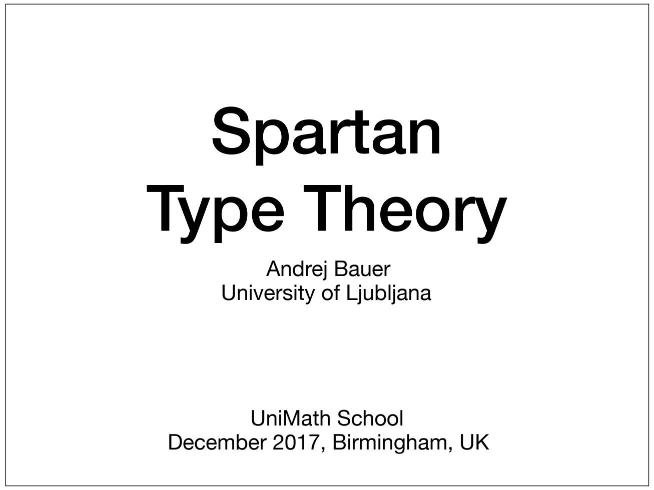

Spartan Type Theory

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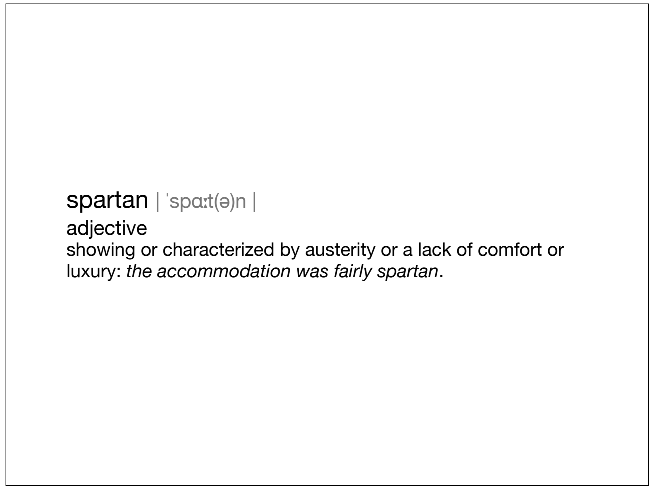
UniMath School
December 2017, Birmingham, UK

1

Welcome everyone. I am honored to have the opportunity to speak here. I was asked to do an introduction to type theory in one hour for people who know nothing about it.
It is an impossible task. Here we go.



spartan | ˈspɑːt(ə)n |
adjective
showing or characterized by austerity or a lack of comfort or luxury: *the accommodation was fairly spartan*.

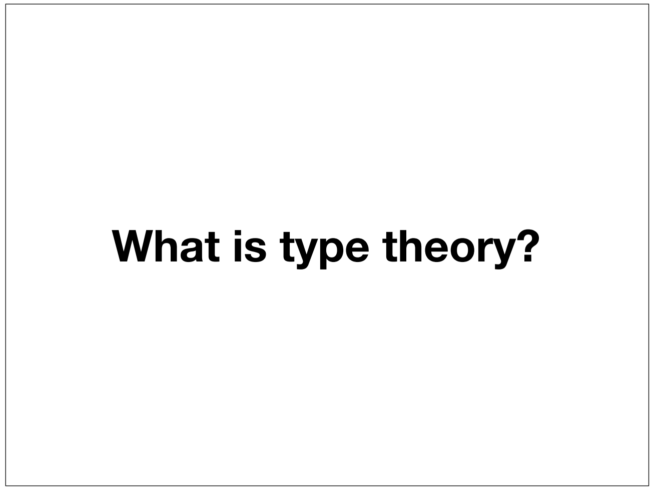
2

Benedikt saved me from the dilemma of choosing a title for my talk.
Through the title he is sending a message to the participants.
He even arranged a snow blizzard.

Vladimir Voevodsky (1966–2017)

3

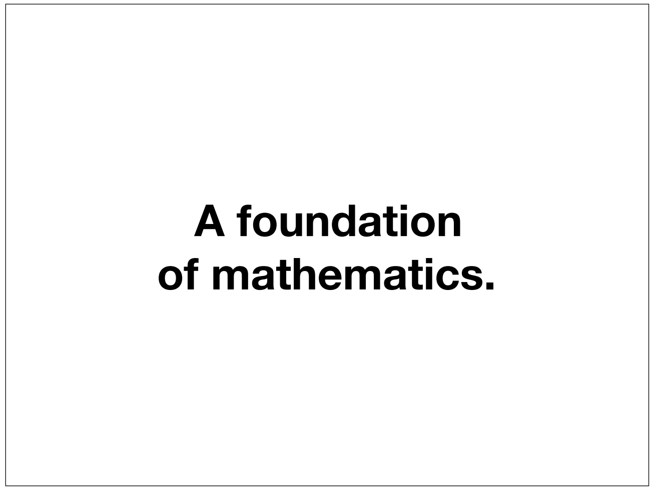
Before we go on, let us remember Vladimir Voevodsky, whose idea this winter school was.
His passing away was a terrible blow, on a professional level, but also on a personal level for many present here. Let us spend a moment in silence in memory of Vladimir.



What is type theory?

4

So, what is type theory?



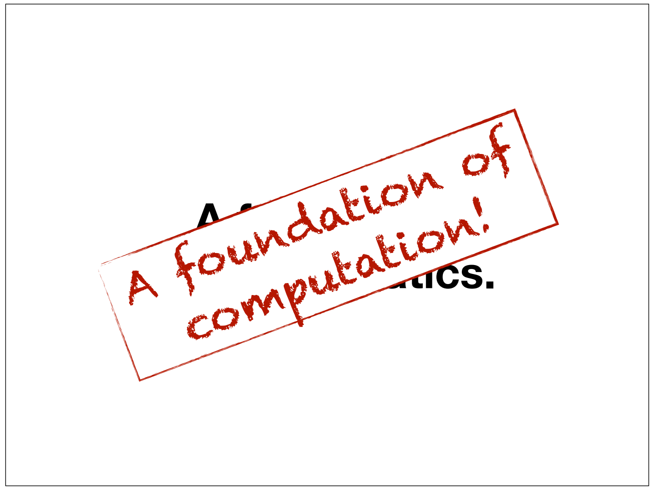
A foundation of mathematics.

5

Some say it is a foundation of mathematics.
If this is the case, then type theory has to stand on its own, without assuming logic or set theory.

But in explaining it I *will* rely on our pre-existing mathematical knowledge, because a foundation is not magic that gets you something out of nothing.

