THE DENOMINATORS OF POWER SUMS OF ARITHMETIC PROGRESSIONS

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Abstract

In a recent paper the authors studied the denominators of polynomials that represent power sums by Bernoulli's formula. Here we extend our results to power sums of arithmetic progressions. In particular, we obtain a simple explicit criterion for integrality of the coefficients of these polynomials. As applications, we obtain new results on the sequence of denominators of the Bernoulli polynomials. A consequence is that certain quotients of successive denominators are infinitely often integers, which we characterize.

1. Introduction

For positive integers n and x, define the *power sum*

$$S_n(x) := \sum_{k=0}^{x-1} k^n = 0^n + 1^n + \cdots + (x-1)^n,$$

and for integers $m \geq 1$ and $r \geq 0$ define the more general *power sum of an arithmetic progression*

$$\mathcal{S}_{m,r}^n(x) := \sum_{k=0}^{x-1} (km+r)^n = r^n + (m+r)^n + \cdots + ((x-1)m+r)^n.$$

In particular, we have $\mathcal{S}_{1,0}^n(x) = S_n(x)$ and, more generally,

$$\mathcal{S}_{m,0}^n(x) = m^n S_n(x). \tag{1}$$

Bazsó et al. [2, 3] considered the generalized Bernoulli formula

$$\mathcal{S}_{m,r}^{n}(x) = \frac{m^n}{n+1}\left(B_{n+1}\Big(x+\frac{r}{m}\Big) - B_{n+1}\Big(\frac{r}{m}\Big)\right), \tag{2}$$

where the nth Bernoulli polynomial $B_n(x)$ is defined by the series

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi)$$

and is given by the formula

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, x^{n-k}, \tag{3}$$

$B_k = B_k(0) \in \mathbb{Q}$ being the kth Bernoulli number. Thus, $\mathcal{S}_{m,r}^{n}(x)$ is a polynomial in x of degree $n+1$ with rational coefficients.

Remark. Bazsó et al. required r and m to be coprime. However, since the forward difference $\Delta B_n(x) := B_n(x+1) - B_n(x)$ equals nx^{n-1} (cf. [7, Eq. (5), p. 18]), the telescoping sum of these differences with $x = k + \frac{r}{m}$ implies (2) at once for any $r/m \in \mathbb{Q}$.

For a polynomial $f(x) \in \mathbb{Q}[x]$, define its *denominator*, denoted by $\operatorname{denom}\big(f(x)\big)$, to be the smallest $d \in \mathbb{N}$ such that $d \cdot f(x) \in \mathbb{Z}[x]$. This includes the usual definition of $\operatorname{denom}(q)$ for $q \in \mathbb{Q}$.

In the classical case of Bernoulli's formula

$$S_n(x) = \frac{1}{n+1}\big(B_{n+1}(x) - B_{n+1}\big),$$

the authors [6, Thms. 1 and 2] determined the denominator of the polynomial $S_n(x)$. From now on, let p denote a prime.

Theorem 1 (Kellner and Sondow [6]). *For $n \geq 1$, denote*

$$\mathbb{D}_n := \operatorname{denom}\big(B_n(x) - B_n\big). \tag{4}$$

Then we have the relation

$$\operatorname{denom}\big(S_n(x)\big) = (n+1)\,\mathbb{D}_{n+1}$$

and the remarkable formula

$$\mathbb{D}_n = \prod_{\substack{p \,\leq\, \mathcal{M}_n \\ s_p(n) \,\geq\, p}} p \quad \text{with} \quad \mathcal{M}_n := \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{n+1}{3}, & \text{if } n \text{ is even,} \end{cases} \tag{5}$$

where $s_p(n)$ denotes the sum of the base-p digits of n, as defined in Section 4. Moreover,

$$\mathbb{D}_n \text{ is odd} \quad \text{if and only if} \quad n = 2^k \ \ (k \geq 0). \tag{6}$$

The first few values of $\mathbb{D}_n$ are (see [11, Seq. A195441])

$$\mathbb{D}_n = 1, 1, 2, 1, 6, 2, 6, 3, 10, 2, 6, 2, 210, 30, 6, 3, 30, 10, 210, 42, 330, \ldots .$$

The sequence $(\mathbb{D}_n)_{n\geq 1}$ and its properties will play a central role in this paper. The denominators $\mathbb{D}_n$ are involved in formulas for related denominators in an essential way. As implied by the product formula (5), it turns out that the values of $\mathbb{D}_n$ obey certain divisibility properties. This culminates in the fact that certain quotients of successive denominators $\mathbb{D}_n$ are infinitely often integers, as we will see.

Here we extend Theorem 1 to the denominator of $\mathcal{S}^n_{m,r}(x)$, as follows.

Theorem 2. *We have*

$$\operatorname{denom}\big(\mathcal{S}^n_{m,r}(x)\big) = \frac{n+1}{\gcd(n+1, m^n)} \cdot \frac{\mathbb{D}_{n+1}}{\gcd(\mathbb{D}_{n+1}, m)}. \tag{7}$$

In particular, $\operatorname{denom}\big(\mathcal{S}^n_{m,r}(x)\big)$ *divides* $\operatorname{denom}\big(S_n(x)\big)$ *and is independent of* r*. Moreover, for any integers* $r_1, r_2 \geq 0$*,*

$$\mathcal{S}^n_{m,r_1}(x) - \mathcal{S}^n_{m,r_2}(x) \in \mathbb{Z}[x].$$

The next theorem shows exactly when $\mathcal{S}^n_{m,r}(x)$ itself lies in $\mathbb{Z}[x]$.

