PARITY PALINDROME COMPOSITIONS

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Abstract

A parity palindrome is a finite sequence of positive integers which when reduced modulo 2 reads the same from back to front as front to back. Compositions that are parity palindromes have 'a surprisingly' nice enumerating function. It will be proved that the number of such compositions of $2n+1$ and also of $2n$ is $2 \cdot 3^{n-1}$. Further refinements and implications are also explored.

1. Introduction

Compositions are representations of integers as ordered sums of integers. This contrasts with partitions, where order is ignored. Thus while there are five partitions of 4:

$$4, 3+1, 2+2, 2+1+1, 1+1+1+1$$

there are eight compositions:

$$4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1.$$

Compositions provide a lovely topic in elementary combinatorics because counting them provides examples of nice formulas and elegant succinct proofs. As an example, there are 2^{n-1} compositions of n. In Section 2, we provide a short survey of these results.

Our subject here is the study of compositions that are *parity palindromes*. This means that, modulo 2, the composition reads the same back to front as front to back.

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We show that there are also elegant theorems for parity palindrome compositions which, surprisingly, are much harder to prove than comparable results for ordinary compositions.

Theorem 1. *The number of parity palindrome compositions (PPC's) of $2n$ equals the number of PPC's of $2n+1$ which in turn equals $2\cdot 3^{n-1}$.*

For example, both 4 $(4, 3+1, 1+3, 2+2, 1+2+1, 1+1+1+1)$ and 5 $(5, 3+1+1, 1+3+1, 1+1+3, 2+1+2, 1+1+1+1+1)$ have $6 = 2\cdot 3^{2-1}$ parity palindrome compositions.

We now define $\mathrm{ppc}(n,s)$ to be the number of parity palindrome compositions of n with s parts.

Theorem 2. *The following identities hold for PPC's depending upon the parity of their arguments:*

$$\mathrm{ppc}(2k+1, 2s+1) = \mathrm{ppc}(2k+2, 2s+1) = \sum_{j=0}^{s} 2^j \binom{k}{s+j}\binom{s}{j} \tag{1.1}$$

$$= \sum_{j=0}^{s} \binom{k+j}{2s}\binom{s}{j},$$

$$\mathrm{ppc}(2k+1, 2s) = 0, \tag{1.2}$$

$$\mathrm{ppc}(2k, 2s) = \sum_{j=0}^{s} 2^s \binom{k-1}{j+s-1}\binom{s}{j} = \sum_{j=0}^{s} \binom{k+j-1}{2s-1}\binom{s}{j}. \tag{1.3}$$

In Section 2, we recount the classical results on ordinary compositions and palindrome compositions. The proofs of these results are remarkably elegant and straightforward.

To make the proofs of Theorems 1 and 2 readable, we have collected a number of combinatorial identities in Section 3. Section 4 provides proofs of Theorems 1 and 2. Section 5 considers further aspects of PPC's. Section 6 is devoted to open questions.

2. The Classical Results

The point of this section is to contrast compositions and palindrome compositions with parity palindrome compositions (PPC's). A palindrome composition is, as the name implies, a composition that reads the same from back to front as from front to back.

Theorem 3. *There are 2^{n-1} compositions of n.*

Theorem 4. *There are $\binom{n-1}{s-1}$ compositions of n with s parts.*

Remark 1. Both of these theorems can be found in the works of Netto [4, p. 120] and MacMahon [2, p. 621], [3, p. 151]. We follow natural recursive proofs.