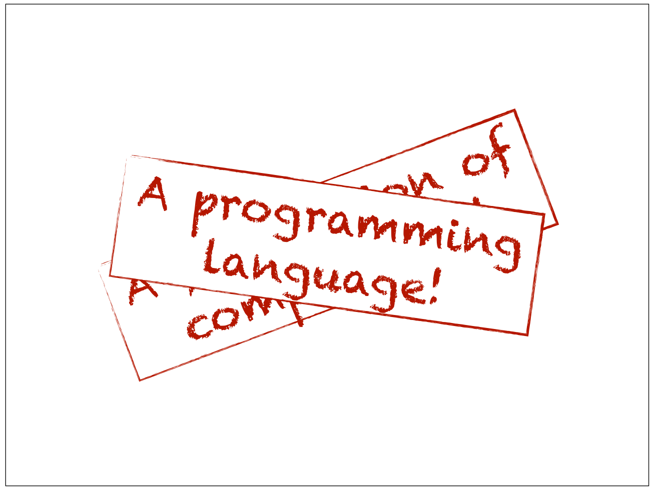
A foundation *organizes* mathematics.
It is a *tool* for doing mathematics (and maybe a *language*).
It provides a *method* for doing mathematics.
It *serves* the mathematician first, and the logician second!



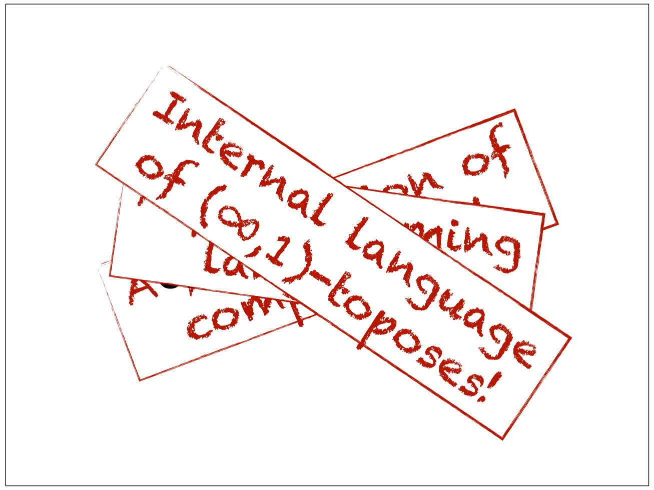
A foundation of computation!

6

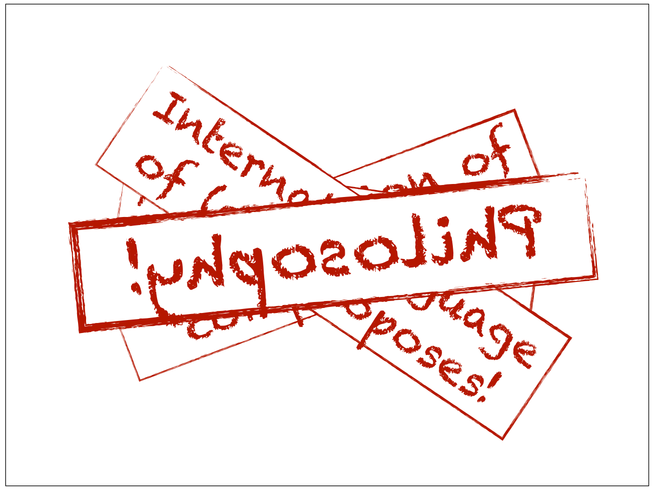
There are several other views of type theory.
We do not have time to go into them, but they are all interesting.



7 Ask your teachers here, they can tell you all about it.



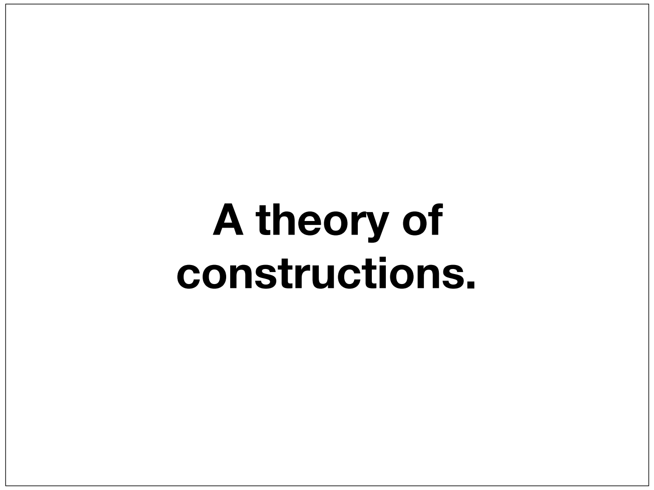
8 Careful though, they may drag you into strange territory.



9

Or they may say incomprehensible things. It’s fun!

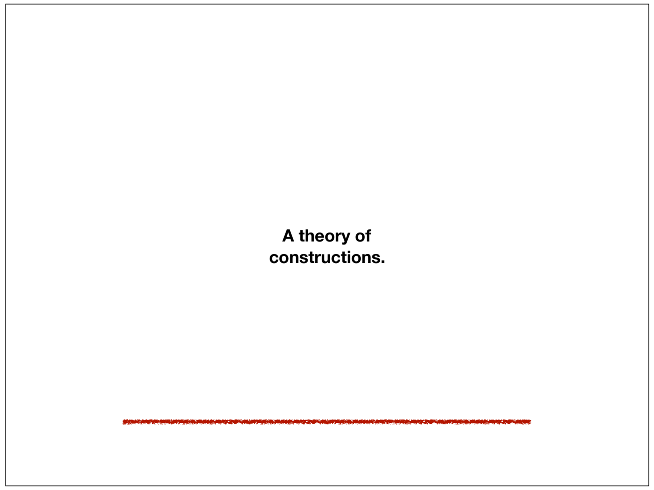
But let me tell you a secret: the informal style of a typical mathematical text is *as close as,* if not closer to type theory than it is to Zermelo-Fraenkel set theory. This is not to say that anything is wrong with set theory, or that it hasn’t had an important role. It’s just an observation, supported by evidence: the most successful computer formalizations of mathematics all rely on type theory. (I include Mizar in this list, for those who wonder.)



A theory of constructions.

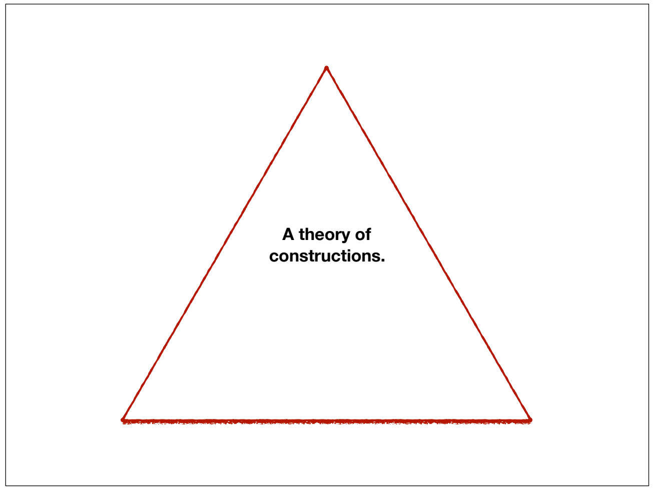
10

If I had to give you a single short phrase describing type theory,
I would say that it is a **theory of constructions**.
It is not about what is true, but about how to build mathematical objects generally.


A theory of constructions.

