FIBONACCI AND CATALAN PATHS IN A WALL

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Abstract. We study the distribution of some statistics (width, number of steps, length, area) defined for paths contained in walls. We present the results by giving generating functions, asymptotic approximations, as well as some closed formulas. We prove algebraically that paths in walls of a given width and ending on the x-axis are enumerated by the Catalan numbers, and we provide a bijection between these paths and Dyck paths. We also find that paths in walls with a given number of steps are enumerated by the Fibonacci numbers. Finally, we give a constructive bijection between the paths in walls of a given length and peakless Motzkin paths of the same length.

1. Introduction and notation

Lattice path theory takes an important place in combinatorics. In the literature, there are many articles that study combinatorial problems on lattice paths (see [15]). Most of the time, lattice paths are defined in $\mathbb{Z}^2$ by a starting point (almost always the origin), and a sequence of vectors (also called steps) lying in a given set S. For instance, paths (also called walks) defined with $S = \{N, S, E, W\}$, where $N = (0, 1)$, $S = (0, -1)$, $E = (1, 0)$, and $W = (-1, 0)$, are widely studied in the quarter plane $\mathbb{N}^2$ (see [9, 17, 18] for instance). Such a path may overlap itself (i.e., vertices and edges can be repeated). The problem of the enumeration of these paths is very interesting to solve whenever boundary constraints are imposed. On the other hand, we can consider lattice paths in $\mathbb{N}^2$ with no overlaps by forcing the paths to go to the right; this is the subclass of *directed paths*. For example, if $S = \{U, D\}$, where $U = (1, 1)$, $D = (1, -1)$, then the paths in $\mathbb{N}^2$ starting at the origin and ending on the x-axis are the famous Dyck paths that are counted with respect to the semilength (number of steps divided by 2) by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see A000108 in Sloane's On-line Encyclopedia of Integer Sequences [27]). Moreover, if we permit steps $H = (1, 0)$, then we obtain the class of Motzkin paths that are enumerated by the sequence A001006 in [27]. We refer to [1, 4, 5, 6, 7, 8, 10, 12, 19, 20, 25, 26] for several works on the enumeration and the generation of such paths (with and without overlaps) with respect to the length and various statistics.

In this work we introduce a new class of paths in $\mathbb{N}^2$ induced by a regular tiling of the first quadrant: the *wall*. More precisely, a *wall* is a tiling of $\mathbb{N}^2$ using tiles (or bricks) of size 1×2 organized as shown in Figure 1. More formally, a wall is a subgrid of $\mathbb{N}^2$ constituted by the segments $(0, b) - (\infty, b)$ for every $b \geq 0$, and $(a, b) - (a, b+1)$ for $a \geq 0$ and $b \geq 0$ of

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the same parity. Note that this tiling can be also viewed as the cell structure of the plant tissues (see [28] for instance), and our work studies the enumeration of paths of the sap [16] according to several parameters defined below.

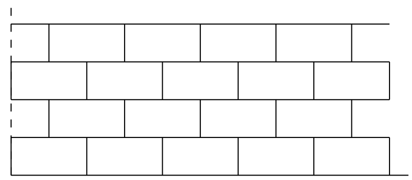

FIGURE 1. The wall tiling of $\mathbb{N}^2$.

A *path* in a wall is a lattice path on the subgrid defined by the wall. More precisely, it is a lattice path in $\mathbb{N}^2$ starting at the origin $(0,0)$ where each step links two adjacent corners of the bricks, by following the sides of the bricks with no overlap and no return to the left (each step touching exactly two corners of some bricks at its beginning and its end). Thus a path consists of steps $N=(0,1)$, $S=(0,-1)$, and $E\in\{E_1=(1,0),E_2=(2,0)\}$ and their connections are constrained by the tiling (E_2 is used on the x-axis and E_1 above). Let $\mathcal{P}$ be the set of all paths in a wall. For instance, Figure 2 shows the two paths $NEEENEESESENEES$ and $NEEENEESESENENEN$, which can also written as $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1E_1S$ and $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1NE_1N$. The first path ends on the x-axis and the second path ends at ordinate 3. Note that some works [13, 14] have investigated the connection between paths and tilings of the plane, but this does not correspond to our definition of the paths in a wall.