Theorem 3. *For* $n \geq 1$*, denote*

$$\mathfrak{D}_n := \operatorname{denom}\big(B_n(x)\big), \quad \mathbf{D}_n := \operatorname{denom}(B_n).$$

Then we have the equivalence

$$\mathcal{S}^n_{m,r}(x) \in \mathbb{Z}[x] \quad \textit{if and only if} \quad \mathfrak{D}_n \mid m$$

as well as the equalities

$$\mathfrak{D}_n = \operatorname{lcm}(\mathbb{D}_n, \mathbf{D}_n) \tag{8}$$

and

$$\mathfrak{D}_n = \operatorname{lcm}\big(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\big), \tag{9}$$

where $\operatorname{rad}(k) := \prod_{p \mid k} p$*.*

The first few values of $\mathfrak{D}_n$ and $\mathbf{D}_n$ are (see [11, Seqs. A144845 and A027642])

$$\mathfrak{D}_n = 2, 6, 2, 30, 6, 42, 6, 30, 10, 66, 6, 2730, 210, 30, 6, 510, 30, 3990, \ldots,$$
$$\mathbf{D}_n = 2, 6, 1, 30, 1, 42, 1, 30, 1, 66, 1, 2730, 1, 6, 1, 510, 1, 798, 1, 330, \ldots .$$

Example 1. Set $m = \mathfrak{D}_n = 2, 6, 2, 30, 6$ for $n = 1, 2, 3, 4, 5$, respectively. Then certainly $\mathfrak{D}_n \mid m$, so $\mathbb{Z}[x]$ contains the polynomials $\mathcal{S}^n_{m,r}(x)$ with $r = 0$ (which satisfy (1)):

$$\begin{aligned}
\mathcal{S}_{2,0}^{1}(x) &= x^2 - x = 2 \cdot \frac{1}{2}\,(x^2 - x) = 2 \cdot S_1(x),\\
\mathcal{S}_{6,0}^{2}(x) &= 6\,(2x^3 - 3x^2 + x) = 6^2 \cdot \frac{1}{6}\,(2x^3 - 3x^2 + x) = 6^2 \cdot S_2(x),\\
\mathcal{S}_{2,0}^{3}(x) &= 2\,(x^4 - 2x^3 + x^2) = 2^3 \cdot \frac{1}{4}\,(x^4 - 2x^3 + x^2) = 2^3 \cdot S_3(x),\\
\mathcal{S}_{30,0}^{4}(x) &= 27000\,(6x^5 - 15x^4 + 10x^3 - x)\\
&= 30^4 \cdot \frac{1}{30}\,(6x^5 - 15x^4 + 10x^3 - x) = 30^4 \cdot S_4(x),\\
\mathcal{S}_{6,0}^{5}(x) &= 648\,(2x^6 - 6x^5 + 5x^4 - x^2)\\
&= 6^5 \cdot \frac{1}{12}\,(2x^6 - 6x^5 + 5x^4 - x^2) = 6^5 \cdot S_5(x)
\end{aligned}$$

as well as those with $r = 1$:

$$\begin{aligned}
\mathcal{S}_{2,1}^{1}(x) &= x^2,\\
\mathcal{S}_{6,1}^{2}(x) &= 12x^3 - 12x^2 + x,\\
\mathcal{S}_{2,1}^{3}(x) &= 2x^4 - x^2,\\
\mathcal{S}_{30,1}^{4}(x) &= 162000x^5 - 378000x^4 + 217800x^3 + 24360x^2 - 26159x,\\
\mathcal{S}_{6,1}^{5}(x) &= 1296x^6 - 2592x^5 + 540x^4 + 1200x^3 - 273x^2 - 170x.
\end{aligned}$$

Remark. Bazsó and Mező [2, Eqs. (7), (8) and Thm. 2, pp. 121–122] defined a very complicated formula $F(n)$ in order to give a somewhat tautological characterization of when $\mathcal{S}_{m,r}^{n}(x) \in \mathbb{Z}[x]$. With their formula they computed a few values of $F(n)$ that apparently equal $\mathfrak{D}_n$, but without recognizing this relation. They were not aware of advanced results like those in our Theorems 2 and 3.

As an immediate by-"product" of our theorems, we obtain a new product formula for $\mathfrak{D}_n$ from (9) by applying Theorem 1. (Other explicit product formulas for this denominator, based on (8), were already given in [6, Thm. 4].)

Corollary 1. *For $n \geq 1$, the denominator of the nth Bernoulli polynomial equals*

$$\mathfrak{D}_n = \prod_{p \,\mid\, n+1} p \quad \times \prod_{\substack{p \,\nmid\, n+1 \\ p \,\leq\, \mathcal{M}_{n+1} \\ s_p(n+1) \,\geq\, p}} p.$$

Remark. The first author [5] has shown that the condition $s_p(n) \geq p$ is sufficient in (5) to define $\mathbb{D}_n$ as a product over all primes:

$$\mathbb{D}_n = \prod_{s_p(n) \,\geq\, p} p. \tag{10}$$

(So one can remove the restrictions $p \le \mathcal{M}_n$ in (5) and $p \le \mathcal{M}_{n+1}$ in Corollary 1.) Moreover, if $n+1$ is composite, then (see [5, Thm. 1])

$$\operatorname{rad}(n+1) \mid \mathbb{D}_n. \tag{11}$$

Finally, we obtain new properties of $\mathbb{D}_n$ and $\mathfrak{D}_n$.