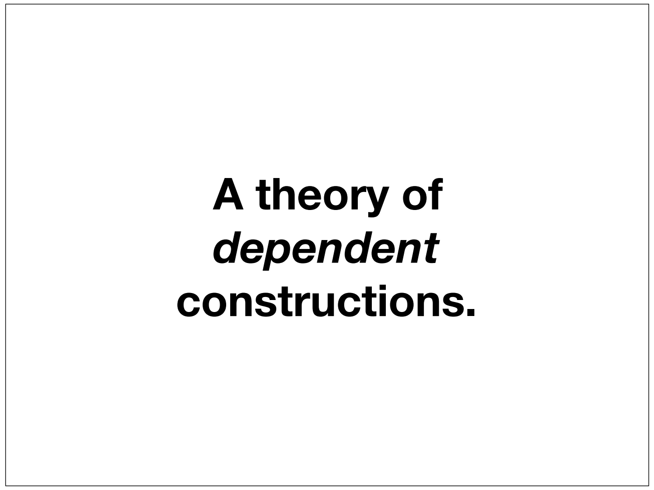
11

Did you know that Euclid’s *Elements* contain both geometric *constructions* and *statements* of truth? Proposition 1 says “to erect an equilateral triangle on a line segment”.


A theory of constructions.

12

Mathematics started with a construction!

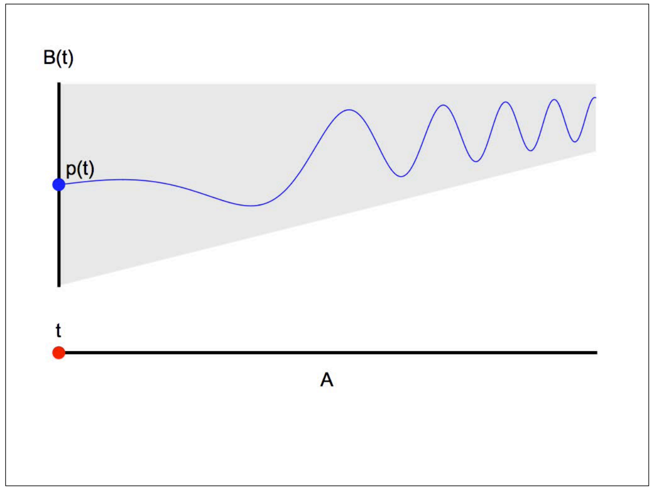


A theory of *dependent* constructions.

13

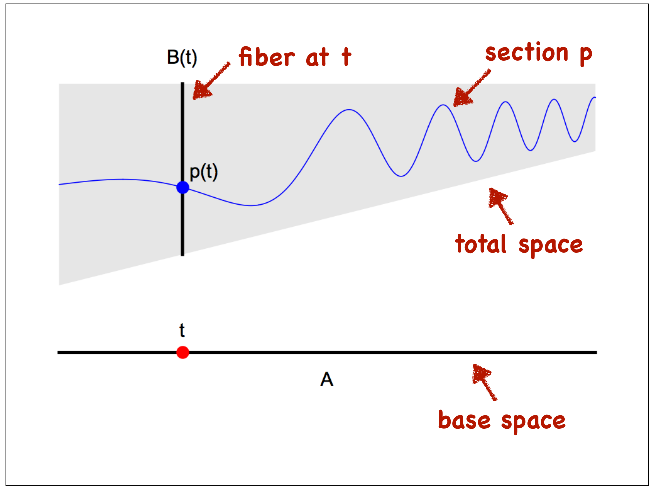
In fact, Euclid’s construction is **dependent**: *given* a segment, construct an equilateral triangle which *depends* on it.

To emphasize this aspect of type theory, let us say that it is a theory of *dependent* constructions.
Dependency is a very general idea, so let us have a closer look at it.



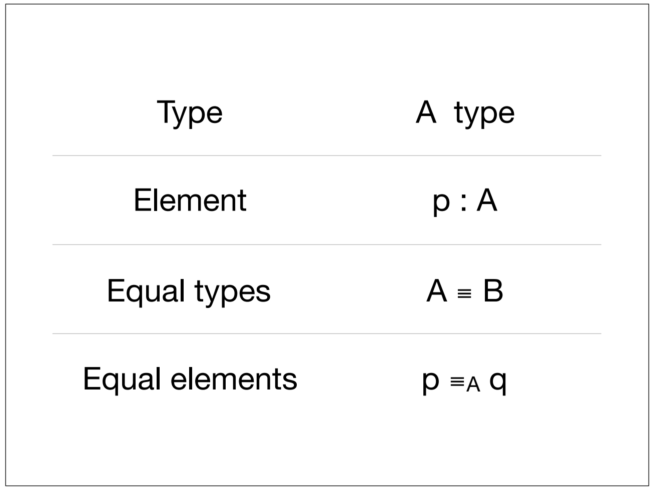
14

We have here a pretty video in which **two** kinds of dependency are presented.
First, the point p(t) depends on the point t. As t moves in space A, the point p(t) moves around.
Second, the space B(t) depends on t as well. This is very important. As t moves, the space B(t) changes.



15

In geometry and topology each piece of the picture has a name.
If you are familiar with it, that is good. If not, do not worry, we will not really use advanced geometry.
We have a **base space**.
The dependent space is called a **fiber**.
We say that **a total space** is **over** the base space.
The moving point gives us a map p from the base space to the total space, called a **section.**
In a moment we will give these items their type-theoretic nomenclature.



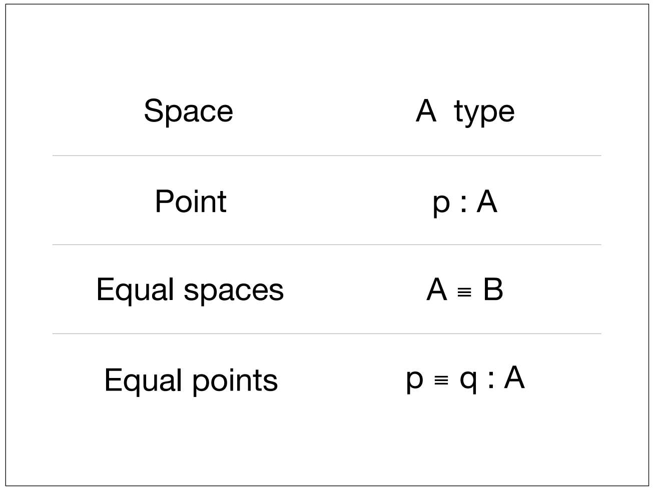
Type	A type
Element	p : A
Equal types	$A \equiv B$
Equal elements	$p \equiv_A q$

16

Because I said there was no logic and no set theory, I have to give you something else to work with. Type theory has **judgments**. These are **not** logical statements, they record the fact that something has been constructed, or that two constructions are equal. For instance, we *cannot* make the judgment “A is not a type”.

If type theory is about making houses and cars from Lego blocks, the judgments correspond to the kid running up to their parents and showing them what they’ve built. Nobody ever runs to their parents to show them that they didn’t build anything.

