implies $\delta_{m+rs,r}(j) = \delta_{m,r}(j)$ for any $s \in \mathbb{Z}$. Then we may alter $m$ to assume $0 \le m < r$.

Similarly, $\delta_{m,r}(j+rs) = \delta_{m,r}(j)$. Then for fixed $r, m$, the value of $\delta_{m,r}(j)$ depends only on ($j$ modulo $r$). Consequently, any block of $r$ consecutive terms in the sums in Inequality (2) yields the same answer, namely the sum over all values in $\mathbb{Z}/r\mathbb{Z}$. Then if $n \ge r$ we may cancel the top $r$ terms of each sum in (2) and replace $n$ by $n - r$. By repeating this process, we may assume $0 \le n < r$.

For real numbers $\alpha, \beta \in [0, 1)$, a quick check shows that

$$\lfloor \alpha + \beta \rfloor - \lfloor \alpha \rfloor - \lfloor \beta \rfloor = \begin{cases} 1 & \text{if } \alpha + \beta \ge 1, \\ 0 & \text{if } \alpha + \beta < 1. \end{cases}$$

Express $j \equiv j' \pmod{r}$ where $0 \le j' < r$. Then since $0 \le m, n < r$, we find,

$$\delta_{m,r}(j) = \begin{cases} 1 & \text{if } m + j' \ge r, \\ 0 & \text{if } m + j' < r. \end{cases}$$

This says that the sequence $\delta_{m,r}$ begins with $r - m$ zeros followed by $m$ ones, and that pattern repeats with period $r$. For instance, when $m = 2$ and $r = 6$ the sequence is

$$\delta_{2,6} = (0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, \ \ldots).$$

For such a sequence, it is not hard to see that for the sums of any "window" of $n$ consecutive terms, the minimal value is provided by the $n$ initial terms. This proves Inequality (2) and completes the proof of the theorem. □

Further examples of divisor-products are provided by sequences that satisfy a linear recurrence of degree 2. The Fibonacci sequence and $(2^n - 1)$ are included in this class of "Lucas sequences."

**Lemma 16.** *For $P, Q \in \mathbb{Z}$ not both zero, define the* Lucas sequence $U = U_{P,Q}$ *by setting $U(0) = 0$ and $U(1) = 1$, and requiring*

$$U(n+2) = P \cdot U(n+1) - Q \cdot U(n) \quad \textit{for every } n \ge 0.$$

*Then $U$ is a divisor-product.*

*Proof.* Factor $x^2 - Px + Q = (x - \alpha)(x - \beta)$ for $\alpha, \beta \in \mathbb{C}$. Then $\alpha, \beta$ are not both zero. Recall the following well-known formulas:

- If $\alpha \ne \beta$ then $U(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for every $n \in \mathbb{Z}^+$,
- If $\alpha = \beta$ then $U(n) = n\alpha^{n-1}$ for every $n \in \mathbb{Z}^+$.

To verify those formulas, note that the sequences $(\alpha^n)$ and $(\beta^n)$ satisfy the recurrence displayed in Lemma 16. Therefore every linear combination $(c\alpha^n + d\beta^n)$ also satisfies that recurrence. When $\alpha \neq \beta$, the stated formula has this form and matches $U(n)$ for $n = 0, 1$. Induction shows that those quantities are equal for every $n$.

When $\alpha = \beta$ show that $\alpha \in \mathbb{Z}$ and use the same method to prove $U(n) = n\alpha^{n-1}$. In this case, $U$ is a divisor-product since Lemma 13 implies that $(n)$, $(\alpha^{n-1})$, and their product are divisor-products.

Suppose $\alpha \neq \beta$. Since $\alpha^n - \beta^n = \prod_{d|n} \Phi_d(\alpha, \beta)$ and $\Phi_1(\alpha, \beta) = \alpha - \beta$, then $U = \mathcal{P}(g)$ where

$$g(n) = \begin{cases} 1 & \text{if } n = 1, \\ \Phi_n(\alpha, \beta) & \text{if } n > 1. \end{cases}$$

Then $U$ is a divisor-product provided $g$ has integer values. Möbius (Lemma 14) implies that every $g(n)$ is a rational number. Since $\alpha, \beta$ are algebraic integers and each $\Phi_n$ has integer coefficients, we know that $g(n)$ is an algebraic integer. Therefore each $g(n) \in \mathbb{Z}$.[7] □

For the Fibonacci sequence $F = U_{1,-1}$, this lemma implies that $F = \mathcal{P}(b)$ where $b = (1, 1, 2, 3, 5, 4, 13, 7, 17, 11, 89, 6, \dots)$ is an integer sequence.