Corollary 2. *The sequences $(\mathbb{D}_n)_{n\ge 1}$ and $(\mathfrak{D}_n)_{n\ge 1}$ satisfy the following conditions:*

(i) We have the relations

$$\begin{aligned} \mathbb{D}_n &= \operatorname{lcm}\big(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\big), \quad \text{if } n \ge 3 \text{ is odd,} \\ \mathfrak{D}_n &= \operatorname{lcm}\big(\mathfrak{D}_{n+1}, \operatorname{rad}(n+1)\big), \quad \text{if } n \ge 2 \text{ is even.} \end{aligned}$$

(ii) We have the divisibilities

$$\begin{aligned} \mathbb{D}_{n+1} &\mid \mathbb{D}_n, \quad \text{if } n \ge 1 \text{ is odd,} \\ \mathfrak{D}_{n+1} &\mid \mathfrak{D}_n, \quad \text{if } n \ge 2 \text{ is even.} \end{aligned}$$

Theorem 4. *For odd $n \ge 1$, the quotients (see [11, Seq. A286516])*

$$\frac{\mathbb{D}_n}{\mathbb{D}_{n+1}} = 1, 2, 3, 2, 5, 3, 7, 2, 3, 5, 11, 1, 13, 7, 15, 2, 17, 3, 19, 5, 7, \ldots$$

are odd, except that

$$\frac{\mathbb{D}_n}{\mathbb{D}_{n+1}} = 2 \quad \text{if and only if} \quad n = 2^k - 1 \ \ (k \ge 2).$$

Moreover, if p is an odd prime and $n = 2^\ell p^k - 1$, then

$$\frac{\mathbb{D}_n}{\mathbb{D}_{n+1}} \in \{1, p\} \quad (k, \ell \ge 1),$$

and more precisely,

$$\frac{\mathbb{D}_n}{\mathbb{D}_{n+1}} = p \qquad (k \ge 1,\ 1 \le \ell < \log_2 p),$$

while

$$\frac{\mathbb{D}_n}{\mathbb{D}_{n+1}} = 1 \qquad (k \ge 1,\ \ell \ge \mathcal{L}_p),$$

where $\mathcal{L}_p > \log_2 p$ is a constant depending on p.

Theorem 5. *For even $n \geq 2$, all terms are odd in the sequence (see [11, Seq. A286517])*

$$\frac{\mathfrak{D}_n}{\mathfrak{D}_{n+1}} = 3, 5, 7, 3, 11, 13, 5, 17, 19, 7, 23, 5, 3, 29, 31, 11, 35, 37, \dots .$$

In particular, if p is an odd prime and $n = p^k - 1$, then

$$\frac{\mathfrak{D}_n}{\mathfrak{D}_{n+1}} = p \quad (k \geq 1).$$

More generally, if $p \neq q$ are odd primes and $n = p^k q^\ell - 1$, then

$$\frac{\mathfrak{D}_n}{\mathfrak{D}_{n+1}} \in \{1, p, q, pq\} \quad (k, \ell \geq 1)$$

with the following cases:

$$\begin{aligned} \frac{\mathfrak{D}_n}{\mathfrak{D}_{n+1}} &= p \quad (k \geq \mathcal{L}'_{p,q},\ 1 \leq \ell < \log_q p), \\ \frac{\mathfrak{D}_n}{\mathfrak{D}_{n+1}} &= q \quad (1 \leq k < \log_p q,\ \ell \geq \mathcal{L}''_{p,q}), \\ \frac{\mathfrak{D}_n}{\mathfrak{D}_{n+1}} &= 1 \quad (k \geq \mathcal{L}'_{p,q},\ \ell \geq \mathcal{L}''_{p,q}), \end{aligned}$$

where $\mathcal{L}'_{p,q} > \log_p q$ and $\mathcal{L}''_{p,q} > \log_q p$ are constants depending on p and q.

Theorems 4 and 5 immediately imply the following result.

Corollary 3. *Statements (i), (ii) (respectively, (iii), (iv)) below hold for infinitely many odd (respectively, even) values of n:*

(i) $\mathbb{D}_n/\mathbb{D}_{n+1} = p$ *for a given prime* $p \geq 2$.

(ii) $\mathbb{D}_n = \mathbb{D}_{n+1}$.

(iii) $\mathfrak{D}_n/\mathfrak{D}_{n+1} = p$ *for a given prime* $p \geq 3$.

(iv) $\mathfrak{D}_n = \mathfrak{D}_{n+1}$.

2. Preliminaries

Let $\mathbb{Z}_p$ be the ring of p-adic integers, $\mathbb{Q}_p$ be the field of p-adic numbers, and $\mathrm{v}_p(s)$ be the p-adic valuation of $s \in \mathbb{Q}_p$ (see [9, Chap. 1.5, pp. 36–37]). If $s \in \mathbb{Z}$, then $p^e \parallel s$ means that $p^e \mid s$ but $p^{e+1} \nmid s$, or equivalently, $e = \mathrm{v}_p(s)$.