Proof of Theorem 3. Split the compositions of n into two classes: (1) those that end in 1, and (2) those that do not. For class (1), remove the ending 1's, thus leaving all the compositions of $n-1$. For class (2), subtract 1 from the last part of each composition, again leaving all the compositions of $n-1$. Thus there are twice as many compositions of n as of $n-1$. Finally since there is one ($= 2^{1-1}$) composition of 1, we see that the result follows. □

Proof of Theorem 4. Let $\mathrm{c}(n, s)$ denote the number of compositions of n into s parts. The exact same argument used in the proof of theorem 3 reveals

$$\mathrm{c}(n, s) = \mathrm{c}(n-1, s-1) + \mathrm{c}(n-1, s).$$

This recurrence together with

$$\mathrm{c}(n, 1) = \mathrm{c}(n, n) = 1,$$

reveals immediately (by mathematical induction) that

$$\mathrm{c}(n, s) = \binom{n-1}{s-1}.$$

□

Let $\mathrm{pc}(n, s)$ denote the number of palindrome compositions of n into s parts, and $\mathrm{pc}(n)$ denote the total number of palindrome compositions of n.

Theorem 5. *For $n \geq 1$, we have* $\mathrm{pc}(n) = 2^{\lfloor \frac{n}{2} \rfloor}$.

Proof. We mimic the proof of Theorem 3. The first and last parts are identical. If they are both 1, delete them leaving a palindrome composition of $n-2$. If they are both greater than 1, subtract 1 from each again leaving a palindrome composition of $n-2$. Thus we have a bijection between the palindrome compositions of n and two copies of those for $n-2$. Hence

$$pc(n) = 2\,\mathrm{pc}(n-2).$$

Noting that $\mathrm{pc}(1) = 1$ and $\mathrm{pc}(2) = 2$, we see that the desired result follows by mathematical induction. □

Theorem 6. *The following identities hold for pc's, again depending on the parity of their arguments:*

$$\mathrm{pc}(2n, 2m) = \mathrm{pc}(2n, 2m-1) = \mathrm{pc}(2n-1, 2m-1) = \binom{n-1}{m-1}, \mathrm{pc}(2n-1, 2m) = 0.$$

Proof. Now we mimic the proof of Theorem 4. As in the proof of Theorem 5, given a palindrome partition of n into s parts, the first and last parts are identical. If they are both 1, delete them leaving a palindrome composition of $n-2$ into $s-2$ parts. If they are both greater than 1, subtract 1 from each leaving a palindrome composition of $n-2$ into s parts. This operation yields a bijection between the palindrome compositions of n into s parts and those of $n-2$ with either $s-2$ or s parts. Hence

$$\mathrm{pc}(n, s) = \mathrm{pc}(n-2, s-2) + \mathrm{pc}(n-2, s).$$

Also,

$$\mathrm{pc}(2n-1, 2m) = 0$$

follows from the fact that any palindrome composition with an even number of parts must necessarily be a composition of an even number.

Finally, we see immediately that $\mathrm{pc}(n, 1) = \mathrm{pc}(n, n) = 1$. Thus the assertions now follow from the recurrences and initial condition by mathematical induction. □

3. Necessary Lemmas

The combinatorial ease of the proofs of Theorems 3-6 is very appealing. A natural bijective proof of recurrences establishes everything.

The treatment of parity palindrome compositions is not nearly so straightforward. To prove Theorems 1 and 2, we require the following results.

Lemma 1. *For* $k \geq 2, \sum_{j=1}^{k-1} (2j-1)3^{k-j} = 3^k - 3k.$

Proof. This is a straightforward mathematical induction exercise. □

Lemma 2. *For* $N \geq 0, \sum_{j=0}^{M} \binom{j}{N} = \binom{M+1}{N+1}.$

Proof. This classical result is also an elementary exercise in mathematical induction. □

Lemma 3. *For* $A \geq 0, k \geq 1$*, the binomial identity* $\sum_{N=1}^{k} (2N-1)\binom{k-N}{A} = \binom{k}{A+2} + \binom{k+1}{A+2}$ *holds.*