We end this section with a few more remarks about divisor-products.

The triangular number sequence $T$ is binomid but not a divisor-product. In fact, $T$ is not even divisible. The sequence $w = (1, c, c, c, c, \dots)$ is a divisor-chain, so it is binomid at every level by Lemma 5. But when $c > 1$ it is not a divisor-product since $w_2w_3$ does not divide $w_1w_6$.

To see that divisibility does not imply binomid, suppose $c > 1$ and define $h(n) = \begin{cases} 1 & \text{if } n = 1, 5, 7, \\ c & \text{otherwise.} \end{cases}$ Then $h$ is divisible. Since $h_1h_2h_3$ does not divide $h_5h_6h_7$, we see that $h$ is not binomid.

It is curious that the Euler function $\varphi(n)$ is a divisor-product. Recall that $\varphi$ is *multiplicative:* $\varphi(ab) = \varphi(a)\varphi(b)$ whenever $a, b$ are coprime. Standard formulas imply that $\varphi$ is divisible as in Definition 12. Then the next result applies to $\varphi$.

**Proposition 17.** *Every multiplicative divisible sequence is a divisor-product.*

*Proof.* If $f$ is a sequence of nonzero integers, let $f = \mathcal{P}(g)$ for a sequence $g$ in $\mathbb{Q}$. If $f$ is multiplicative, then $g(n) = \begin{cases} \frac{f(p^m)}{f(p^{m-1})} & \text{if } n = p^m > 1 \text{ is a prime power,} \\ 1 & \text{otherwise.} \end{cases}$

If $f$ is also divisible, every $g(n)$ is an integer and $f$ is a divisor-product. □

[7] To avoid theorems about algebraic integers, we may use the theory of symmetric polynomials. If $p \in \mathbb{Z}[x, y]$ and $p(x, y) = p(y, x)$, then $p \in \mathbb{Z}[\sigma_1, \sigma_2]$, where $\sigma_1 = x + y$ and $\sigma_2 = xy$ are the elementary symmetric polynomials. Note that $\sigma_1(\alpha, \beta) = \alpha + \beta = P$ and $\sigma_2(\alpha, \beta) = \alpha\beta = Q$.

**Remark 18.** Suppose $f = \mathcal{P}(g)$ is a divisor-product with $f(1) = 1$. Then:

(1) $f$ is multiplicative if and only if $g(n) = 1$ whenever $n$ is not a prime power;

(2) $f$ is homomorphic if and only if $g(n) = 1$ whenever $n$ is not a prime power and $g(p^m) = g(p)$ for every prime power $p^m > 1$;

(3) $f$ is a GCD sequence (see Definition 19 below) if and only if $g(m)$ and $g(n)$ are coprime whenever $m \nmid n$ and $n \nmid m$.

Property (3) was pointed out in [6].

## 4. Related Topics

### 4.1. GCD Sequences

We use notation motivated by lattice theory:

$$a \wedge b = \gcd(a, b) \quad \text{and} \quad a \vee b = \operatorname{lcm}(a, b).$$

Those operations are defined on $\mathbb{Z}$, except that $0 \wedge 0$ is undefined.

**Definition 19.** An integer sequence $f$ is a *GCD sequence* if

$$f(m \wedge n) = f(m) \wedge f(n) \quad \text{for every } m, n \in \mathbb{Z}^+.$$

Using earlier notation, this says that for every $m, n$ with $d = \gcd(m, n)$ we have:

$$\gcd(f_m, f_n) = f_d.$$

Other authors use different names for sequences with this property. Hall [14] and Kimberling [18] calls them *strong divisibility* sequences. Granville [13] considers sequences that satisfy a linear recurrence, and refers to those with this GCD property as *strong LDS's* (linear division sequences). Knuth and Wilf [19] use the term *regularly divisible*. Dziemiańczuk and Bajguz [6] call them *GCD-morphic* sequences.