A *statistic* on the set $\mathcal{P}$ is a function $\mathtt{w}$ from $\mathcal{P}$ to $\mathbb{N}$. Below, we define three important statistics for our study. The *width* of a path P, denoted $\mathtt{width}(P)$, is the abscissa of its last point. For instance Figure 2 shows two paths of width 10. The *length* of a path P, denoted $\mathtt{length}(P)$, is the length of the path considering as a curve in $\mathbb{R}^2$. Figure 2 shows two paths of length 16 and 17, respectively. The *number of steps* of a path P, denoted $\mathtt{nbstep}(P)$, is the number of steps in the path (or equivalently the number of connections of two corners). Figure 2 shows two paths with 15 and 16 steps, respectively.

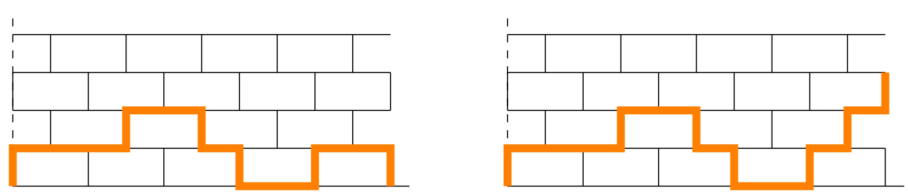

FIGURE 2. Two paths in a wall $NEEENEESESENEES$ and $NEEENEESESENENEN$. The left path ends on the x-axis, its width is 10, it has 15 steps, and its length is 16. The right path ends at ordinate 3, its width is 10, it has 16 steps, and its length is 17.

Now, let us assume that the statistic $\mathtt{w}$ returns either the width, or the length, or the number of steps of a path. For $k \geq 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, resp. $h_k = h_k(z)$), where the coefficient of z^n in the series expansion is the number of paths $P \in \mathcal{P}$ such that $\mathtt{w}(P) = n$, ending at ordinate k with an up-step N (resp., with a down-step S, resp., with a horizontal-step E). Let f_k^0 (resp. f_k^1) be the generating function consisting of the terms z^n in f_k such that $n + k \equiv 0 \mod 2$ (resp., $n+k \equiv 1 \mod 2$). Similarly, we define g_k^0, g_k^1, h_k^0, and h_k^1. Obviously, we have $f_k = f_k^0 + f_k^1$, $g_k = g_k^0 + g_k^1$, and $h_k = h_k^0 + h_k^1$ for any $k \geq 0$.

Also, we introduce the bivariate generating functions for $i \in \{0, 1\}$,

$$F^i(u, z) = \sum_{k\geq 0} u^k f_k^i(z), \quad G^i(u, z) = \sum_{k\geq 0} u^k g_k^i(z), \text{ and } \quad H^i(u, z) = \sum_{k\geq 0} u^k h_k^i(z).$$

For short, we use the notation $F^i(u)$, $G^i(u)$, and $H^i(u)$, $i \in \{0, 1\}$, for these functions.

We will use all these notations for the three following sections of this study according to the choice of the statistic $\mathtt{w}$ ($\mathtt{w} = \mathtt{width}$ in Section 3, $\mathtt{w} = \mathtt{nbstep}$ in Section 4, and $\mathtt{w} = \mathtt{length}$ in Section 5).

Outline of the paper. In this paper, we investigate the enumeration problem of the paths defined by the wall with respect to several parameters. In Section 2, we count paths of a given width (ending on a given abscissa) according to the type of the last step and the ordinate of the last point. We provide an asymptotic approximation for the expected ordinate of the last point, and we prove that such paths ending on the x-axis are counted by the well known Catalan numbers. We exhibit a bijection between these paths and Dyck paths. Note that this last result was already found by Odlyzko [22] in the context of the enumeration of fountains with a given number of coins on the basis. *En passant*, Odlyzko also enumerates fountains with n coins, which allows us to say that paths ending on the x-axis in the wall and having a given number of bricks below the path, are counted by the infinite continued fraction

$$\cfrac{1}{1 - \cfrac{z}{1 - \cfrac{z^2}{1 - \cfrac{z^3}{\ddots}}}}.$$

In this section we also enumerate the paths ending on the x-axis with a given area and width. We note that the total area of the paths is related with the path length in binary trees.