The Bernoulli numbers satisfy the following properties (cf. [4, Chap. 9.5, pp. 63–68]). The first few nonzero values are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42}, \tag{12}$$

while $B_n = 0$ for odd $n \geq 3$. For even $n \geq 2$ the von Staudt–Clausen theorem states that the denominator of B_n equals

$$\mathbf{D}_n = \prod_{p-1 \mid n} p \quad (n \in 2\mathbb{N}). \tag{13}$$

Thus, all nonzero Bernoulli numbers have a squarefree denominator. Moreover, for even $n \geq 2$ the p-adic valuation of the *divided Bernoulli number* B_n/n is

$$\mathrm{v}_p\left(\frac{B_n}{n}\right) = \begin{cases} -(\mathrm{v}_p(n)+1), & \text{if } p-1 \mid n, \\ \geq 0, & \text{else.} \end{cases} \tag{14}$$

Now let m, n, and r be positive integers. The Bernoulli polynomials satisfy as Appell polynomials the general relation

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(y)\, x^{n-k}, \tag{15}$$

of which (3) is a special case, as well as the reflection formula

$$B_n(1-x) = (-1)^n B_n(x) \tag{16}$$

(see [8, Chap. 3.5, pp. 114–115]). Further, denote by $\mathcal{B}^n_{m,r}$ the number

$$\mathcal{B}^n_{m,r} := m^n \left(B_n\left(\frac{r}{m}\right) - B_n \right) = \sum_{k=0}^{n-1} \binom{n}{k} B_k\, m^k r^{n-k}. \tag{17}$$

Almkvist and Meurman [1, Thm. 2, p. 104] showed that

$$\mathcal{B}^n_{m,r} \in \mathbb{Z}. \tag{18}$$

Actually, (18) holds for all $r \in \mathbb{Z}$ (cf. [4, Thm. 9.5.29, pp. 70–71]). We also point out an analog to (15) for $r_1, r_2 \in \mathbb{Z}$, namely,

$$\mathcal{B}^n_{m,r_1+r_2} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}^k_{m,r_1} r_2^{n-k} + \mathcal{B}^n_{m,r_2}.$$

The integers $\mathcal{B}^n_{m,r}$ satisfy a useful divisibility property, which we need later on. The following lemma is part of [4, Thm. 11.4.12, pp. 327–329], but we give here a clearer and simpler proof.

Lemma 1. *If $m, n \geq 1$, $r \in \mathbb{Z}$, a prime $p \nmid m$, and $0 \leq e \leq \mathrm{v}_p(n)$, then*

$$\mathcal{B}^n_{m,r} \equiv 0 \pmod{p^e}. \tag{19}$$

Proof. It suffices to prove the case $e = \mathrm{v}_p(n)$. If $e = 0$, then we are trivially done. So let $p^e \parallel n$ with

$$n > e = \mathrm{v}_p(n) \geq 1.$$

We split the proof into two cases as follows.

Case $p \mid r$: From (12) and (17) we deduce that

$$\begin{aligned} \mathcal{B}^n_{m,r} &= \sum_{k=0}^{n-1} \binom{n}{k} B_k \, m^k r^{n-k} \\ &= r^n + \sum_{k=1}^{n-1} n \binom{n-1}{k-1} B_{n-k} \, m^{n-k} \frac{r^k}{k}. \end{aligned}$$

Since $p \mid r$, we have $\mathrm{v}_p(r^n) \geq n$ and $\mathrm{v}_p(r^k/k) \geq 1$ for all $k \geq 1$. If $B_{n-k} \neq 0$, then $\mathrm{v}_p(B_{n-k}) \geq -1$, since the denominator is squarefree. In this case we obtain

$$\mathrm{v}_p\left(n \, B_{n-k} \, \frac{r^k}{k} \right) \geq e.$$

Considering all summands, we finally infer that (19) holds.

Case $p \nmid r$: Since $n \geq 2$, we have by (3) and (16) that

$$B_n(1) - B_n = \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0,$$

which we use in the second step below. Set $u := m/r \in \mathbb{Z}_p^\times$. As in the first case above, we derive that

$$\begin{aligned} r^{-n} \, \mathcal{B}^n_{m,r} &= \sum_{k=0}^{n-1} \binom{n}{k} B_k \, u^k \\ &= \sum_{k=0}^{n-1} \binom{n}{k} B_k \cdot (u^k - 1) \\ &= -\frac{n}{2}(u-1) + \sum_{\substack{k=2 \\ 2 \mid k}}^{n-1} n \binom{n-1}{k-1} \frac{B_k}{k} \, (u^k - 1). \end{aligned}$$

In both cases $p = 2$ and $p \geq 3$, we have

$$\frac{n}{2}(u-1) \equiv 0 \pmod{p^e}.$$

Since (14) implies $B_k/k \in \mathbb{Z}_p$ if $k \geq 2$ is even and $p-1 \nmid k$, we get

$$r^{-n}\,\mathcal{B}^n_{m,r} \equiv \sum_{\substack{k=2\\ 2\,\mid\,k\\ p-1\,\mid\,k}}^{n-1} n\binom{n-1}{k-1}\frac{B_k}{k}\,(u^k-1) \pmod{p^e}.$$

Now fix one k of the above sum. We then have the decomposition

$$k = a\,(p-1)\,p^t = a\,\varphi(p^{t+1}),$$

where $p \nmid a$, $t = \mathrm{v}_p(k)$, and $\varphi(\cdot)$ is Euler's totient function. By assumption u is a unit in $\mathbb{Z}_p$ and so is $\hat{u} := u^a \in \mathbb{Z}_p^\times$. Euler–Fermat's theorem shows that

$$u^k \equiv \hat{u}^{\varphi(p^{t+1})} \equiv 1 \pmod{p^{t+1}}.$$

Thus, $\mathrm{v}_p(u^k-1) \geq t+1$. Since $\mathrm{v}_p(B_k/k) = -(t+1)$ by (14), we achieve finally that

$$\mathrm{v}_p\left(n\,\frac{B_k}{k}\,(u^k-1)\right) \geq e-(t+1)+(t+1) = e,$$

implying that

$$r^{-n}\,\mathcal{B}^n_{m,r} \equiv 0 \pmod{p^e}$$

and showing the result. □

3. Proof of Theorem 2

Before giving the proof of Theorem 2, we need several lemmas with some complementary results. The next lemma easily shows a related partial result toward Theorem 2, while the full proof of this theorem requires much more effort.