Examples of GCD sequences include the Fibonacci sequence and $(a^n - b^n)_{n \geq 1}$ for integers $a \neq b$. More generally, Carmichael [4] proved in 1913 that every Lucas sequence (as in Lemma 16) is GCD.

In 1936 Ward [25] used prime factorizations to prove that every GCD sequence is binomid, a result also proved in [19]. Here is a stronger result.

**Theorem 20.** *Every GCD sequence is a divisor-product.*

This result and Theorem 15 imply that GCD sequences are binomid at every level. The proof of Theorem 20 is not included here. Kimberling [18] proved it by showing that Möbius Inversion (Lemma 14) applied to a GCD sequence always produces integers. The proof by Dziemiańczuk and Bajguz [6] is quite different. A third argument can be given by first reducing to the case of sequences of type $f(n) = c^{a(n)}$.

If $f$ and $g$ are GCD sequences, then so are $f \wedge g$ and $f \circ g$. For instance, when $F$ is the Fibonacci sequence, we find that $(2^{F_n} - 1)_{n \geq 1}$ is a GCD sequence. This sequence was mentioned in [1].

### 4.2. ComboSum Sequences

An inductive proof that all binomial coefficients $\binom{n}{k}$ are integers uses the formula

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

We extend that recurrence formula to our context. Other authors have pointed out some of these ideas, as seen in [15], [21], and [7].

**Lemma 21.** *For a sequence $f$ of nonzero numbers, suppose $1 \leq k \leq n$. If $f_{n+1} = uf_{n-k+1} + vf_k$ for some $u$ and $v$, then* $\begin{bmatrix} n+1 \\ k \end{bmatrix}_f = u \cdot \begin{bmatrix} n \\ k \end{bmatrix}_f + v \cdot \begin{bmatrix} n \\ k-1 \end{bmatrix}_f$.

*Proof.* Given $\frac{f_{n+1}}{f_k f_{n-k+1}} = u \cdot \frac{1}{f_k} + v \cdot \frac{1}{f_{n-k+1}}$, multiply by $\frac{\langle n \rangle !_f}{\langle k-1 \rangle !_f \langle n-k \rangle !_f}$ to obtain the stated conclusion. □

**Definition 22.** An integer sequence $f$ is a *ComboSum* sequence if for every $m, n$

$$f_m \wedge f_n \mid f_{m+n}.$$

By elementary number theory, that condition is equivalent to saying

$$f_{m+n} = u \cdot f_m + v \cdot f_n \text{ for some } u, v \in \mathbb{Z}.$$

Certainly any GCD sequence is a ComboSum. In particular, the Lucas sequences $U_{P,Q}$ have the ComboSum property.

**Proposition 23.** *Every ComboSum sequence is binomid.*

*Proof.* If $f$ is a ComboSum and $0 < k < n$, then $f_{n+1} = uf_{n-k+1} + vf_k$, for some $u, v \in \mathbb{Z}$, and Lemma 21 applies. An inductive proof shows that $f$ is binomid. □

Every ComboSum sequence is divisible. For when $m = n$, Definition 22 implies $f(n) \mid f(2n)$. An inductive argument shows that $f(n) \mid f(kn)$ for every $k \in \mathbb{Z}^+$.

Sequence $(1, 2, 2, 2, \dots)$ is a ComboSum sequence that is not a divisor-product.

**Open Question 24.** Must a ComboSum sequence be binomid at every level?

### 4.3. Polynomial Sequences

A polynomial $f \in \mathbb{C}[x]$ is called *integer-valued* if $f(n) \in \mathbb{Z}$ for every $n = 1, 2, 3, \ldots$ Which integer-valued polynomials provide sequences that satisfy the various conditions discussed above?

**Proposition 25.** *If $f$ is an integer-valued polynomial such that $f(n) \mid f(2n)$ for infinitely many $n \in \mathbb{Z}^+$, then $f(x) = bx^d$ for some $b, d \in \mathbb{Z}$ with $d \geq 0$.*

We omit the proof. This result helps determine all the polynomial sequences that are divisible. It is more difficult to determine which polynomials are binomid.

Recall that the Pascal column polynomial $C_m(x) = \binom{x+m-1}{m}$ is integer-valued, degree $m$, and $C_m(1) = 1$. By Theorem 15, $C_m$ is binomid.