In Section 3, we count paths of a given number of steps according to the type of the last step and the ordinate of the last point. We prove that such paths are counted by the Fibonacci numbers. We exhibit a bijection between these paths and binary words avoiding two consecutive ones.

Finally, in Section 4, we make an analogous study for paths having a given length. We prove that such paths are counted by the generalized Catalan number, which are known to also count RNA structures. We exhibit a bijection between these paths and peakless

Motzkin paths. Note that the study made in this last section is equivalent to the study of paths of a given length in the honeycomb (i.e., hexagonal) lattice (it suffices to expand all bricks of the wall into hexagonal cells).

2. Paths of a given width

In this part, we count paths in $\mathcal{P}$ of a given width, i.e., ending at a given abscissa.

By convention, we fix $f_0^0 = 1$ for considering the empty path consisting of the origin $(0,0)$ only. Observing that a step $N = (0,1)$ in a path cannot end on a corner (n,k), $k \geq 1$, with $n+k \equiv 0 \mod 2$, we deduce $f_k^0 = 0$ for $k \geq 1$. Of course, a path ending with N cannot end at ordinate 0, which implies $f_0^1 = 0$. A path ending with N at ordinate $k=1$ ends necessarily at an even abscissa, and then, it equals either N or QN, where Q is a path ending with a horizontal step $(2,0)$, which implies $f_1^1 = 1 + h_0^0$. Finally, a path ending with N on a corner (n,k), $k>1$, $n+k \equiv 1 \mod 2$, follows necessarily an horizontal step $(1,0)$ that ends at $(n,k-1)$, which implies $f_k^1 = h_{k-1}^0$, $k>1$.

Other recurrence relations for g_k^i and h_k^i, $k \geq 0$, $i \in \{0,1\}$, can be obtained *mutatis mutandis*. Thus we obtain the following equations:

$$\begin{cases} f_0^0 = 1 \text{ and } f_k^0 = 0, & k \geq 1, \\ f_0^1 = 0, f_1^1 = 1 + h_0^0, \text{ and } f_k^1 = h_{k-1}^0, & k > 1, \end{cases}, \quad \begin{cases} g_k^0 = h_{k+1}^1, & k \geq 0, \\ g_k^1 = 0, & k \geq 0, \end{cases} \quad \text{and}$$

$$\begin{cases} h_0^0 = z^2 + z^2(h_0^0 + g_0^0) \text{ and } h_k^0 = z(h_k^1 + f_k^1), & k \geq 1, \\ h_0^1 = 0 \text{ and } h_k^1 = z(h_k^0 + g_k^0), & k \geq 1. \end{cases} \tag{1}$$

Summing the recursions in (1), we have:

$$\begin{cases} F^0(u) &= 1, \\ F^1(u) &= u + \sum\limits_{k\geq 1} u^k h_{k-1}^0 = u + uH^0(u), \end{cases} \quad \begin{cases} G^0(u) &= \sum\limits_{k\geq 0} u^k h_{k+1}^1 = \frac{1}{u} H^1(u), \\ G^1(u) &= 0, \end{cases},$$

$$\begin{cases} H^0(u) &= h_0^0 + z \sum\limits_{k\geq 1} u^k (f_k^1 + h_k^1) \\ &= h_0^0 + z(F^1(u) + H^1(u)), \\ H^1(u) &= z \sum\limits_{k\geq 0} u^k (g_k^0 + h_k^0) - z(h_0^0 + g_0^0) \\ &= z(H^0(u) + G^0(u)) - \frac{h_0^0 - z^2}{z}. \end{cases}$$

Solving the above functional equations, we deduce

$$F^0(u) = 1 \quad \text{and} \quad F^1(u) = \frac{u\left(h_0^0 z - u + z\right)}{u^2 z - u + z},$$

$$G^1(u) = 0 \quad \text{and} \quad G^0(u) = -\frac{h_0^0 uz + h_0^0 z^2 + z^2 - h_0^0}{\left(u^2 z - u + z\right) z},$$

$$H^0(u) = \frac{z\left(-u^2 + h_0^0\right)}{u^2 z - u + z} \quad \text{and} \quad H^1(u) = -\frac{u\left(h_0^0\, uz + h_0^0\, z^2 + z^2 - h_0^0\right)}{\left(u^2 z - u + z\right) z}.$$

In order to compute h_0^0, we use the kernel method [2, 3, 24, 25] on $F^1(u)$. This method consists in cancelling the denominator of $F^1(u)$ by finding u as an algebraic function r of z. Therefore, if we substitute u with r in the numerator then it necessarily equals zero (in order to counterbalance the cancellation of the denominator), which induces the value of h_0^0.