Lemma 2. *We have*

$$\begin{aligned}\mathrm{denom}\big((n+1)\mathcal{S}^n_{m,r}(x)\big) &= \mathrm{denom}\big(m^n(B_{n+1}(x)-B_{n+1})\big)\\ &= \frac{\mathbb{D}_{n+1}}{\gcd(\mathbb{D}_{n+1},m)}.\end{aligned}$$

Proof. By rewriting $\mathcal{S}^n_{m,r}(x)$ as given in (2), and using (3) and (15), we easily derive that

$$\begin{aligned}(n+1)\mathcal{S}^n_{m,r}(x) &= m^n\sum_{k=0}^{n}\binom{n+1}{k}B_k\Big(\frac{r}{m}\Big)\,x^{n+1-k}\\ &= m^n\sum_{k=0}^{n}\binom{n+1}{k}\Big(B_k\Big(\frac{r}{m}\Big)-B_k+B_k\Big)\,x^{n+1-k}\end{aligned}$$

$$= \sum_{k=0}^{n} \binom{n+1}{k} m^{n-k} \underbrace{m^k \left(B_k\left(\frac{r}{m}\right) - B_k\right)}_{\mathcal{B}_{m,r}^k \in \mathbb{Z} \text{ by } (18)} x^{n+1-k} \tag{20}$$

$$+ m^n \left(B_{n+1}(x) - B_{n+1}\right). \tag{21}$$

By applying the simple observation that if $f(x) \in \mathbb{Z}[x]$ and $g(x) \in \mathbb{Q}[x]$, then

$$\operatorname{denom}\big(f(x) + g(x)\big) = \operatorname{denom}\big(g(x)\big), \tag{22}$$

we infer that

$$\operatorname{denom}\big((n+1)\mathcal{S}_{m,r}^n(x)\big) = \operatorname{denom}\big(m^n(B_{n+1}(x) - B_{n+1})\big).$$

Finally, from (4) and (5) we deduce that

$$\operatorname{denom}\big(m^n(B_{n+1}(x) - B_{n+1})\big) = \frac{\mathbb{D}_{n+1}}{\gcd(\mathbb{D}_{n+1}, m^n)} = \frac{\mathbb{D}_{n+1}}{\gcd(\mathbb{D}_{n+1}, m)},$$

the latter equation holding because $\mathbb{D}_{n+1}$ is squarefree. This completes the proof. □

Lemma 3. *For positive integers $k \le n$, define the rational number*

$$c_{n,k} := \frac{1}{k}\binom{n}{k-1}.$$

Then we have the following properties:

(i) Symmetry:

$$c_{n,k} = c_{n,n+1-k}.$$

(ii) Denominator:

$$\operatorname{denom}(c_{n,k}) \mid \gcd(n+1, k), \quad \operatorname{denom}(c_{n,k}) \le \frac{n+1}{2}.$$

(iii) Integrality:

If $k = 1$ or $k = n$ or $n+1$ is prime, then $c_{n,k} \in \mathbb{Z}$.

Proof. We first observe that

$$c_{n,k} = \frac{1}{k}\binom{n}{k-1} = \frac{1}{n+1}\binom{n+1}{k}, \tag{23}$$

which shows the symmetry in (i). From (23) it also follows that $\operatorname{denom}(c_{n,k})$ must divide both of the integers $n+1$ and k. Thus,

$$\operatorname{denom}(c_{n,k}) \mid \gcd(n+1,k).$$

Since $k < n+1$, we then infer that $\operatorname{denom}(c_{n,k}) \leq (n+1)/2$. This shows (ii). If $k = 1$ or $k = n$, then $c_{n,k} = 1$. If $n+1$ is prime, then $\gcd(n+1,k) = 1$ as $k \leq n$, so $\operatorname{denom}(c_{n,k}) = 1$. This proves (iii). $\square$

Lemma 4. *If $m, n, r \geq 1$ and $0 \leq k \leq n$, then*

$$\frac{m^n}{n+1}\binom{n+1}{k}\left(B_k\left(\frac{r}{m}\right) - B_k\right) \in \mathbb{Z}. \tag{24}$$

Proof. If $k = 0$, then the quantity in (24) vanishes by $B_0(x) - B_0 = 0$. For $1 \leq k \leq n$, we can rewrite the quantity in (24) by (17) and (23) as

$$c_{n,k} \times m^{n-k} \times \mathcal{B}_{m,r}^k, \tag{25}$$

where $\mathcal{B}_{m,r}^k \in \mathbb{Z}$ by (18) and $c_{n,k} = \frac{1}{k}\binom{n}{k-1} \in \mathbb{Q}$. We have to show that (25) lies in $\mathbb{Z}$. If $k = 1$ or $k = n$ or $n+1$ is prime, then $c_{n,k} \in \mathbb{Z}$ by Lemma 3. We can now assume that $n \geq 3$, $1 < k < n$, and $d := \operatorname{denom}(c_{n,k}) > 1$. For each prime power divisor $p^e \parallel d$ we consider two cases, which together imply the integrality of (25).

`Case` $p \nmid m$: Since $d \mid k$, we have $p^e \mid \mathcal{B}_{m,r}^k$ by Lemma 1.