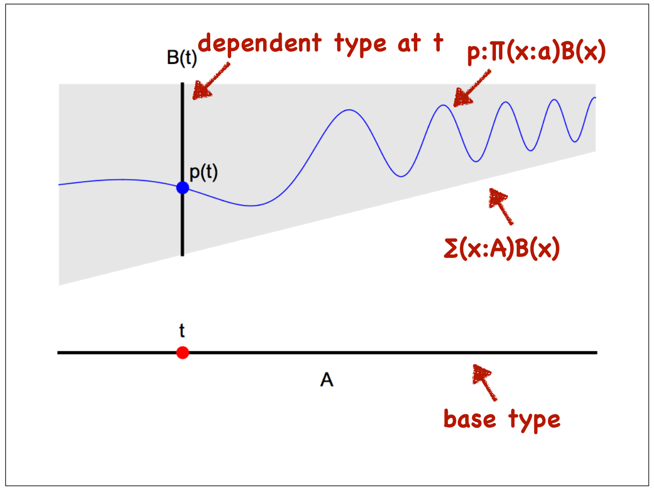
We may and should discuss whether equality is part of “logic”. We will be discussing equality for the rest of the week.



Space	A type
Point	p : A
Equal spaces	A ≡ B
Equal points	p ≡ q : A

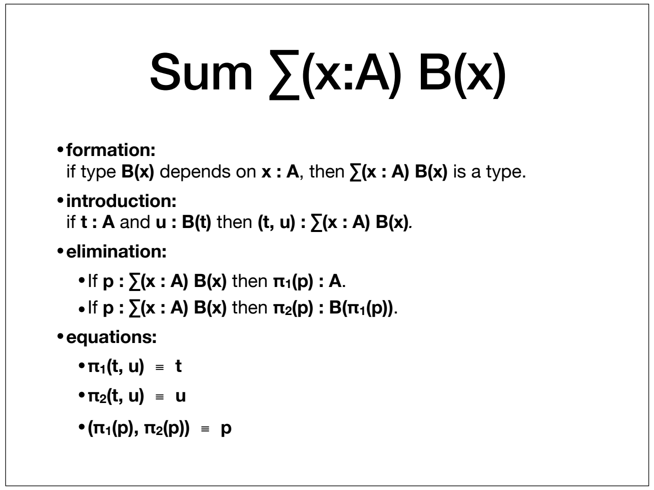
17

As I already mentioned, there are many views of type theory, and so there are many ways of imagining what types and elements are.
To connect this back to our geometric picture, here is the geometric explanation.
Think of types as “spaces”, but take this word in the most general sense you can. It’s like a topological space. It’s like a homotopy type. It’s like a datatype. It’s like a computable space. Some of these explanations may mean nothing to you, but all have been suggested by people.
It is customary then to speak of *points* of a *space*. I emphasize again that a point never exists without being in a space, and in can never be in two spaces at the same time.



18

Going back to our picture, the type-theoretic terminology is as follows.
Instead of fibers we have dependent types.
The total space becomes a construction known as “dependent sum”.
Sections are elements of “dependent products”.



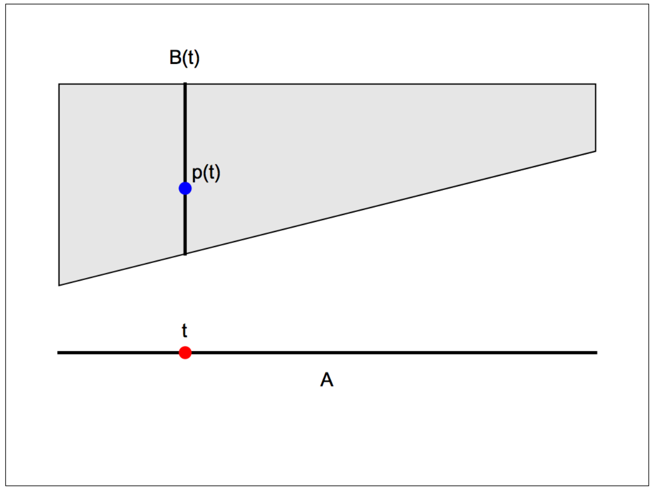
Sum $\sum(x{:}A)\ B(x)$

- **formation:**
  if type **B(x)** depends on **x : A**, then **$\sum$(x : A) B(x)** is a type.
- **introduction:**
  if **t : A** and **u : B(t)** then **(t, u) : $\sum$(x : A) B(x)**.
- **elimination:**
  - If **p : $\sum$(x : A) B(x)** then **π_1(p) : A**.
  - If **p : $\sum$(x : A) B(x)** then **π_2(p) : B(π_1(p))**.
- **equations:**
  - **$\pi_1(t, u) \equiv t$**
  - **$\pi_2(t, u) \equiv u$**
  - **$(\pi_1(p), \pi_2(p)) \equiv p$**

19

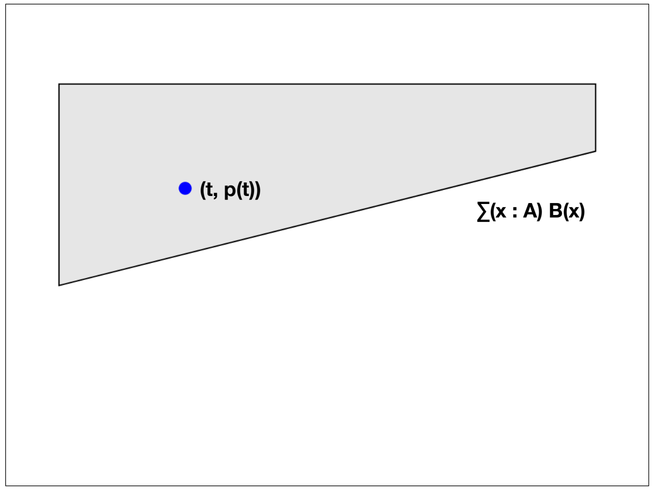
Let us look at dependent sums. Every type theoretic construction follows a certain pattern:

- first we state explain how to form the type, the *formation rule*
- then we give rules for making elements of the type, these are *introduction rules*
- elimination rules explain how we *use* or *"eliminate"* elements of the type
- equations explain what happens when we create an element and take it apart, or vice versa



20

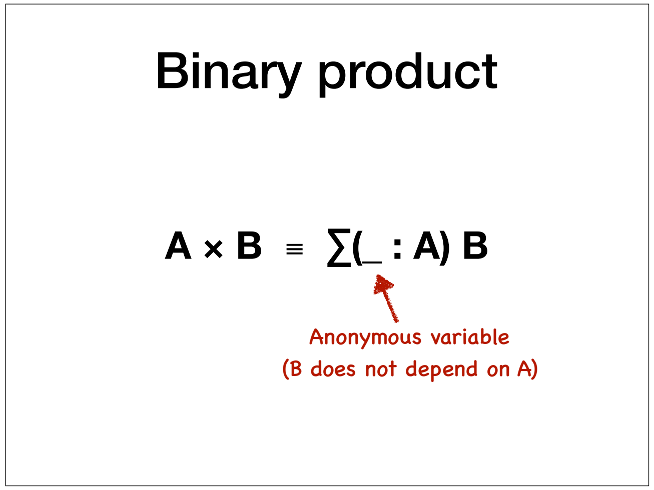
We said that every point is always precisely in one space.
Consider a point p(t) in the fiber over t. It is a member of B(t). It is *not* a member of the dependent sum!