Thus, we factorize the denominator $u^2z - u + z = z(u-r)(u-s)$ with

$$r = \frac{1-\sqrt{1-4z^2}}{2z} \quad \text{and} \quad s = \frac{1+\sqrt{1-4z^2}}{2z}.$$

Note that r is the generating function of Catalan numbers, which ensures us that we remain in the ring of formal power series.

Hence we obtain

$$h_0^0 = \frac{r-z}{z}.$$

Now, substituting h_0^0 with its value in the above generating functions, and simplifying by $(u-r)$ in the numerator and the denominator, we obtain the following:

Theorem 1. *We have*

$$F^0(u) = 1, \quad F^1(u) = \frac{u}{z(s-u)}, \quad G^0(u) = \frac{r-z}{z^2(s-u)}, \quad G^1(u) = 0, \text{ and}$$

$$H^0(u) = \frac{r+u}{s-u}, \quad H^1(u) = \frac{u(r-z)}{z^2(s-u)}.$$

Finally, the bivariate generating function $S(z,u)$, where the coefficient of $z^n u^k$ is the number of paths of width n ending at ordinate k, satisfies

$$S(z,u) = \frac{r(1+u)}{z^2(s-u)}.$$

The first terms of the series expansion of $S(z,u)$ are

$$\begin{aligned}1 + u + \left(u^2+u\right)z + \left(u^3+u^2+2\,u+2\right)z^2 + \left(u^4+u^3+3\,u^2+3\,u\right)z^3 + \\ + \left(u^5+u^4+4\,u^3+4\,u^2+5\,u+5\right)z^4 + \left(u^6+u^5+5\,u^4+5\,u^3+9\,u^2+9\,u\right)z^5 + \\ + \left(u^7+u^6+6\,u^5+6\,u^4+14\,u^3+14\,u^2+14\,u+14\right)z^6 + \cdots\end{aligned}$$

Using the Vieta relations $rs = 1$ and $r + s = 1/z$, we deduce the following.

Corollary 1. *We have*

$$[u^k]F^0(u) = [k=0], \ k \geq 0, \quad [u^k]F^1(u) = \frac{r^k}{z}, \ k \geq 1,$$

$$[u^k]G^0(u) = \frac{(r-z)r^{k+1}}{z^2}, \ k \geq 0,$$

$$[u^k]H^0(u) = \frac{r^{k+1}}{z},\ k \geq 1, \quad [u^k]H^1(u) = \frac{(r-z)r^k}{z^2},\ k \geq 1,$$

$$[u^k]S(z,u) = \frac{r^{k+1}(r+1)}{z^2} - \frac{[k=0]}{z},\ k \geq 0,\ \textit{and}$$

all other coefficients are equal to zero.

In order to provide closed forms for the coefficients of $z^n u^k$ in the previous generating functions, we need to obtain a closed form for the coefficient of z^n in r^k, $k \geq 0$. The quantity r satisfies the functional equation $r = z(1+r^2) = z\phi(r)$ with $\phi(t) = 1 + t^2$. From Lagrange inversion, (see [21] for instance), we have

$$[z^n]r^k = \frac{k}{n}[t^{n-k}]\phi(t)^n = \frac{k}{n}[t^{n-k}](1+t^2)^n.$$

Thus we have

$$[z^n]r^k = 0 \text{ if } n-k \not\equiv 0 \mod 2 \quad \text{and} \quad [z^n]r^k = \frac{k}{n}\binom{n}{\frac{n-k}{2}} \text{ otherwise.}$$

Therefore, we obtain the following.