Proof. This is slightly trickier than the previous results. Begin by considering

$$\begin{aligned}\sum_{N=1}^{k}(2N-1)\binom{k-N}{A} &= \sum_{N=0}^{k-1}(2(k-N)-1)\binom{k-(k-N)}{A}\\ &\text{(reversing summation order)}\\ &= \sum_{N=0}^{k-1}(2k+1-(2N+2))\binom{N}{A}\\ &= (2k+1)\sum_{N=0}^{k-1}\binom{N}{A} - 2\sum_{N=0}^{k-1}(N+1)\binom{N}{A}\\ &= (2k+1)\sum_{N=0}^{k-1}\binom{N}{A} - 2(A+1)\sum_{N=0}^{k-1}\binom{N+1}{A+1}\\ &= \binom{k}{A+2}+\binom{k+1}{A+2},\end{aligned}$$

where the final line follows from Lemma 2 and algebraic simplifications. □

We only require the first entry in the next result, but both results are needed for the mathematical induction proof.

Lemma 4. *For* $k \geq 1$*,*

$$S_1(k) := \sum_{j,s\geq 0}\binom{k+j}{2s}\binom{s}{j} = 2\cdot 3^{k-1}$$

$$S_2(k) := \sum_{j,s\geq 0}\binom{k+j}{2s-1}\binom{s}{j} = 4\cdot 3^{k-1}.$$

Proof. We proceed by induction on k. We have

$$S_1(1) = \binom{1}{0}+\binom{2}{2}\binom{1}{1} = 2 = 2\cdot 3^{1-1};$$

$$S_2(1) = \binom{1}{1}\binom{1}{0}+\binom{2}{1}\binom{1}{1}+\binom{3}{3}\binom{2}{2} = 4 = 4\cdot 3^{1-1}.$$

Now assume both identities are true for each value of k less than a given k. Thus

$$\begin{aligned}S_1(k) &= \sum_{j,s\geq 0}\left(\binom{k+j-1}{2s}+\binom{k+j-1}{2s-1}\right)\binom{s}{j}\\ &= S_1(k-1)+S_2(k-1)\\ &= 2\cdot 3^{k-2}+4\cdot 3^{k-2}\\ &= 6\cdot 3^{k-2} = 2\cdot 3^{k-1},\end{aligned}$$

and

$$\begin{aligned}
S_2(k) &= \sum_{j\geq 0, s\geq 1}\left(\binom{k+j-1}{2s-1}+\binom{k+j-1}{2s-2}\right)\binom{s}{j}\\
&= S_2(k-1)+\sum_{s,j\geq 0}\binom{k+j-1}{2s}\binom{s+1}{j}\\
&= S_2(k-1)+\sum_{s,j\geq 0}\binom{k+j-1}{2s}\left(\binom{s}{j}+\binom{s}{j-1}\right)\\
&= S_2(k-1)+S_1(k-1)+\sum_{s,j\geq 0}\binom{k+j}{2s}\binom{s}{j}\\
&= S_2(k-1)+S_1(k-1)+S_1(k)\\
&= 4\cdot 3^{k-2}+2\cdot 3^{k-2}+2\cdot 3^{k-1}\\
&= 2\cdot 3^{k-2}(2+1+3)\\
&= 4\cdot 3^{k-1},
\end{aligned}$$

and the lemma follows by mathematical induction. □

Lemma 5. *For $k \geq 0$ and $s \geq 0$,* $\sum_{j=0}^{s} 2^j \binom{k}{s+j}\binom{s}{j} = \sum_{j=0}^{s}\binom{k+j}{2s}\binom{s}{j}$.