For $m \in \mathbb{Z}^+$ consider $H_m(x) = \binom{mx}{m}$. Then $H_m(x)$ is an integer-valued polynomial of degree $m$, and $H_m(1) = 1$. It is straightforward to verify that $\langle n \rangle!_{H_m} = \frac{(mn)!}{(m!)^n}$, so that $\left[ {n \atop k} \right]_{H_m} = \binom{mn}{mk}$. Since binomial coefficients are integers, $H_m$ is binomid.

**Theorem 26.** *Let $f$ be a binomid polynomial sequence with $f(1) = 1$. If $\deg(f) \leq 2$ then $f$ is one of* $1,\ x,\ \binom{x+1}{2},\ x^2,\ \binom{2x}{2}$.

The proof involves many details and will appear in a separate paper. Higher degree cases seem much more difficult.

**Open Problem 27.** Find all binomid polynomial sequences of degree 3.

### 4.4. Linear Recurrences

Suppose $f$ is an integer sequence satisfying a linear recurrence of order 2:

$$f(n+2) = P \cdot f(n+1) - Q \cdot f(n) \quad \text{for every } n \geq 1, \tag{3}$$

where $P, Q \in \mathbb{Z}$. Suppose the associated polynomial is

$$p(x) = x^2 - Px + Q = (x - \alpha)(x - \beta), \text{ for } \alpha, \beta \in \mathbb{C}.$$

If $f(0) = 0$ then $f$ is a constant multiple of sequence $U_{P,Q}$ of Lemma 16, and $f$ enjoys most of the properties mentioned above. It is divisible, a divisor-product, a GCD sequence, a ComboSum, and is binomid at every level.

If $f(0) \neq 0$, can $f$ still satisfy some of those properties? If $Q = 0$ then $f$ satisfies a recurrence of order 1: $f(n+1) = Pf(n)$ for $n \geq 2$. Then $f(n)$ has the form $a \cdot P^{n-1}$ (for $n \geq 2$) and it is not hard to determine which of those properties $f$ satisfies. We assume below that $Q \neq 0$.

For sequences $f$ satisfying a linear recurrence (of any order), Kimberling [17] proved that if $f$ is a GCD sequence with $f(0) \neq 0$, then $f$ must be periodic. For the order 2 case, all the periodic GCD sequences are listed in [16]. The next result uses a much weaker hypothesis.

**Proposition 28.** *Suppose $f$ satisfies Recurrence (3) above, and $Q \cdot f(0) \neq 0$. If $f$ is divisible, then $\alpha/\beta$ is a root of unity.*

*Proof Outline.* If a divisibility sequence $f$ satisfies a linear recurrence, Hall [14] noted that every prime factor of any $f(n)$ also divides $Q{\cdot}f(0)$. (In fact, for an order 2 recurrence $f(n) \mid Q^n \cdot f(0)$ for every $n$.) Then the set of all $f(n)$ involves only finitely many different prime factors.

Ward [28] showed that if $f$ is non-degenerate (meaning that $\alpha/\beta$ is not a root of unity), then the values $f(n)$ involve infinitely many prime factors. This completes the proof. □

A version of this proposition is valid for all linearly recurrent sequences, not just those of order 2. To prove this, apply the generalization of Ward's Theorem established by Laxton [20].

Proposition 28 can be used to make a complete list of divisible sequences that satisfy an order 2 recurrence. In addition to the Lucas sequences and exponential sequences, there are a few periodic cases with periods 1, 2, 3, 4 or 6. It is worth noting that A. Granville [13] has studied dvisible sequences in much greater depth.

In summary, among all sequences $f$ satisfying Recurrence (3), we can list all those that are GCD, or divisor-product, or ComboSum, since each of those properties implies divisibility. The situation is more difficult for binomid sequences.

**Open Questions 29.**

(1) Which sequences $f$ satisfying Recurrence (3) are binomid? If such $f$ is binomid (or binomid at every level), must $f$ be divisible?

(2) What if we allow sequences satisfying a linear recurrence of order $> 2$?

Recall that the sequence $T(n) = \binom{n+1}{2}$ is binomid, but is not divisible and is not binomid at level 2. This $T$ satisfies a linear recurrence of order 3 with polynomial $p(x) = (x-1)^3$.

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