Theorem 2. *The number $f(n,k)$ of paths of width n ending at ordinate k with a step N is given by*

$$f(n,k) = \frac{k}{n+1}\binom{n+1}{\frac{n+1-k}{2}} \quad \textit{if } n+k \not\equiv 0 \mod 2 \ \textit{ and } \ 0 \ \textit{otherwise.}$$

The number $g(n,k)$ of paths of width n ending at ordinate k with a step S is given by

$$g(n,k) = \frac{k+2}{n+2}\binom{n+2}{\frac{n-k}{2}} - \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}} \quad \textit{if } n+k \equiv 0 \mod 2 \ \textit{ and } \ 0 \ \textit{otherwise.}$$

Theorem 3. *The number $s(n,k)$ of paths of width n ending at ordinate k is given by*

$$s(n,k) = \frac{k+2}{n+2}\binom{n+2}{\frac{n-k}{2}} \quad \textit{if } n+k \equiv 0 \mod 2 \ \textit{and}$$

$$s(n,k) = \frac{k+1}{n+2}\binom{n+2}{\frac{n-k+1}{2}} \quad \textit{otherwise.}$$

From Theorem 2 and Theorem 3, we can easily deduce a closed form for the number $h(n,k)$ of paths of width n ending at ordinate k with a horizontal step. As a byproduct of Theorem 3, if we set $k = 0$ and $n = 2m$ is even, then $s(2m,0) = \frac{1}{m+1}\binom{2m+2}{m} = \frac{1}{m+2}\binom{2m+2}{m+1}$, which corresponds to the $(m+1)$-th Catalan number (see A000108). Figure 3 shows the 14 paths of width 6 ending on the x-axis (i.e., ending at ordinate $k = 0$).

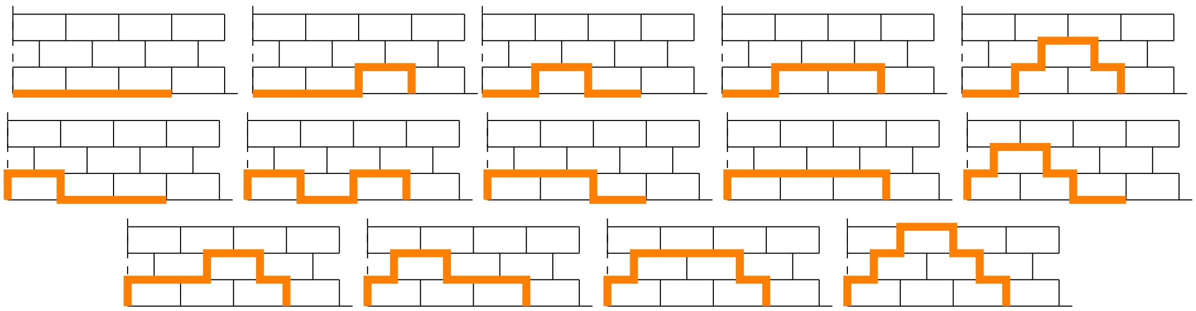

FIGURE 3. The 14 paths of width 6 ending on the x-axis (Catalan number).

Corollary 2. *The generating function for the number of paths of a given width is*

$$S(z,1) = \frac{2r}{z^2(s-1)} = \frac{1 - z - 2z^2 - (1-z)\sqrt{1-4z^2}}{z^3(-1+2z)},$$

and the n-th coefficient in the series expansion is given by

$$2 \cdot \binom{n+1}{\lfloor n/2 \rfloor},$$

which corresponds to twice the coefficients of A037952 in [27].

The first terms of the series expansion are

$$2 + 2z + 6z^2 + 8z^3 + 20z^4 + 30z^5 + 70z^6 + 112z^7 + 252z^8 + 420z^9 + \cdots$$

By calculating $\partial_u(S(z,u))|_{u=1}$, and using classical methods [11, 23] for an asymptotic approximation of the coefficient of z^n, we obtain the following.

