THE PISANO PERIOD OF FIBONACCI K-SECTIONS

A PREPRINT

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March 22, 2019

ABSTRACT

We show how to transform the period length of the Fibonacci numbers modulo m into the period length of any Fibonacci k-section (F_{kn}) modulo m. Then we establish properties for the k-section period length and examine the behaviour for certain values of k.

***K*eywords** Fibonacci number, Pisano period

1 Introduction

Let F be the Fibonacci numbers such $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$, $F_1 = 1$. This simple recurrence have many intriguing properties and we will investigate one related to the periodicity when it's taken modulo m. The periodicity of $F_n \mod m$ was known by Lagrange (1774) and since then the periodicity and many other properties have been proven. We examine the period length of $F_{kn} \mod m$, where k and m are positive integers. The period length is often referred to as the Pisano period, $\pi(m)$. By the Fibonacci k-section we mean F_{kn}[1]. *Example:* If we let $k = 2$ we get the Fibonacci bi-section F_{2n} (A001906)[2]. The main result is a transform from $\pi(m)$ into the Pisano period for any Fibonacci k-section, $\pi_k(m)$. The author found related sequences in sequencedb.net [5] which eventually led to this article.

We will very briefly remind the reader of the Fibonacci entry points, the Pisano period and the zeroes of the Pisano period. The Fibonacci entry point, $z(m) = k$, is the smallest k such that $m|F_k$. *Example:* The Fibonacci entry point of 7 is 8 since $F_8 = 21$ is the first Fibonacci number divisible by 7. The Pisano period $\pi(m) =$ the period length of $F_n \mod m$. *Example*: $F_n \mod 4 =$ A079343: 0,1,1,2,3,1,0,1,1,2,3,1,0,1,1,2,3,1,0,... for which the period length is 6. Hence $\pi(4) = 6$. The number of zeros in a Pisano period is denoted $\omega(m)$. *Example*: The period above (A079343) has one zero, so $\omega(4) = 1$.

2 The Pisano period of Fibonacci k-sections

The function $F_n \mod m$ is periodic for all m [1]. Now, let's examine the Fibonacci bi-section modulo m. Taking $F_{2n} \mod m$ will also generate a periodic sequence. Instead of taking modulo on every term of F_n we take modulo on every second (bolded in example below).

f	$f \mod 7$
F_n	1,**1**,2,**3**,5,**1**,6,**0**,6,**6**,5,**4**,2,**6**,1,**0**,1,**1**,2...
F_{2n}	1,3,1,0,6,4,6,0,1,3,1...

We introduce a function, $\pi_2(m)$, to describe the period length of $F_{2n} \mod m$. Since F_{2n} selects every second term, $\pi_2(m)$ should be half of $\pi(m)$ as long as $\pi(m)$ is even, which is true for all $m > 2$ [1]. In the example above $\pi(7) = 16$

[1]Not to be confused with the generalized k-Fibonacci numbers
[2]All A-numbers can be found at https://oeis.org/AXXXXXX, there is also a list of sequences in the appendix.

and $\pi_2(7) = 8$. The sequence formed by $\pi_2(m)$ can be found in the OEIS under the somewhat surprising name "Arnold cat map" (A281230). What happens when the period is uneven? Let us examine $f \mod 2$ in this case $\pi(2) = 3$.

f	$f \mod 2$
F_n	1,**1**,0,**1**,1,**0**,...
F_{2n}	1,1,0,...

As seen above, every other (bolded) term is found in F_{2n} - but since the skipping 'misses' the terminating zero at position 3, we continue for yet another period before the cycle is closed at position 6. This closing/alignment can be thought of as how many times we need to repeat a string of length 2 (ab) and length 3 (abc) before they align:

abc**a**bc**a**bc...
ab**a**b**a**b**a**b**a**...

