Squares in $(1^2+1)\cdots(n^2+1)$

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1 Introduction

The study of sequences containing infinitely many squares is a common topic in number theory. It has been conjectured [1], and checked for $n \le 10^{3200}$, that

$$P_n = \prod_{k=1}^{n} (k^2+1)$$

is not a square for $n > 3$. We prove this conjecture in full.

As an easy consequence we deduce that the sequence $x_n := \tan \sum_{k=1}^n \tan^{-1}(k)$ doesn't vanish for $n > 3$, which is the main result of [1]. Indeed, as $\sum_{k=1}^n \tan^{-1}(k)$ is the argument of the Gaussian integer $\prod_{k=1}^n (1+ki) = r + si$, we have that if $x_n = 0$ then $s = 0$, so $\prod_{k=1}^n (1+k^2) = r^2$, which is impossible for $n > 3$.

There exists a wide literature about the greatest prime factor, say Q_n, of the product P_n. We observe that the early estimates $Q_n/n \to \infty$ ([3]) or $Q_n \gg n \log n$ ([4]) easily imply that P_n is not a square for n large enough after the first remark in the proof of Theorem 1.

It should be noted, however, that our proof is completely elementary. Actually, the most sophisticated tool used in the proof is the Chebyshev's upper bound inequality for prime numbers. In particular we avoid the use of the asymptotic $\sum_{p \not\equiv 1 \pmod 4} \frac{\log p}{p} \sim \frac{1}{2} \log n$ used in the above mentioned estimates of Q_n.

2 The result

Theorem 1 *If $n > 3$, then $P_n = \prod_{k=1}^n (k^2+1)$ is not a square.*

Proof. Through the proof, p denotes a rational prime. If P_n were a square and $p|P_n$ then $p^2|P_n$. There are two possibilities: If $p^2|k^2+1$ for some $k \le n$ then $p \le \sqrt{n^2+1} < 2n$. Otherwise, there exist $j, k,\ j < k \le n$ such that $p|j^2+1$ and $p|k^2+1$ and then $p|(k-j)(k+j)$ which also implies that $p < 2n$. Then, if P_n is a square we can write

$$P_n = \prod_{p<2n} p^{\alpha_p}.$$

Since $P_n > n!^2$, if we write $n! = \prod_{p \le n} p^{\beta_p}$ we have that

$$\sum_{p \le n} \beta_p \log p < \frac{1}{2} \sum_{p<2n} \alpha_p \log p. \tag{1}$$

We observe that $\alpha_2 = \lceil n/2 \rceil$ since $k^2+1 \equiv 1$ or $2 \pmod 4$ depending whether k is odd or even. Also it is well known that if an odd prime p divides k^2+1 then $p \equiv 1 \pmod 4$. In this case, since each interval of length p^j contains two solutions of $x^2+1 \equiv \pmod{p^j}$, we have

$$\alpha_p = \sum_{j \le \log(n^2+1)/\log p} \#\{k \le n,\ p^j | k^2+1\} \le \sum_{j \le \log(n^2+1)/\log p} 2\lceil n/p^j \rceil. \tag{2}$$

On the other hand

$$\beta_p = \sum_{j \le \log n/\log p} \#\{k \le n,\ p^j | k\} = \sum_{j \le \log n/\log p} \lfloor n/p^j \rfloor. \tag{3}$$

Thus, if $p \equiv 1 \pmod 4$ we have

$$\begin{aligned} \alpha_p/2 - \beta_p &\le \textstyle\sum_{j \le \frac{\log n}{\log p}} \left(\lceil n/p^j \rceil - \lfloor n/p^j \rfloor \right) + \sum_{\frac{\log n}{\log p} < j \le \frac{\log(n^2+1)}{\log p}} \lceil n/p^j \rceil \\ &\le \textstyle\sum_{j \le \frac{\log n}{\log p}} 1 + \sum_{\frac{\log n}{\log p} < j \le \frac{\log(n^2+1)}{\log p}} 1 \le \frac{\log(n^2+1)}{\log p}. \end{aligned}$$

We use this in (1) to write

$$\sum_{\substack{p\le n\\ p\not\equiv 1\ (4)}} \beta_p \log p \le \frac{1}{2}\lceil n/2\rceil \log 2 + \log(n^2+1)\pi(n;1,4) + \frac{1}{2}\sum_{n<p<2n} \alpha_p \log p. \quad (4)$$

The estimates $\alpha_p \le 2$ if $p > n$ and

$$\beta_p \ge \frac{n}{p-1} - \frac{p}{p-1} - \frac{\log n}{\log p} \ge \frac{n-1}{p-1} - \frac{\log(n^2+1)}{\log p} \quad \text{if} \quad p \le n$$

can be obtained easily from (2) and (3). Next we put these estimates in (4) to get

$$(n-1)\sum_{\substack{p\le n\\ p\not\equiv 1\ (4)}} \frac{\log p}{p-1} \le (n+1)\frac{\log 2}{4} + \log(n^2+1)\pi(n) + \sum_{n<p<2n} \log p.$$

Now we use the Chebyshev inequalities $\sum_{p\le n} \log p \le \log 4n$ and $\sum_{n<p<2n} \log p \le n\log 4$ and $\pi(n) \le 2\log 4 \frac{n}{\log n} + \sqrt{n}$ (see for example [2]) to obtain

$$\sum_{\substack{p\le n\\ p\not\equiv 1\ (4)}} \frac{\log p}{p-1} \le \frac{n+1}{n-1}\left(\frac{\log 2}{4} + \log 4\right) + \frac{\log(n^2+1)}{n-1}\left(2\log 4\frac{n}{\log n} + \sqrt{n}\right).$$

The limit of the right hand side is $\frac{41}{4}\log 2$. Actually, that quantity is < 7.14 for $n \ge 702007$. Adding over enough primes $p \not\equiv 1 \pmod 4$ we can see that for $n \ge 702007$

$$\sum_{\substack{p\le n\\ p\not\equiv 1\ (4)}} \frac{\log p}{p-1} > 7.14, \quad (5)$$

which proves the theorem for $n \ge 702007$.

Finally we have to check that P_n is not a square for $4 \le n < 702007$.

$4^2+1 = 17$. The next time that the prime 17 divides k^2+1 is for $k = 17-4 = 13$. Hence P_n is not a square for $4 \le n \le 12$.

$10^2+1 = 101$. The next time that the prime 101 divides k^2+1 is for $k = 101-10 = 91$. Hence P_n is not a square for $10 \le n \le 90$.

$36^2 + 1 = 1297$. The next time that the prime 1297 divides $k^2 + 1$ is for $k = 1297 - 36 = 1261$. Hence P_n is not a square for $36 \leq n \leq 1260$.

$860^2 + 1 = 739601$. The next time that the prime 739601 divides $k^2 + 1$ is for $k = 739601 - 860 = 738741$. Hence P_n is not a square for $860 \leq n \leq 738740$.

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References

[1] T. Amdeberhan et al., *Arithmetical properties of a sequence arising from an arctangent sum*, J. Number Theory (2007), doi:10.1016/j.jnt.2007.05.008

[2] G. Hardy and E. Wright. *An Introduction to the Theory of Numbers.* Oxford University Press, 1980.

[3] E. Landau. *Handbuch űber die Lehre von der Verteilung der Primzahlen,* 1 (1909), 559-561.

[4] T. Nagell. *Zur Arithmetik der Polynome.* Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg. (1922), 179-194.