21

The corresponding point in the dependent sum is the pair (t, p(t)).

Note that the dependent sum is a space on its own. It is not a dependent type over A.



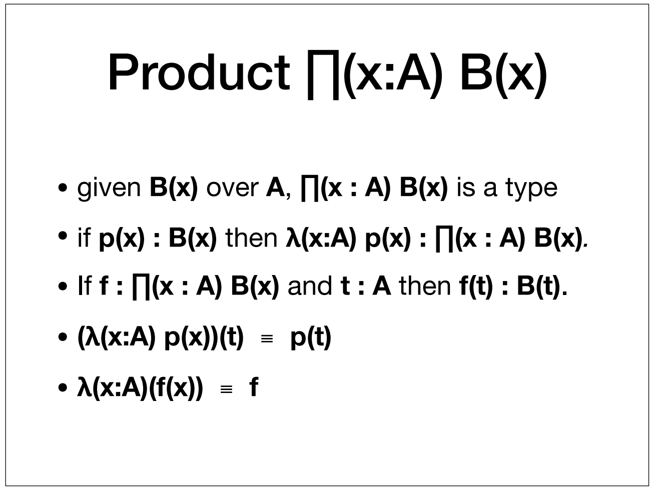
Binary product

$$\mathbf{A \times B} \equiv \mathbf{\sum(\_ : A)\ B}$$

Anonymous variable
(B does not depend on A)

22

A special case of dependent sum is the usual cartesian product. We get it when **B(x)** does not depend on **A**, i.e., we have a **constant** dependent type.



Product $\prod(x:A)\ B(x)$

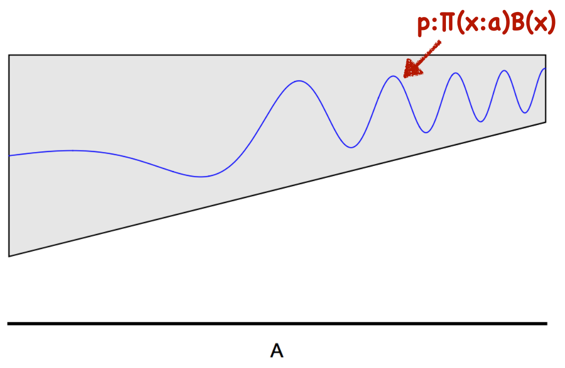
- given **B(x)** over **A**, **$\prod$(x : A) B(x)** is a type
- if **p(x) : B(x)** then **λ(x:A) p(x) : $\prod$(x : A) B(x)**.
- If **f : $\prod$(x : A) B(x)** and **t : A** then **f(t) : B(t)**.
- **(λ(x:A) p(x))(t)** $\equiv$ **p(t)**
- **λ(x:A)(f(x))** $\equiv$ **f**

23

Dependent products are spaces of sections. It is difficult to draw a picture of a product, but it is easy to understand what its elements are: they are maps.

How is a map constructed? If for every fiber B(x) we have a point p(x) in it, then we can form a map. The λ thing should be read as: “the map which takes x to p(x)”.

To use a map, we apply it to an argument.



24

It is difficult to draw a picture of the product, but it is easy to see one of its elements: a section over A.

Function space

$$\mathbf{A \to B \;\equiv\; \textstyle\prod(\_ : A)\, B}$$

25

If we form a dependent product over the constant dependent type we get the (non-dependent) function space. The elements are ordinary maps from A to B.

So far we have not given any ways of constructing a type from scratch. We need some *primitive* types from which all other constructions start.

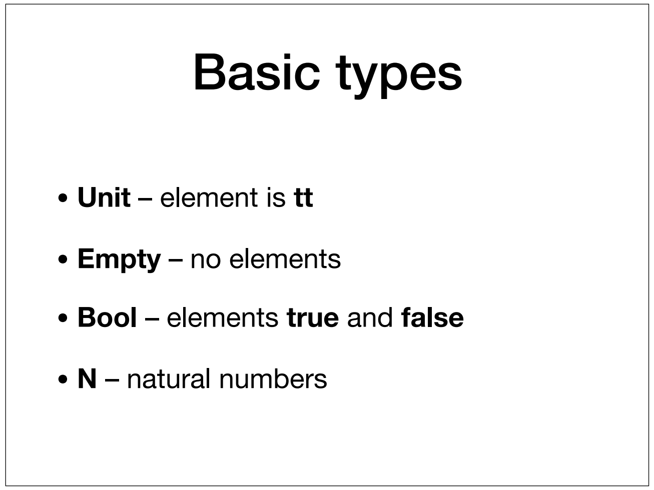
Universe

- There is a type **Type**.
- The elements of **Type** are types.
- A dependent type is a map **B : A → Type**.
- Beware of paradoxes:
  there can be no type of all types!
- $\mathbf{Type_0 : Type_1 : Type_2 : \ldots}$

26

If you think about it, types themselves get constructed too. But anything that is constructed should have a type, so there should be a type of types. Such a type is called a *universe.*
Its elements are types. A dependent type is a map from the base space to the universe.

This is all very neat, but we have to beware of paradoxes. An early version of type theory stated that the universe was its own element, and that turned out to lead to the same sort of trouble as the set of all sets. If you need more than one universe you can make a hierarchy of them.



Basic types

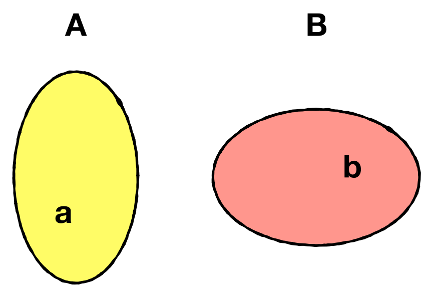
- **Unit** – element is **tt**
- **Empty** – no elements
- **Bool** – elements **true** and **false**
- **N** – natural numbers

27

We do not have the time to carefully give the rules for other primitive types, so let us just list them.

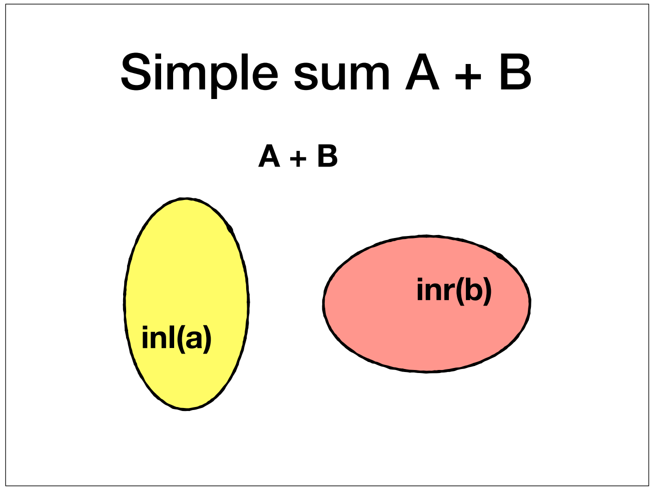
- the unit type has just one element, called the “unit” (to confuse people), in Coq it is **tt**
- the empty type has no elements. It is instructive to give the elimination principle for **Empty**.
- Bool is the type with two elements, called “true” and “false”. Beware: this is not logic! It is just a type. We might have called its elements “left” and “right”.
- The natural numbers are also a basic type. We will look at it shortly.