`Case` $p \mid m$: We show that $p^e \mid m^{n-k}$, or equivalently,

$$n+1 > e+k. \tag{26}$$

As $p^e \mid k$, by symmetry in Lemma 3 we also have $p^e \mid n+1-k$, so $e < n+1-k$ and (26) holds. This completes the proof. $\square$

Proof of Theorem 2. To prove the last statement, it suffices to show that for $r \geq 0$

$$\mathcal{S}_{m,r}^n(x) - \mathcal{S}_{m,0}^n(x) \in \mathbb{Z}[x]. \tag{27}$$

By (20) and (21) we have

$$\mathcal{S}_{m,r}^n(x) = \frac{m^n}{n+1}\big(B_{n+1}(x) - B_{n+1}\big) + h(x), \tag{28}$$

where $(n+1)h(x) = f(x) \in \mathbb{Z}[x]$ as given by (20). By Lemma 4 it turns out that the coefficients of $h(x)$ are already integral, and thus $h(x) \in \mathbb{Z}[x]$. Since by (2)

$$\mathcal{S}_{m,0}^n(x) = \frac{m^n}{n+1}\big(B_{n+1}(x) - B_{n+1}\big),$$

relation (27) follows.

Applying the rule (22) to (28) and using (4) along with the fact that $B_{n+1}(x)$ is monic, we then infer that

$$\operatorname{denom}\bigl(\mathcal{S}_{m,r}^n(x)\bigr) = \operatorname{denom}\left(\frac{m^n}{(n+1)\mathbb{D}_{n+1}}\right). \tag{29}$$

We have to show that (29) implies (7).

In the following trivial cases we are done: case $m = 1$; cases $n = 1, 3$, since $\mathbb{D}_2 = \mathbb{D}_4 = 1$; and case $n = 2$, since $n + 1 = 3$ and $\mathbb{D}_3 = 2$.

So let $m \geq 2$ and $n \geq 4$. If a prime power $p^e \parallel n + 1$, then $e < n$. Consequently, we deduce that

$$\gcd(n + 1, m^n) = \gcd(n + 1, m^{n-1}). \tag{30}$$

Then by splitting m^n into $m^{n-1} \cdot m$ in (29) and applying (30) and the fact that $\mathbb{D}_{n+1}$ is squarefree, we infer that (7) holds.

Since $\operatorname{denom}\bigl(S_n(x)\bigr) = (n+1)\,\mathbb{D}_{n+1}$ by Theorem 1, we see at once that (7) implies

$$\operatorname{denom}\bigl(\mathcal{S}_{m,r}^n(x)\bigr) \mid \operatorname{denom}\bigl(S_n(x)\bigr).$$

As a result of (29), the denominator of $\mathcal{S}_{m,r}^n(x)$ is independent of r. This completes the proof of Theorem 2. $\square$

4. Proofs of Theorem 3 and Corollary 2

Before we give the proofs, we need some definitions and lemmas. Recall that, given a prime p, any positive integer n can be written in base p as a unique finite *p-adic expansion*

$$n = \alpha_0 + \alpha_1\, p + \cdots + \alpha_t\, p^t \quad (0 \leq \alpha_j \leq p - 1).$$

This expansion defines the *sum-of-digits function*

$$s_p(n) := \alpha_0 + \alpha_1 + \cdots + \alpha_t,$$

which satisfies the congruence

$$s_p(n) \equiv n \pmod{p-1}. \tag{31}$$

Actually, these properties hold for any integer base $b \geq 2$ in place of a prime p.

The following lemma (see [9, Chap. 5.3, p. 241]) shows the relation between $s_p(n)$ and $s_p(n + 1)$.

Lemma 5. *If $n \geq 1$ and p is a prime, then*

$$s_p(n + 1) = s_p(n) + 1 - (p - 1)\ \mathrm{v}_p(n + 1).$$

In particular,

$$s_p(n+1) \le s_p(n) \qquad \textit{if and only if} \quad p \mid n+1,$$

while

$$s_p(n+1) = s_p(n)+1 \quad \textit{if and only if} \quad p \nmid n+1. \tag{32}$$

Lemma 6. *If $n \ge 1$, then*

$$\operatorname{lcm}(\mathbb{D}_n, \mathbf{D}_n) \mid \operatorname{lcm}\big(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\big).$$

Proof. Set $L_n := \operatorname{lcm}\big(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\big)$. Since $\mathbb{D}_n$ and $\mathbf{D}_n$ are squarefree by (5) and (13), we show that $p \mid \operatorname{lcm}(\mathbb{D}_n, \mathbf{D}_n)$ implies $p \mid L_n$. Moreover, since $\operatorname{rad}(n+1) \mid L_n$, we may assume that $p \nmid n+1$.

If $p \mid \mathbb{D}_n$, then by (5) we have $s_p(n) \ge p$. Applying (32) followed by (10), we obtain $p \mid \mathbb{D}_{n+1}$, and finally $p \mid L_n$.

Since $\mathbf{D}_1 = 2$ by (12) and $\mathbf{D}_n = 1$ for odd $n \ge 3$, we have $\mathbf{D}_n \mid L_n$ for odd $n \ge 1$. So take $n \ge 2$ even. If $p \mid \mathbf{D}_n$, then $p-1 \mid n$ by (13), so also $p-1 \mid s_p(n)$ by (31). Thus $s_p(n) \ge p-1$. As $p \nmid n+1$ by assumption, (32) implies $s_p(n+1) \ge p$, so $p \mid \mathbb{D}_{n+1}$ by (10). Finally $p \mid L_n$. This proves the lemma. ◻

Lemma 7. *If $n \ge 1$, then*

$$\operatorname{lcm}\big(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\big) \mid \operatorname{lcm}(\mathbb{D}_n, \mathbf{D}_n).$$

Proof. As $\mathbb{D}_1 = \mathbb{D}_2 = 1$, and $\mathbf{D}_1 = 2$ by (12), the case $n = 1$ holds. So assume $n \ge 2$ and set $L_n := \operatorname{lcm}(\mathbb{D}_n, \mathbf{D}_n)$.