Proof. We begin by using a binomial identity:

$$\begin{aligned}
\sum_{j=0}^{s} 2^j \binom{k}{s+j}\binom{s}{j} &= 2^s \sum_{j=0}^{s} 2^{-j}\binom{k}{2s-j}\binom{s}{j} \text{ (by } j\to s-j)\\
&= 2^s\binom{k}{2s}\sum_{j=0}^{s}\frac{(-2s)_j(-s)_j 2^{-j}}{(k-2s+1)_j j!}\\
&\text{(where } (A)_j = A(A+1)\cdots(A+j-1))\\
&= \binom{k}{2s}\sum_{j=0}^{s}\frac{(-s)_j(k+1)_j(-1)^j}{j!(k-2s+1)_j}\\
&\text{(by [1, p. 10, eq.(1)] with } z=-1)\\
&= \sum_{j=0}^{s}\frac{(k+j)!}{(2s)!(k-2s+j)!}\binom{s}{j}\\
&= \sum_{j=0}^{s}\binom{k+j}{2s}\binom{s}{j}.
\end{aligned}$$

□

4. Proofs of Theorems 1 and 2

We have relegated much of the mechanics of our proofs to the previous section in order to make the work in this section more transparent. We begin with a direct proof of Theorem 1. We then proceed to Theorem 2.

The first paragraph in the proof of Theorem 1 is the critical one in this section. Everything else relies on exactly this argument and the recurrences it produces. We shall denote by $\mathrm{ppc}(n)$ the number of parity palindrome compositions of n.

Proof of Theorem 1. Let us analyze the first and last summands in a parity palindrome composition (=PPC) of $2k$. First note that there is also the one term PPC, namely $2k$ itself. Otherwise the first and last summands can be either (1) $(2i-1)$ and $(2j-1)$ with the remaining parts being a PPC for $2k-2i-2j+2$, or (2) $2i$ and $2j$ with remaining parts being a PPC for $2k-2i-2j$. Hence

$$\begin{aligned}
\mathrm{ppc}(2k) &= 1 + \sum_{i,j\geq 1} \mathrm{ppc}(2k-2i-2j+2) + \sum_{i,j\geq 1} \mathrm{ppc}(2k-2i-2j) \\
&= 1 + \sum_{\substack{N\geq 2 \\ i+j=N \\ i,j\geq 1}} \mathrm{ppc}(2k-2N+2) + \sum_{\substack{N\geq 2 \\ i+j=N \\ i,j\geq 1}} \mathrm{ppc}(2k-2N) \\
&= 1 + \sum_{N=2}^{k+1} (N-1)\,\mathrm{ppc}(2k-2N+2) + \sum_{N=2}^{k} (N-1)\,\mathrm{ppc}(2k-2N) \\
&= 1 + \sum_{N=1}^{k} N\,\mathrm{ppc}(2k-2N) + \sum_{N=1}^{k} (N-1)\,\mathrm{ppc}(2k-2N) \\
&= 1 + \sum_{N=1}^{k} (2N-1)\,\mathrm{ppc}(2k-2N).
\end{aligned}$$

We now proceed, by strong mathematical induction, to prove $\mathrm{ppc}(2k) = 2\cdot 3^{k-1}$. When $k=1, \mathrm{ppc}(2) = 2 = 2\cdot 3^0$ where the PPC's are 2 and $1+1$.

Now assume $\mathrm{ppc}(2i) = 2\cdot 3^{i-1}$ for all $i<k$ for a particular k. Then

$$\begin{aligned}
\mathrm{ppc}(2k) &= 1 + \sum_{N=1}^{k} (2N-1)\,\mathrm{ppc}(2k-2N) \\
&= 1 + \sum_{N=1}^{k-1} (2N-1)\,\mathrm{ppc}(2k-2N) + (2k-1)\,\mathrm{ppc}(0) \\
&= 1 + \sum_{N=1}^{k-1} (2N-1)\cdot 2\cdot 3^{k-N-1} + (2k-1 \\
&= 1 + 2\cdot 3^{k-1} - 2k + 2k - 1 = 2\cdot 3^{k-1}.
\end{aligned}$$

Thus the case for $2k$ is proved via strong mathematical induction.