The period completes at position 6 after 3 repetitions, then restart realigned at position 7 (both aligned 'a' are bolded above). Mathematically, we find the least common multiple of $\pi(2) = 3$ and $k = 2$ (alignment) and then divide the multiple by $k = 2$ to get the number of repetitions (there is 1 term in each repetition). Hence the Pisano period for the Fibonacci bi-section, $F_{2n} \mod m$ is

$$\pi_2(m) = \frac{\operatorname{lcm}(\pi(m), 2)}{2} \text{ for } k \geq 1$$

In fact, this can be generalized for all k-sections which leads us to our main result.

Theorem 1. *The Pisano period for $F_{kn} \mod m$ is*

$$\pi_k(m) = \frac{\operatorname{lcm}(\pi(m), k)}{k} \text{ for } k \geq 1 \tag{1}$$

Proof. Let a be the periodic sequence generated by $F_n \mod m$ then the period length of a is $\pi(m)$. Note that $F_{kn} \mod m$ is a subset of a, specifically $b = a(kn)$. A full period in b will occur when kn is the least common multiple with $\pi(m)$. Finally we need to divide this multiple by k to take account that we are only interested in every k-th term. $\square$

We can rewrite $\pi_k(m)$ in terms of the greatest common divisor using the identity $\operatorname{lcm}(a, b) = \frac{|ab|}{\gcd(a,b)}$

$$\pi_k(m) = \frac{\pi(m)}{\gcd(\pi(m), k)} \text{ for } k \geq 1 \tag{2}$$

The second representation is useful to understand some of the properties below.

3 Properties of the period length of Fibonacci k-sections

Many of the corollaries below investigate the behaviour of special values for k and rely directly on well established theorems.

Lemma 2. *$\pi(m)$ is even for $m > 2$*

Lemma 3. $a|b \implies \pi(a)|\pi(b)$

Lemma 4. $\pi(m) = \omega(m)z(m)$

Lemma 5. *$\pi(m)|2p - (\frac{p}{5})$ where $(\frac{p}{5})$ is the Legendre symbol.*

Lemma 6. *$\omega(m)$ equals either 1, 2 or 4*

Lemma 7. *$\pi(p^2)$ equals either $p\pi(p)$ or $\pi(p)$*

Lemma 8. *$\pi(p^e) = p^{e-1}\pi(p)$ if $\pi(p) \neq \pi(p^2)$*

Lemma 9. *$\gcd(\pi(p), p) = 1$ for $p \neq 5$*

Most of the lemmas are standard results that can be found on Wikipedia [1] another excellent source on the Pisano period is Marc Renault's page [2]. We also recommend Walls article [3] for more details. For the best possible descriptions and references on the actual sequences oeis.org is the go to place [4].

Corollary 10. $\pi_k(m) \leq \pi(m)$.

Proof. The denominator of (2) is always $\geq 1 \iff \pi_k(m) \leq \pi(m)$. □

Corollary 11. *$\pi_{\pi(m)}(m) = 1$ and more generally $\pi_{a\pi(bm)}(m) = 1$ for all $a, b \geq 1$.*

Proof. For $a > 1$ we are basically skipping a periods ($a\pi(m)$) at a time and the period itself will only have one term, 0. In (2) it's obvious that the denominator $\gcd(\pi(m), a\pi(m)) = \pi(m)$. For $b > 1$ we rely on lemma 3 and since $m|bm$ we can be sure that $\pi(m)|\pi(bm)$, so again, $\gcd(\pi(m), \pi(bm)) = \pi(m)$ □

Corollary 12. *$\pi_{\frac{\pi(m)}{2}}(m) = 2$ for $m > 2$.*

Proof. By lemma 2 $\pi(m)$ is even for all $m > 2$, hence divisible by 2. Therefore the period $\pi_k(m)$ when $k = \frac{\pi(m)}{2}$ always contains two remainders, the one in the middle of the period and the last at $\pi(m)$. □

3.1 Relation to the Fibonacci entry point and the zeros the Pisano period

By lemma 4 we know that $\pi(m) = z(m)\omega(m)$ where $z(m)$ is the Fibonacci entry point and $\omega(m)$ is the number of zeroes in the Pisano period $\pi(m)$. Recall that $\omega(m)$ can only be 1,2 or 4 (lemma 6). The relationship will help us explain the following results.