If $n+1$ is not prime, then (11) implies $\operatorname{rad}(n+1) \mid L_n$. Otherwise, $p = n+1 = \operatorname{rad}(n+1)$ is an odd prime and so n is even. By (13) we have $p \mid \mathbf{D}_n$, so $\operatorname{rad}(n+1) \mid L_n$.

It remains to show that $\mathbb{D}_{n+1} \mid L_n$. As $\mathbb{D}_{n+1}$ is squarefree by (5), it suffices to show for any prime $p \mid \mathbb{D}_{n+1}$ that $p \mid L_n$. By (5) again we have $s_p(n+1) \ge p$, and as $\operatorname{rad}(n+1) \mid L_n$ we may assume that $p \nmid n+1$. Then by (32) we obtain $s_p(n) = s_p(n+1) - 1 \ge p-1$. If $s_p(n) \ge p$, then $p \mid \mathbb{D}_n$ by (10), so $p \mid L_n$. Otherwise, $s_p(n) = p-1$ and so $p-1 \mid n$ by (31). Moreover, n must be even, as n odd would imply $p = 2$, contradicting $p \nmid n+1$. Hence $p \mid \mathbf{D}_n$ by (13), and finally $p \mid L_n$. This completes the proof. ◻

Proof of Theorem 3. To show the equivalence, we have to prove that

$$\operatorname{denom}\big(\mathcal{S}_{m,r}^n(x)\big) = 1 \quad \text{if and only if} \quad \mathfrak{D}_n \mid m.$$

By (7), we have $\operatorname{denom}\big(\mathcal{S}_{m,r}^n(x)\big) = 1$ if and only if

$$\frac{n+1}{\gcd(n+1, m^n)} = \frac{\mathbb{D}_{n+1}}{\gcd(\mathbb{D}_{n+1}, m)} = 1,$$

which in turn is true if and only if $n+1 \mid m^n$ and $\mathbb{D}_{n+1} \mid m$. Moreover,

$$n+1 \mid m^n \quad \text{if and only if} \quad \operatorname{rad}(n+1) \mid m.$$

Indeed, $p \mid n+1 \mid m^n$ implies $p \mid m$, proving the "$\Rightarrow$" direction. Conversely, if $p \mid \operatorname{rad}(n+1) \mid m$, then $p^n \mid m^n$. But $p^e \parallel n+1$ with $e \leq n$, so finally $n+1 \mid m^n$. It follows that

$$\operatorname{denom}\bigl(\mathcal{S}_{m,r}^n(x)\bigr) = 1 \quad \text{if and only if} \quad \operatorname{lcm}\bigl(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\bigr) \mid m.$$

By Lemmas 6 and 7, together with the proof of [6, Thm. 4], we have

$$\operatorname{lcm}\bigl(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\bigr) = \operatorname{lcm}(\mathbb{D}_n, \mathbf{D}_n) = \mathfrak{D}_n.$$

This proves the theorem. □

Proof of Corollary 2. (i), (ii) If $n \geq 3$ is odd, then $\mathbf{D}_n = 1$. Hence, (8) and (9) yield $\mathbb{D}_n = \operatorname{lcm}\bigl(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\bigr)$. Together with $\mathbb{D}_1 = \mathbb{D}_2 = 1$, this implies that $\mathbb{D}_{n+1} \mid \mathbb{D}_n$ for all odd $n \geq 1$, as desired.

Similarly, for even $n \geq 2$, we have $\mathfrak{D}_{n+1} = \mathbb{D}_{n+1}$ by (8). Then (9) gives $\mathfrak{D}_n = \operatorname{lcm}\bigl(\mathfrak{D}_{n+1}, \operatorname{rad}(n+1)\bigr)$, so $\mathfrak{D}_{n+1} \mid \mathfrak{D}_n$, as claimed. □

5. Proofs of Theorems 4 and 5

Let $a, b \geq 2$ be integers that are multiplicatively independent, that is, $a^e \neq b^f$ for all integers $e, f \geq 1$. Senge and Straus [10, Thm. 3] showed that for a given constant A the number of integers n satisfying

$$s_a(n) + s_b(n) < A$$

is finite. Steward [12, Thm. 1, p. 64] proved the effective lower bound

$$s_a(n) + s_b(n) > \frac{\log\log n}{\log\log\log n + C} - 1 \tag{33}$$

for $n > 25$, where $C > 0$ is an effectively computable constant depending on a and b. This bound leads to the following lemma.

Lemma 8. *If $p \neq q$ are primes, then*

$$\lim_{k \to \infty} s_p\bigl(q^k\bigr) = \infty.$$

In particular, there exists a positive integer $L_{p,q} > \log_q p$ such that

$$s_p\bigl(q^k\bigr) \geq p \quad (k \geq L_{p,q}).$$

Proof. By taking $a = p$, $b = q$, $k \geq 5$, and $n = q^k > 25$, we derive from (33) that

$$s_p(q^k) > \frac{\log(k \log q)}{\log\log(k \log q) + C} - 2 =: f(k)$$

with some constant $C > 0$ depending on p and q. Since $f(k)$ is strictly increasing for all sufficiently large k, we infer that $\lim_{k\to\infty} s_p\bigl(q^k\bigr) = \infty$. Therefore, there exists a positive integer $L_{p,q}$ such that $s_p\bigl(q^k\bigr) \geq p$ for $k \geq L_{p,q}$. On the other hand, since $s_p(m) = m$ for $0 \leq m < p$, we have that $s_p(q^k) < p$ for $1 \leq k < \log_q p$, implying that $L_{p,q} > \log_q p$, as claimed. □

Proof of Theorem 4. If $n = 1$, then $\mathbb{D}_1/\mathbb{D}_2 = 1$. By (6), if $n \geq 3$ is odd and $n+1$ is not a power of 2, then $\mathbb{D}_n$ and $\mathbb{D}_{n+1}$ are both even. Since by (5) they are squarefree, $\mathbb{D}_n/\mathbb{D}_{n+1}$ must be odd.