Finally, we need to show that

$$\mathrm{ppc}(2k+1) = 2 \cdot 3^{k-1} \text{ for } k \geq 1, \mathrm{ppc}(1) = 1.$$

It is not hard to see that ppc(2k+1) = ppc(2k). There is a simple bijection between the two sets of compositions. Namely, all the PPC's for 2k+1 must be of odd length. Subtract 1 from the middle entry (or delete the middle entry if it is a 1). The reverse map is clear and establishes the bijection. □

Proof of Theorem 2. We proceed step by step. First, to see that

$$\mathrm{ppc}(2k+2, 2s+1) = \mathrm{ppc}(2k+1, 2s+1),$$

we note that because the number of parts is odd, the parity of the number considered must be the same as the middle part of the PPC in question. We now provide a bijection between the PPC's of $2k+2$ into $2s+1$ parts and those of $2k+1$ into $2s+1$ parts. Namely for $2k+2$, we subtract 1 from the middle part, and the inverse map adds 1 to the middle part. For example, $\mathrm{ppc}(5,3) = \mathrm{ppc}(6,3) = 4$, and the bijection is as follows:

1 1 3	1 2 3
1 3 1	1 4 1
3 1 1	3 2 1
2 1 2	2 2 2

Next we shall prove that

$$\mathrm{ppc}(2k+1, 2s+1) = \sum_{j=0}^{s} 2^j \binom{k}{s+j} \binom{s}{j}.$$

The argument given in the first paragraph of the proof of Theorem 1 may be copied here to show that

$$\mathrm{ppc}(2k+1, 2s+1) = \sum_{N=1}^{k} (2N-1)\, \mathrm{ppc}(2k+1-2N, 2s-1).$$

The only addition to the argument is the observation that when the first and last part are removed from the PPC, then the number of parts is reduced by 2.

We now proceed by induction on s. When $s = 0$, the assertion is 1=1.

Now assume the assertion is true up to but not including a given s:

$$\begin{aligned}
\operatorname{ppc}(2k+1,2s+1) &= \sum_{N=1}^{k}(2N-1)\operatorname{ppc}(2k+1-2N,2s-1)\\
&= \sum_{N=1}^{k}(2N-1)\sum_{j=0}^{s-1}2^j\binom{k-N}{s-1+j}\binom{s-1}{j}\\
&= \sum_{j=0}^{s-1}2^j\binom{s-1}{j}\sum_{N=1}^{k}(2N-1)\binom{k-N}{s-1+j}\\
&= \sum_{j=0}^{s-1}2^j\binom{s-1}{j}\left(\binom{k}{s+1+j}+\binom{k+1}{s+1+j}\right) \text{ (by Lemma 3)}\\
&= \sum_{j=0}^{s-1}2^j\binom{s-1}{j}\left(2\binom{k}{s+1+j}+\binom{k}{s+j}\right)\\
&= \sum_{j=0}^{s-1}2^{j+1}\binom{s-1}{j}\binom{k}{s+1+j}+\sum_{j=0}^{s-1}2^j\binom{s-1}{j}\binom{k}{s+j}\\
&= \sum_{j=0}^{s}2^j\binom{s-1}{j-1}\binom{k}{s+j}+\sum_{j=0}^{s-1}2^j\binom{s-1}{j}\binom{k}{s+j}\\
&= \sum_{j=0}^{s}2^j\left(\binom{s-1}{j-1}+\binom{s-1}{j}\right)\binom{k}{s+j}\\
&= \sum_{j=0}^{s}2^j\binom{s}{j}\binom{k}{s+j},
\end{aligned}$$

and the assertion follows by strong mathematical induction.

The second line of Theorem 2 is a restatement of Lemma 5.

The assertion

$$\operatorname{ppc}(2k+1,2s)=0$$

follows immediately from the fact that a PPC with an even number of parts must be a composition of an even number because part i and $2s+1-i$ have the same parity.