Corollary 13. *$\pi_{z(m)}(m) = \omega(m)$.*

Proof. Since $\pi(m) = z(m)\omega(m)$, we know that $\gcd(\pi(m), z(m)) = z(m)$. Via (2) we get

$$\pi_{z(m)}(m) = \frac{\pi(m)}{\gcd(\pi(m), z(m))} = \frac{\pi(m)}{z(m)} = \omega(m)$$

□

Corollary 14. *$\pi_{\omega(m)}(m) = z(m)$.*

Proof. Analogous to the proof to corollary 13. □

3.2 Properties related to prime numbers

Corollary 15. *If $p \neq 5$ is a prime then $\pi_{2(p-(\frac{p}{5}))}(p) = 1$.*

Proof. Again recall (2), the only case when $\pi_k(p) = 1$ is when the denominator $\gcd(\pi(p), k) = \pi(p)$ which implies that k is a multiple of $\pi(p)$. When p is a prime then (lemma 5)

$$\begin{cases} \pi(p)|p-1, & \text{if } p = \pm 1 \mod 5 \\ \pi(p)|2(p+1), & \text{if } p = \pm 2 \mod 5 \end{cases}$$

In both cases $\pi(p)|2(p - (\frac{p}{5}))$, we have derived the result, since $\gcd(\pi(p), 2(p - (\frac{p}{5}))) = \pi(p)$. □

Corollary 16. *If $p \neq 5$ is a prime then $\pi_p(p) = \pi(p)$.*

Proof. By lemma 9 we have $\gcd(\pi(p), p) = 1$ that can directly be inserted into (2), giving us the result. □

3.3 Relationship to prime powers

Corollary 17. *$\pi_p(p^2) = \pi_p(p) = \pi(p)$*

Proof. It's known that $\pi(p^2)$ is either $p\pi(p)$ or $\pi(p)$ (lemma 7), the latter will happen if p is a Wall-Sun-Sun prime (non are known but conjectured to exists). Let's examine both cases.

In the first case, $\pi(p^2) = p\pi(p)$, we get by (2)

$$\pi_p(p^2) = \frac{p\pi(p)}{\gcd(p\pi(p), p)} = \pi(p)$$

In the second case, $\pi(p^2) = \pi(p)$, we get by (2)

$$\pi_p(p^2) = \frac{\pi(p)}{\gcd(\pi(p), p)} = \pi(p)$$

So regardless if p is a Wall-Sun-Sun prime the result is the same, $\pi(p)$. □

For simplicity, the remaining corollary assume that $\pi(p) \neq \pi(p^2)$, that p is not an Wall-Sun-Sun prime. First recall corollary 11 which states that $\pi_{\pi(m)}(m) = 1$, we will now expand on this a bit by considering primes and power of primes.

Corollary 18. *If* $\pi(p) \neq \pi(p^2)$ *then* $\pi_{\pi(p)}(p^e) = p^{e-1}$.

Proof. By lemma 8 and lemma 9 we know that $\pi(p^e) = p^{e-1}\pi(p)$ and $\gcd(p, \pi(p)) = 1$ substituting into (2) we get

$$\pi_{\pi(p)}(p^e) = \frac{p^{e-1}\pi(p)}{\gcd(p^{e-1}\pi(p), \pi(p))} = p^{e-1}$$

□

3.4 Summary of properties

The first column is a reference to the theorem/corollary, the second the expression, the third what the expression equals which also equals the period length of the last column where k is put back into $F_{kn} \mod m$.

