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ON SOME PROPERTIES OF GENERALIZED NARAYANA NUMBERS

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Abstract. In this paper, we consider a three-dimensional generalization of the Narayana numbers. We find a generating function for these generalized Narayana numbers and show another way to get explicit formula. Also we obtain new properties of these numbers for the composition of generating functions.

Key words: Narayana numbers, generalized Narayana numbers, explicit formula, generating function, composition.

The Narayana triangle is a classic combinatorial number triangle, which has many different applications (Narayana 1979; Petersen 2015; Stanley 2015). In this paper, we use the definition of this number triangle given in OEIS (Sloane). The Narayana triangle is defined by the following generating function:

$$F(x,y)=\frac{1-x(y+1)-\sqrt{\left(1-x(y+1)\right)^2-4x^2y}}{2x}=\sum_{n>0}\sum_{m>0}N(n,m)x^ny^m,$$

where the Narayana numbers are defined by

$$N(n,m)=\frac{1}{n}\binom{n}{m-1}\binom{n}{m}.$$

The generating function $F(x,y)$ satisfies the functional equation

$$-xF(x,y)^2+(-xy-x+1)F(x,y)-xy=0.$$

The Narayana triangle describes a large number of combinatorial sets. For example, the Narayana numbers are related to the Catalan numbers by the known formula

$$N(n,1)+N(n,2)+\cdots+N(n,n)=C_n.$$

From this formula, we can conclude that the Narayana numbers describe classes of subsets for the combinatorial sets described by the Catalan numbers. There are more than 200 combinatorial sets that are described by the Catalan numbers (Stanley 2015). For instance, it can be some classes of permutations, lattice paths, bracket expressions, Dyck words, trees, etc. One of the combinatorial interpretations for the Narayana numbers is the set of Dyck n-paths with m peaks (Petersen 2015). Also there are a large number of generalizations of the Narayana numbers (Barry 2011; Barry, Hennessy 2012; Mansour, Sun 2009; Alexeev, Tikhomirov 2017). In this paper, we consider the following three-dimensional generalization of the Narayana numbers:

$$N_e(n,m,k)=\frac{k}{n}\binom{n}{m-k}\binom{n}{m}. \tag{1}$$

The study of these numbers was first given in (Guy 2000), where lattice paths were investigated. However, an explicit formula and a generating function were not obtained. The presented explicit formula (1) is given in (Callan; Defant). Also in these papers the connection of the formula (1) with the power of the generating function of the Narayana numbers is described.

In this paper, we present a new proof for the explicit formula of the generalized Narayana numbers and we obtain a multivariate generating function with three variables for these numbers. In addition, we obtain new properties of the generalized Narayana numbers for the composition of generating functions.

Theorem 1. The generalized Narayana numbers $N_e(n,m,k)$ are defined by the generating function

$$F_e(x,y,z)=\frac{1}{1-zF(x,y)}=\sum_{n>0}\sum_{m>0}\sum_{m>0}N_e(n,m,k)x^n y^m z^k$$

The use of the composition of generating functions with one variable is described in detail in (Stanley 2011; Kruchinin D.V., Kruchinin V.V. 2013; Kruchinin V.V., Kruchinin D.V. 2013). In the next theorem we give another important property of the generalized Narayana numbers, which is associated with the operation of the composition of generating functions.

Theorem 2. Suppose the composition of generating functions

$$A(x)=G\big(F(x,y)\big)=\sum_{n\geq 0}\sum_{m\geq 0}a(n,m)x^n y^m,$$

where

$$G(x)=\sum_{n\geq 0}g(n)x^n$$

and $F(x,y)$ is the generating function for the Narayana numbers. Then the coefficients $a(n,m)$ of the composition are

$$a(n,m)=\sum_{k=0}^{n+m}g(k)N_e(n,m,k)$$

or

$$a(n,m)=\frac{1}{n}\binom{n}{m}\sum_{k=1}^{m}k\binom{n}{m-k}g(k).$$

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References

Alexeev N., Tikhomirov A. Singular values distribution of squares of elliptic random matrices and type B Narayana polynomials // Journal Theoretical Probability. 2017. N. 30. P. 1170–1190.

Barry P. On a generalization of the Narayana triangle // Journal of Integer Sequences. 2011. N. 14. Article 11.4.5.

Barry P., Hennessy A. Generalized Narayana polynomials, Riordan arrays, and lattice paths // Journal of Integer Sequences. 2012. N. 15. Article 12.4.8.

Callan D. Generalized Narayana numbers. URL: https://oeis.org/A281260/a281260.pdf.

Defant C. An identity involving Narayana numbers URL: https://math.stackexchange.com/questions/1498014/an-identity-involving-narayana-numbers.

Guy R.K., Catwalks, sandsteps and Pascal pyramids // Journal of Integer Sequences. 2000. N. 3. Article 00.1.6.

Kruchinin D.V., Kruchinin V.V. Application of a composition of generating functions for obtaining explicit formulas of polynomials // Journal of Mathematical Analysis and Applications. 2013. N. 404. P. 61–171.

Kruchinin V.V., Kruchinin D.V. Powers of generating functions and its applications. Russia, Tomsk: TUSUR, 2013. 236 p. (in rus)

Mansour T., Sun Y. Identities involving Narayana polynomials and Catalan numbers // Discrete Mathematics. 2009. N. 309. P. 4079–4088.

Narayana T.V. Lattice path combinatorics with statistical applications. Canada: University of Toronto Press, 1979. P. 100–101.

Petersen T.K. (2015) Narayana numbers // Eulerian numbers. USA: Birkhauser Advanced Texts Basler Lehrbucher, 2015. P. 19–45.

Sloane N.J.A. The on-line encyclopedia of integer sequences. URL: http://oeis.org.

Stanley R.P. Enumerative combinatorics. Volume 1. 2nd ed. USA: Cambridge University Press, 2011. 640 p.

Stanley R.P. Catalan numbers. New York: Cambridge University Press, 2015. 215 p.