Corollary 3. *An asymptotic for the expected ordinate of the last point in all paths of a given width is given by*

$$\sqrt{\frac{\pi n}{2}} \approx 1.253314137\sqrt{n}.$$

We end this section by exhibiting a constructive bijection ϕ between paths of width $2n$ ending on the x-axis in $\mathcal{P}$ and Dyck paths with $n+1$ up steps. Recall that $E_1 = (1,0)$ and $E_2 = (2,0)$. Let $\mathcal{P}^1$ be the set of paths in the wall starting at $(1,1)$, ending at ordinate 1, and never going to the x-axis. If Q is a path in $\mathcal{P}^1$, then we define the path $\bar{Q}$ in $\mathcal{P}$ obtained from Q after a translation by the vector $(-1,-1)$ (note that some occurrences of E_1E_1 in Q can be transformed by the translation into a step E_2). For instance, if $Q = E_1E_1NE_1E_1S$ then $\bar{Q} = E_2NE_1E_1S$.

Proposition 1. *Let us consider the map ϕ recursively defined from $\mathcal{P}$ to the set $\mathcal{D}$ of Dyck paths as follows. For $P \in \mathcal{P}$, we set:*

$$\phi(P) = \begin{cases} UD & \text{if } P = \epsilon, \\ UD\phi(Q) & \text{if } P = E_2\ Q \text{ with } Q \in \mathcal{P}, \\ U\phi(\bar{Q})D & \text{if } P = NE_1QE_1S \in \mathcal{P},\ \text{and } Q \in \mathcal{P}^1, \\ U\phi(\bar{Q})D\phi(R) & \text{if } P = NE_1QE_1SE_2R \in \mathcal{P},\ \text{with } R \in \mathcal{P} \text{ and } Q \in \mathcal{P}^1. \end{cases}$$

Then, the map ϕ is a bijection.

Due to the recursive definition, the image by ϕ of a path ending on the x-axis of width $2n$ in $\mathcal{P}$ is a Dyck path of semilength $n+1$, and it is easy to see that ϕ is a bijection. For instance, the image of $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1E_1S$ (of width 10) is the Dyck path $U\phi(E_2NE_1E_1S)D\phi(NE_1E_1S) = UUD\phi(NE_1E_1S)D\phi(NE_1E_1S) = UUDUUDDDUUDD$ (of length 12), see Figure 4. We also refer to Figure 5 for an illustration of the last three cases used in the definition of ϕ.

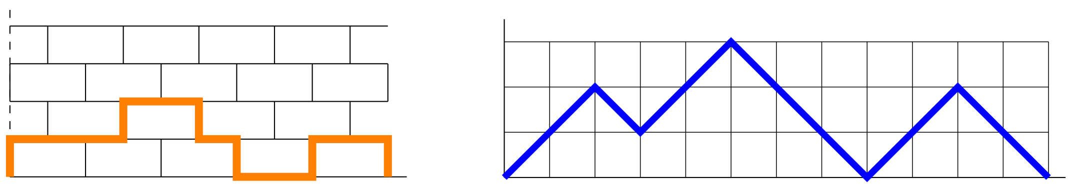

FIGURE 4. The path $NEEENEESESENEES$ of width 10 and its image by ϕ, the Dyck path $UUDUUDDDUUDD$ of length 12,

2.1. **The total area.** The *area* of a path P ending on the x-axis is defined as two times the number of bricks between the path and the x-axis. It is denoted by $\texttt{area}(P)$. For example, the path in Figure 2 has area 10. In this part, we enumerate the paths in $\mathcal{P}$ of a given width, ending on the x-axis, and with a given area.

The following theorem provides a functional equation satisfied by the generating function

$$F(z,q) = \sum_{P \in \mathcal{P}} z^{\texttt{width}(P)} q^{\texttt{area}(P)}.$$

Theorem 4. *The bivariate generating function $F(z,q)$, where the coefficient of z^nq^k is the number of paths of width n ending on the x-axis and area k, satisfies the functional equation*

$$F(z,q) = 1 + z^2F(z,q) + z^2q^2F(zq,q) + z^4q^2F(zq,q)F(z,q). \tag{2}$$

Proof. In the bijection introduced in Proposition 1 we note that any non-empty path can be decomposed as E_2Q, NE_1QE_1S, or $NE_1QE_1SE_2R$, where $Q \in \mathcal{P}^1$ and $R \in \mathcal{P}$, see Figure 5 for a pictorial representation. For the paths P of the form E_2Q, the contribution is $z^2F(z,q)$ since the area of P and Q are the same, and the width of P is equal to that of Q plus two. For the paths P of the form NE_1QE_1S, the contribution is $z^2q^2F(zq,q)$ since the area of P is $2+\texttt{width}(Q)+\texttt{area}(Q)$, and the width of P is $2+\texttt{width}(Q)$. The contribution of the third case is obtained *mutatis mutandis*. Therefore, fom this decomposition follows the functional equation.