Simple sum A + B



28

Let us mention one other construction. Given types A and B, we can form their *simple* sum, also called coproduct.

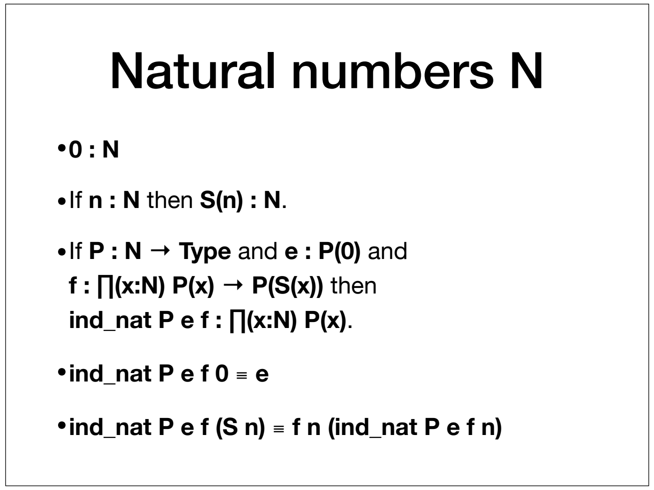


Simple sum A + B

29

Once again, an element a of A cannot be an element of A + B at the same time. We *tag* it to indicate that it is an element of A + B, where “inl” stands for “left injection” and “inr” for “right injection”. It is the tagging that distinguishes the simple sum from a union: it’s possible to make the sum of two equal summands A + A without overlapping them.

In set theory this construction is known as disjoint union, where we usually tag the elements by making pairs of the form (0, a) and (1, b). Using primitive tags inl and inr is more principled.

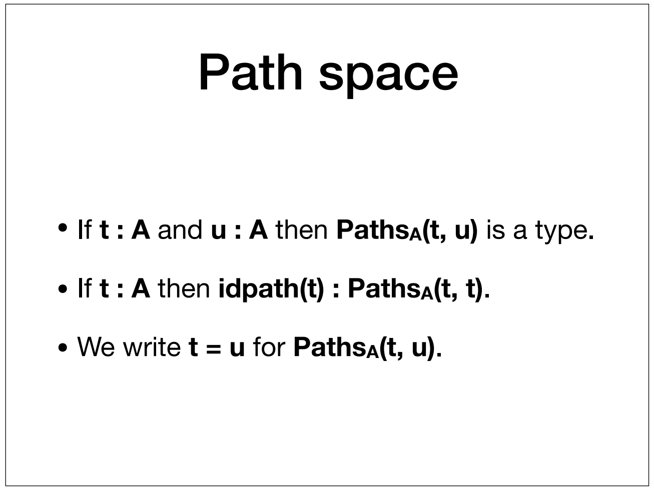


Natural numbers N

- **0 : N**
- If **n : N** then **S(n) : N**.
- If **P : N → Type** and **e : P(0)** and **f : ∏(x:N) P(x) → P(S(x))** then **ind_nat P e f : ∏(x:N) P(x)**.
- **ind_nat P e f 0** ≡ **e**
- **ind_nat P e f (S n)** ≡ **f n (ind_nat P e f n)**

30

The natural numbers are shown in this slide. I am not going to drag you through the rules. Instead I am going to give an exercise: relate the first three rules to Peano axioms, and explain what the equations say in terms of programming.



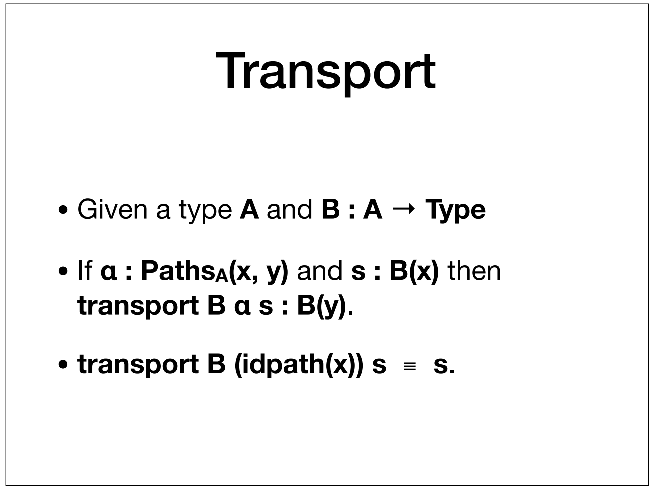
Path space

- If **t : A** and **u : A** then **$\mathbf{Paths_A}$(t, u)** is a type.
- If **t : A** then **idpath(t) : $\mathbf{Paths_A}$(t, t)**.
- We write **t = u** for **$\mathbf{Paths_A}$(t, u)**.

31

We now come to the most important type.

We shall adapt our view of what a type is: types do not tell us just

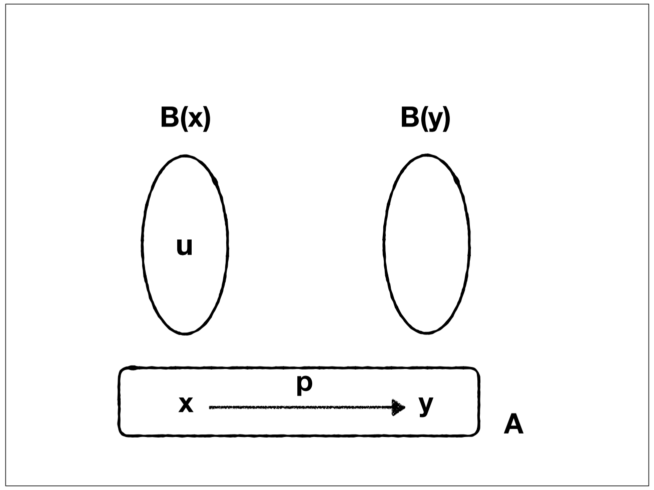


Transport

- Given a type **A** and **B : A → Type**
- If **α : $\mathbf{Paths_A}$(x, y)** and **s : B(x)** then **transport B α s : B(y)**.
- **transport B (idpath(x)) s ≡ s.**

32

The notion of fibration, transfer



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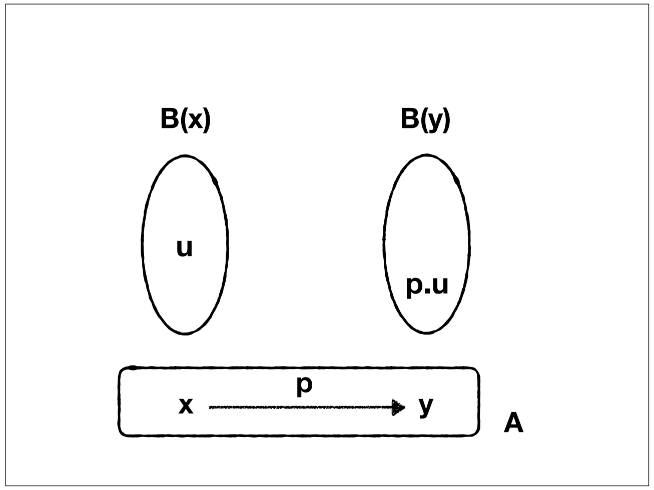