Next we must show that

$$\operatorname{ppc}(2k,2s)=\sum_{j=0}^{s}2^j\binom{k-1}{j+s-1}\binom{s}{j}.$$

As before, we see that the argument given in the first paragraph of the proof of Theorem 1 (except that we must note that the removal of the first and last parts reduces $2s$ to $2s-2$). Hence

$$\mathrm{ppc}(2k,2s)=\sum_{N=1}^{k}(2N-1)\,\mathrm{ppc}(2k-2N,2s-2).$$

We proceed by strong mathematical induction on s. When $s=1$, we see that

$$\mathrm{ppc}(2k,2)=2k-1,$$

the relevant PPC's being $(1,2k-1),(2,2k-2),\ldots,(2k-1,1)$. On the other side,

$$\sum_{j=0}^{1}2^j\binom{k-1}{j}\binom{1}{j}=1+2(k-1)=2k-1.$$

Thus the case $s=1$ is established.

Now we assume the truth of our assertion for all values less than a particular s. It follows that

$$\begin{aligned}
\mathrm{ppc}(2k,2s)&=\sum_{N=1}^{k}(2N-1)\,\mathrm{ppc}(2k-2N,2s-2)\\
&=\sum_{N=1}^{k}(2N-1)\sum_{j=0}^{s-1}2^j\binom{k-N-1}{j+s-2}\binom{s-1}{j}\\
&=\sum_{j=0}^{s-1}\binom{s-1}{j}\sum_{N=1}^{k}(2N-1)2^j\binom{k-N-1}{j+s-2}\\
&=\sum_{j=0}^{s-1}2^j\binom{s-1}{j}\left(\binom{k-1}{j+s}+\binom{k}{j+s}\right)\quad\text{(by Lemma 3)}\\
&=\sum_{j=0}^{s-1}2^j\binom{s-1}{j}\left(2\binom{k-1}{j+s}+\binom{k-1}{j+s-1}\right)\\
&=\sum_{j=0}^{s-1}2^{j+1}\binom{s-1}{j}\binom{k-1}{j+s}+\sum_{j=0}^{s-1}2^j\binom{s-1}{j}\binom{k-1}{j+s-1}\\
&=\sum_{j=0}^{s}2^j\binom{s-1}{j-1}\binom{k-1}{j+s-1}+\sum_{j=0}^{s}2^j\binom{s-1}{j}\binom{k-1}{j+s-1}\\
&=\sum_{j=0}^{s}2^j\binom{k-1}{s+j-1}\left(\binom{s-1}{j-1}+\binom{s-1}{j}\right)\\
&=\sum_{j=0}^{s}2^j\binom{k-1}{j+s-1}\binom{s}{j}.
\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{j=0}^{s} 2^j \binom{k-1}{j+s-1}\binom{s}{j} &= \sum_{j=0}^{s} 2^j \left(\binom{k}{j+s} - \binom{k-1}{j+s}\right)\binom{s}{j} \\ &= \sum_{j=0}^{s} \binom{k+j}{2s}\binom{s}{j} - \sum_{j=0}^{s} \binom{k+j-1}{2s}\binom{s}{j} \\ &= \sum_{j=0}^{s} \left(\binom{k+j}{2s} - \binom{k+j-1}{2s}\right)\binom{s}{j} \\ &= \sum_{j=0}^{s} \binom{k+j-1}{2s-1}\binom{s}{j}.\end{aligned}$$

□

5. Some Further Simplifications

Corollary 1. *For* $T \geq 1, \sum_{k=1}^{T} \text{ppc}(2k, 2s) = \text{ppc}(2T+1, 2s+1)$.

First proof. We have

$$\begin{aligned}\sum_{k=1}^{T} \text{ppc}(2k, 2s) &= \sum_{k=1}^{T}\sum_{j=0}^{s} 2^j \binom{k-1}{j+s-1}\binom{s}{j} \\ &= \sum_{j=0}^{s} 2^j \binom{T}{j+s}\binom{s}{j} \quad \text{(by Lemma 2)} \\ &= \text{ppc}(2T+1, 2s+1).\end{aligned}$$