#	$\pi_k(m)$	Equals	Period length of
Pisano period	$\pi_1(m)$	$\pi(m)$	$F_n \mod m$
theorem 1	$\pi_k(m)$	$\text{lcm}(\pi(m), k)/k$	$F_{kn} \mod m$
corollary 11	$\pi_{a\pi(bm)}(m)$	1	$F_{a\pi(bm)n} \mod m$
corollary 12	$\pi_{\frac{\pi(m)}{2}}(m)$	2	$F_{\frac{\pi(m)}{2}n} \mod m$
corollary 13	$\pi_{z(m)}(m)$	$\omega(m)$	$F_{z(m)n} \mod m$
corollary 14	$\pi_{\omega(m)}(m)$	$z(m)$	$F_{\omega(m)n} \mod m$
corollary 15	$\pi_{2(p-(\frac{p}{5}))}(p)$	1	$F_{2(p-\frac{p}{5})n} \mod p$
corollary 16	$\pi_p(p)$	$\pi(p)$	$F_{pn} \mod p$
corollary 17	$\pi_p(p^2)$	$\pi(p)$	$F_{pn} \mod p^2$
corollary 18	$\pi_{\pi(p)}(p^e)$	p^{e-1}	$F_{\pi(p)n} \mod p^e$

4 Conclusion

We have used a simple transform between the regular Pisano period into the Pisano period of Fibonacci k-sections where we have reapplied well established theorems. Perhaps the perspective of k-sections can complement the understanding of the Pisano period. At least $\pi_k(m)$ provides a handy transform from $\pi(m)$ into the Pisano period of any Fibonacci k-section.

References

[1] https://en.wikipedia.org/wiki/Pisano_period

[2] http://sites.math.rutgers.edu/~zeilberg/essays683/renault.html

[3] Wall, D. D. “Fibonacci Series Modulo m.” The American Mathematical Monthly, vol. 67, no. 6, 1960, pp. 525–532. JSTOR, www.jstor.org/stable/2309169.

[4] https://oeis.org

[5] http://sequencedb.net

Appendix: List of sequences

Initial terms for some of the sequences mentioned in this article.

$F(n)$=A000045: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, ...

$F(2n)$=A001906: 0, 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, 17711, 46368, 121393, 317811, 832040,...

$z(m)$=A001177: 1, 3, 4, 6, 5, 12, 8, 6, 12, 15, 10, 12, 7, 24, 20, 12, 9, 12, 18, 30, 8, 30, 24, 12, 25,...

$\omega(m)$=A001176: 1, 1, 2, 1, 4, 2, 2, 2, 2, 4, 1, 2, 4, 2, 2, 2, 4, 2, 1, 2, 2, 1, 2, 2, 4, 4, 2, 2, 1, 2,...

$\pi(m)$=A001175: 1,3,8,6,20,24,16,12,24,60,10,24,28,48,40,24,36,24,18,60,16,30,48,24,100,84,72,48,14,120..

$\pi_2(m)$=A281230: 1,3,4,3,10,12,8,6,12,30,5,12,14,24,20,12,18,12,9,30,8,15,24,12,50,42,36,24,7,60,...

$\pi_3(m)$=1,1,8,2,20,8,16,4,8,20,10,8,28,16,40,8,12,8,6,20,16,10,16,8,100,28,24,16,14,40,...

$\pi_4(m)$=1,3,2,3,5,6,4,3,6,15,5,6,7,12,10,6,9,6,9,15,4,15,12,6,25,21,18,12,7,30,...

$\pi_5(m)$=1,3,8,6,4,24,16,12,24,12,2,24,28,48,8,24,36,24,18,12,16,6,48,24,20,84,72,48,14,24,...

$\pi_6(m)$=1,1,4,1,10,4,8,2,4,10,5,4,14,8,20,4,6,4,3,10,8,5,8,4,50,14,12,8,7,20,...

$\pi_7(m)$=1,3,8,6,20,24,16,12,24,60,10,24,4,48,40,24,36,24,18,60,16,30,48,24,100,12,72,48,2,120,...