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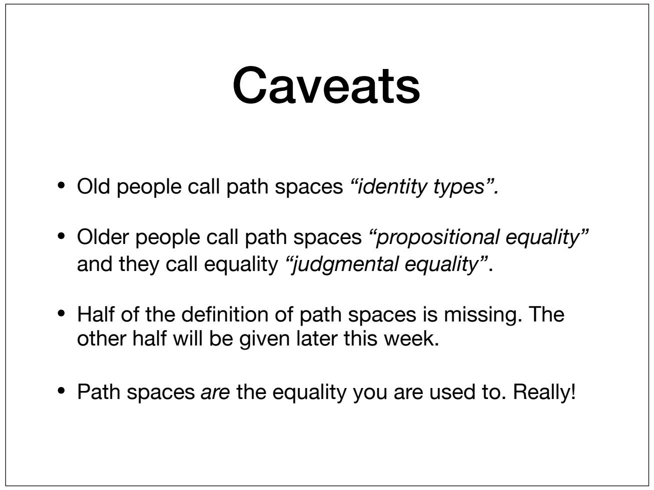


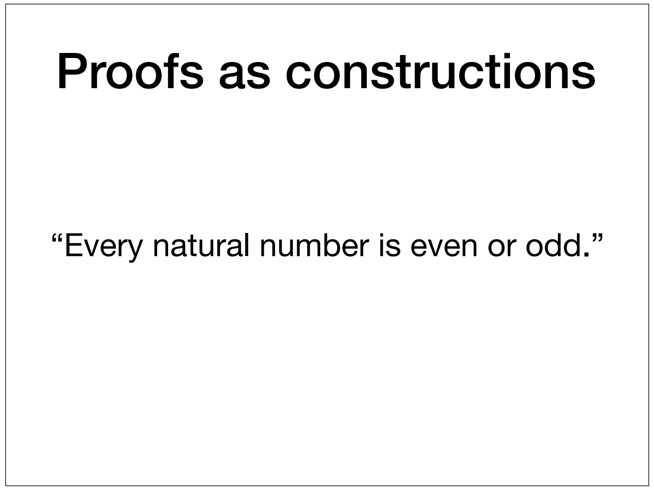
Caveats

- Old people call path spaces *“identity types”.*
- Older people call path spaces *“propositional equality”* and they call equality *“judgmental equality”*.
- Half of the definition of path spaces is missing. The other half will be given later this week.
- Path spaces *are* the equality you are used to. Really!

35

We have only scratched the surface. You will find out the rest about paths spaces throughout the week. But let me also say this (thanks to Peter Lumsdaine for warning me about it): the path spaces correspond not only to, well, spaces of paths, but also to equality as usually used in mathematics.

The equalities between types, $A \equiv B$ and $p \equiv_A q$, are *not* the ones you usually want or need, and in fact they are *invisible* in Coq. The difficulty is that in most mathematics you only see a very trivial kind of path spaces. Martín Escardó will speak more about this point.

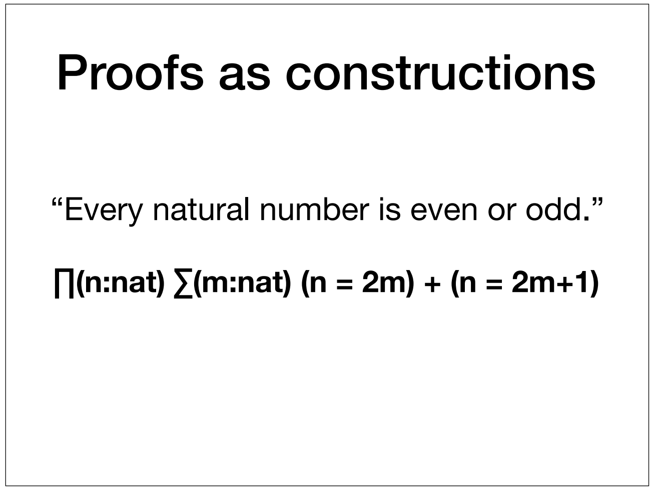


Proofs as constructions

“Every natural number is even or odd.”

36

We banished logic, but we do not want to give it up forever.
There is an ingenious way of doing logic by construction. We can prove logical statements by constructing elements of types.
For instance, consider the statement “every number is even or odd”. How can we “prove” it by constructing something?



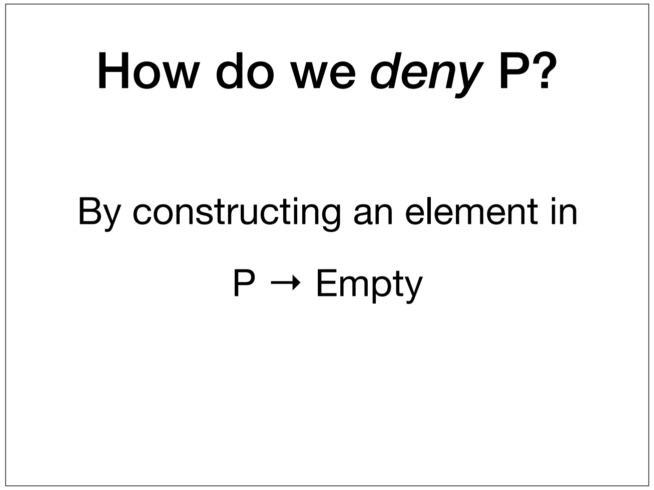
Proofs as constructions

"Every natural number is even or odd."

$\prod(n:\text{nat})\ \sum(m:\text{nat})\ (n = 2m) + (n = 2m+1)$

37

Ask what an element of this type does. It is a map, which takes a number n and gives a pair. The pair tells you how to decompose n into a lowest bit and the rest of it. In essence, it *computes* the fact that every number is even or odd.



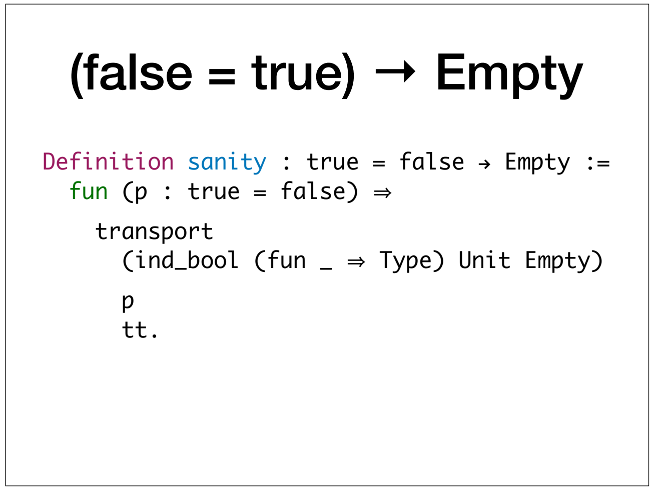
How do we *deny* P?