Likewise, if $n = 2^k - 1$ for some $k \geq 2$, then $\mathbb{D}_n/\mathbb{D}_{n+1}$ must be twice an odd number. If an odd prime p divides $\mathbb{D}_n$, then $s_p(n) \geq p$ by (5). Since $p \nmid 2^k = n+1$, we infer by (32) that $s_p(n+1) > p$. Hence by (10) the prime p also divides $\mathbb{D}_{n+1}$, so indeed $\mathbb{D}_n/\mathbb{D}_{n+1} = 2$.

Now, let $n = 2^\ell p^k - 1$ with p an odd prime and $k, \ell \geq 1$. Then we have $\operatorname{rad}(n+1) = 2p$, and by (6) that $\mathbb{D}_n$ and $\mathbb{D}_{n+1}$ are both even. Thus, $\mathbb{D}_n/\mathbb{D}_{n+1} \in \{1, p\}$ by Corollary 2 part (i). We consider two cases.

Case $1 \leq \ell < \log_2 p$: Since $s_p(n+1) = s_p(2^\ell) < p$, we infer that $p \nmid \mathbb{D}_{n+1}$ by (10) implying $\mathbb{D}_n/\mathbb{D}_{n+1} = p$.

Case $\ell > \log_2 p$: Lemma 8 implies a constant $\mathcal{L}_p := L_{p,2} > \log_2 p$ such that $s_p(n+1) = s_p(2^\ell) \geq p$ for all $\ell \geq \mathcal{L}_p$. Hence $p \mid \mathbb{D}_{n+1}$ by (10) and $\mathbb{D}_n/\mathbb{D}_{n+1} = 1$. This proves the theorem. □

Proof of Theorem 5. It is shown in [6, Thm. 4] that $\mathfrak{D}_n$ is even and squarefree for all $n \geq 1$. (This also follows from (8) for even $n \geq 2$, since $2 \mid \mathbf{D}_n$, and from (9) for odd $n \geq 1$, since $2 \mid \operatorname{rad}(n+1)$, all terms in question being squarefree.) Hence if $n \geq 2$ is even, so that $\mathfrak{D}_{n+1} \mid \mathfrak{D}_n$, then the quotient must be odd.

Let p be an odd prime. If $n = p^k - 1$ for some $k \geq 1$, then we have $\operatorname{rad}(n+1) = p$ and $s_p(n+1) = s_p(p^k) = 1 < p$. Thus $p \nmid \mathbb{D}_{n+1}$ by (10). Since n is even, we have $\mathfrak{D}_{n+1} = \mathbb{D}_{n+1}$ by (8) and so $p \nmid \mathfrak{D}_{n+1}$. By Corollary 2 part (i) we finally obtain $\mathfrak{D}_n/\mathfrak{D}_{n+1} = p$.

Now, let $p \neq q$ be odd primes and $n = p^k q^\ell - 1$ with $k, \ell \geq 1$. We then have $\operatorname{rad}(n+1) = pq$ and by Corollary 2 part (i) that $\mathfrak{D}_n/\mathfrak{D}_{n+1} \in \{1, p, q, pq\}$. Note that $s_p(n+1) = s_p(q^\ell)$ and $s_q(n+1) = s_q(p^k)$. By Lemma 8 we define $\mathcal{L}'_{p,q} := L_{q,p} > \log_p q$ and $\mathcal{L}''_{p,q} := L_{p,q} > \log_q p$. We consider the following statements by using (10):

If $1 \leq \ell < \log_q p$, then $s_p(q^\ell) < p$ and $p \nmid \mathbb{D}_{n+1}$. Otherwise, if $\ell \geq \mathcal{L}''_{p,q}$, then $s_p(q^\ell) \geq p$ and $p \mid \mathbb{D}_{n+1}$.

If $1 \leq k < \log_p q$, then $s_q(p^k) < q$ and $q \nmid \mathbb{D}_{n+1}$. Otherwise, if $k \geq \mathcal{L}'_{p,q}$, then $s_q(p^k) \geq q$ and $q \mid \mathbb{D}_{n+1}$.

All three cases of the theorem follow from the arguments given above, since $\mathfrak{D}_{n+1} = \mathbb{D}_{n+1}$. This completes the proof of the theorem. $\square$

6. Conclusion

The numbers

$$\mathcal{B}^n_{m,r} = m^n \left(B_n\left(\frac{r}{m}\right) - B_n \right),$$

shown by Almkvist and Meurman to be integers, play here a key role in proofs. By their result, the polynomial $B_n(x) - B_n$, with an extra factor, takes integer values at rational arguments $x = r/m$. In the present paper, the numbers $\mathcal{B}^n_{m,r}$ reveal their natural connection with the power sums of arithmetic progressions $\mathcal{S}^n_{m,r}(x)$. Moreover, the divisibility properties of $\mathcal{B}^n_{m,r}$ are important in attaining our results in Theorems 2 and 3.

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