□

Second proof. The central part in PPC's of $2T+1$ into $2s+1$ parts must also be odd. If this central odd part, $2i+1$, is removed, the result is a PPC of $(2T+1)-(2i+1)$

into $2s$ parts, hence

$$\begin{aligned}\operatorname{ppc}(2T+1,2s+1) &= \sum_{i=0}^{T} \operatorname{ppc}((2T+1)-(2i+1),2s)\\ &= \sum_{i=0}^{T} \operatorname{ppc}(2(T-i),2s)\\ &= \sum_{i=0}^{T} \operatorname{ppc}(2i,2s)\\ &= \sum_{i=1}^{T} \operatorname{ppc}(2i,2s).\end{aligned}$$

□

Corollary 2. *Theorem 2 implies Theorem 1.*

Proof. Clearly,

$$\operatorname{ppc}(n) = \sum_{s=0}^{n} \operatorname{ppc}(n,s).$$

Hence

$$\begin{aligned}\operatorname{ppc}(2n) &= \sum_{s=0}^{2n} \operatorname{ppc}(2n,s)\\ &= \sum_{s=0}^{n} \operatorname{ppc}(2n,2s) + \sum_{s=0}^{n-1} \operatorname{ppc}(2n,2s+1)\\ &= \sum_{s=0}^{n}\sum_{j=0}^{s} \binom{n+j-1}{2s-1}\binom{s}{j} + \sum_{s=0}^{n-1}\sum_{j=0}^{s} \binom{n+j-1}{2s}\binom{s}{j}\\ &= \sum_{s,j\geq 0} \binom{n+j}{2s}\binom{s}{j}\\ &= 2\cdot 3^{n-1}\end{aligned}$$

by Lemma 4. Next,

$$\begin{aligned}\operatorname{ppc}(2n+1) &= \sum_{s=0}^{2n+1} \operatorname{ppc}(2n+1,s)\\ &= \sum_{s=0}^{n} \operatorname{ppc}(2n+1,2s+1) \text{ (because } \operatorname{ppc}(2n+1,2s)=0)\\ &= \sum_{s,j\geq 0} \binom{k+j}{2s}\binom{s}{j} = 2\cdot 3^{n-1}.\end{aligned}$$

□

We close this section with a surprising relation between compositions with parts having k colors and parity palindrome compositions.

Let $T_k(n)$ denote the number of compositions of n with k colors.

Theorem 7. *For $n, k \geq 1$, we have $T_k(n) = k(k+1)^{n-1}$.*

Proof. Clearly $T_k(1) = k$. Also $T_k(n) = (k+1)T_k(n-1)$ owing to the following natural bijection. Namely, split the compositions of n into $(k+1)$ classes as follows.

The first k classes end with $1_i; 1 \leq i \leq k$. Removing the 1_i leaves the compositions of $n-1$. The final class consists of compositions of n with the last part $m_j > 1$. Replacing m_j by $(m-1)_j$ yields again all the compositions of $n-1$. Thus we have divided the compositions of n into $k+1$ disjoint subsets each with $T_k(n-1)$ elements. Hence

$$T_k(n) = (k+1)T_k(n-1),$$

and the desired result follows by induction. □

Corollary 3. *For $n \geq 1$, we have $T_2(n) = \mathrm{ppc}(2n+1) = \mathrm{ppc}(2n)$.*

Proof. This follows immediately from Theorems 1 and 7 and the fact that each of these functions equals $2 \cdot 3^{n-1}$. □

6. Conclusion

A theme of this paper is the contrast between the simplicity of the proofs given in Section 2 for Theorems 3-6, and the seemingly necessary intricacy of the proofs of Theorems 1 and 2. Especially striking is the similarity and simplicity of the statements of Theorems 1 and 5. Clearly Theorem 1 (if not Theorem 2) cries out for a much more combinatorial bijective proof, perhaps involving Corollary 3.

One is tempted to consider other moduli besides 2 and to invoke palindromes modulo k. So far we have not found results with the elegance of Theorems 1 and 2.

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