By constructing an element in

$P \to \text{Empty}$

38

Bow how do we show that something *isn't* the case?
For that we can use the empty type.

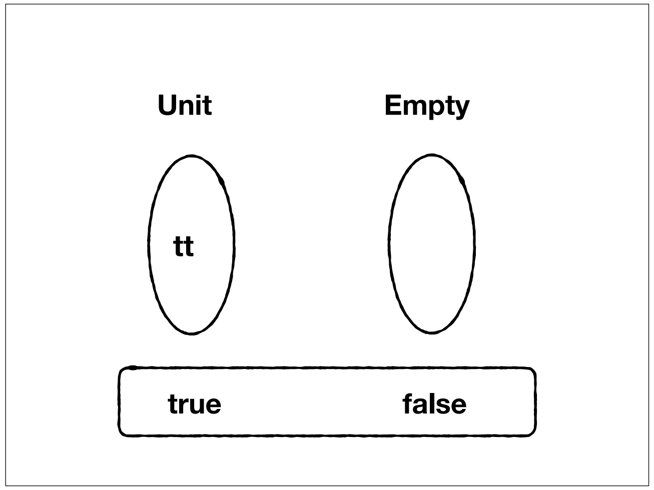


(false = true) → Empty

```
Definition sanity : true = false → Empty :=
  fun (p : true = false) ⇒
    transport
      (ind_bool (fun _ ⇒ Type) Unit Empty)
      p
      tt.
```

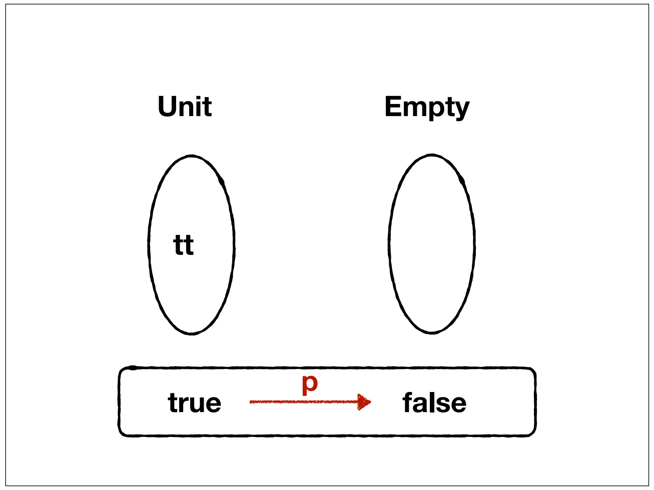
39

Let us walk through an example by showing that true isn't false. We need to construct map which takes a path p from true to false and turns it into an element of Empty.
Hopefully by mid-week you will be able to understand how such a map is defined in Coq, shown here. Let us draw the picture that this construction encodes.



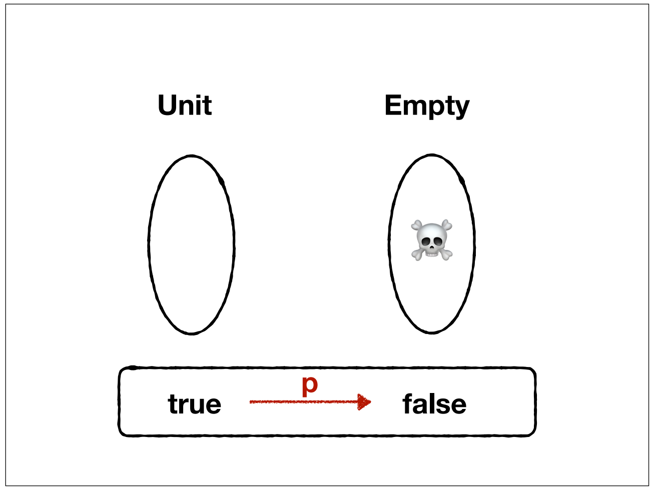
40

We start with a dependent type over Bool.
The fiber at true is the Unit type. The fiber at false is the Empty type.
(We did not actually say how such a dependent type is obtained, but let us say we can get it.)



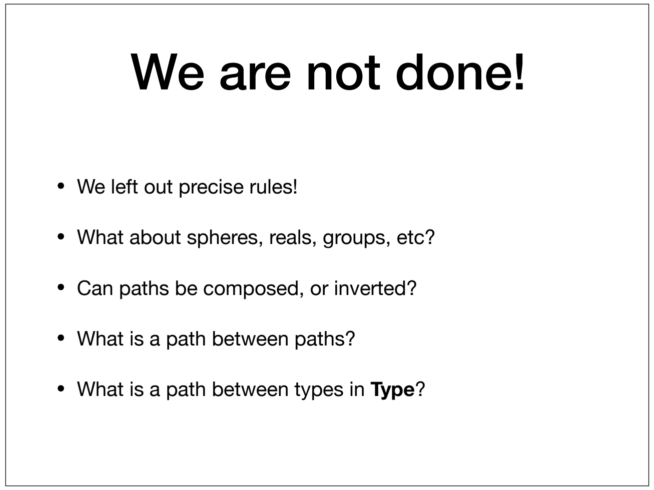
41

What happens if there is a path from true to false? Can we make an element of Empty?



42

Yes, we just transport tt across p.



We are not done!

- We left out precise rules!
- What about spheres, reals, groups, etc?
- Can paths be composed, or inverted?
- What is a path between paths?
- What is a path between types in **Type**?

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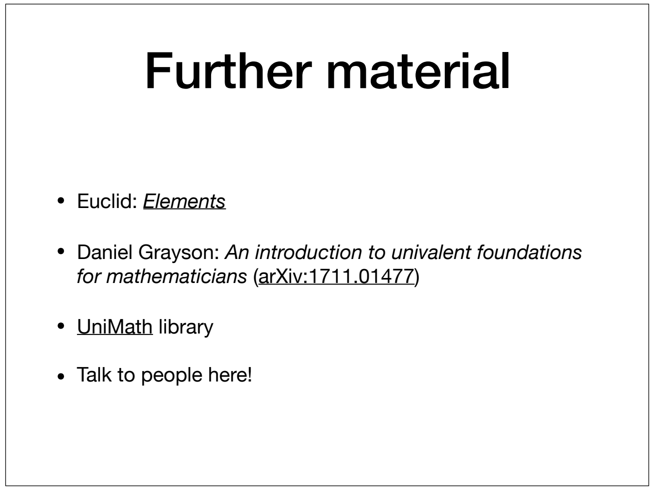
I have to stop here. Bear in mind that this was only the introductory talk, which attempted to explain one possible intuition about type theory. You will learn other ways of looking at type theory throughout the week.

And please ask everyone many questions. Here is a list to get you started.



Further material

- Euclid: *Elements*
- Daniel Grayson: *An introduction to univalent foundations for mathematicians* (arXiv:1711.01477)
- UniMath library
- Talk to people here!

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If you would like some further reading, I recommend that you look at Euclid's elements and see what proportion is construction, and what proportion are statements.

A very good introduction to univalent foundations was written by Daniel Grayson. He should have been here, but had an open heart surgery a couple of weeks ago.

Of course, for the brave there is always the Coq code.

Thank you.