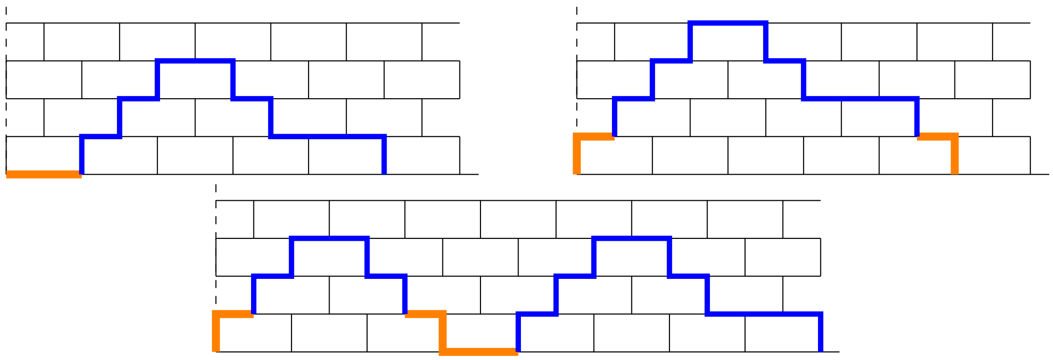

FIGURE 5. Decomposition of a path into E_2Q, or NE_1QE_1S, or $NE_1QE_1SE_2R$.

□

Theorem 5. *An expression for the generating function $F(z,q)$ is given by the continued fraction*

$$F(z,q)=\cfrac{1}{-z^2+\cfrac{1}{1+\cfrac{q^2z^2}{-q^2z^2+\cfrac{1}{1+\cfrac{q^4z^2}{-q^4z^2+\cfrac{1}{\ddots}}}}}}.$$

Proof. From Theorem 4 we have

$$F(z,q)=\frac{1+z^2q^2F(zq,q)}{1-z^2-z^4q^2F(zq,q)}=\cfrac{1}{-z^2+\cfrac{1}{1+z^2q^2F(zq,q)}}.$$

Iterating this expression yields the desired result. □

The first terms of the continued fraction are as follows:

$$1+(1+q^2)z^2+(1+2q^2+q^4+q^6)z^4+(\mathbf{1+3q^2+3q^4+3q^6+2q^8+q^{10}+q^{12}})z^6+ (1+4q^2+6q^4+7q^6+7q^8+5q^{10}+5q^{12}+3q^{14}+2q^{16}+q^{18}+q^{20})z^8+\cdots$$

The paths corresponding to the boldface in the expansion are displayed in Figure 3. Note that by considering $1+q\cdot F(\sqrt{q},\sqrt{q})$, we obtain the generating function for the paths without steps on the x-axis having a given number of bricks below the path, and we retrieve the results of Odlyzko [22] that enumerates fountains with n coins with the continued fraction

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{z^2}{1-\cfrac{z^3}{\ddots}}}}.$$

Corollary 4. *The generating function of the total area in all paths in $\mathcal{P}$ of a given width and ending on the x-axis is*

$$\frac{1-3z^2-(1-z^2)\sqrt{1-4z^2}}{z^4(1-4z^2)}.$$

The n-th coefficient of the series expansion is asymptotically equivalent to

$$(1+(-1)^n)2^{n+1}.$$

Proof. Let $G := \partial_q(F(z,q))|_{q=1}$. Then, by differentiating (2) with respect to q, we obtain

$$G = z^2G + (2z^2 + 2z^4F(z,1) + z^4G)F(z,1) + z^2(1 + z^2F(z,1))H, \tag{3}$$

where $H = \partial_q(F(qz,q))|_{q=1}$. From Theorem 1 we have

$$F(z,1) = \frac{1-2z^2-\sqrt{1-4z^2}}{2z^4} = \sum_{n\geq 0} C_{n+1}z^{2n},$$

where C_n is the n-th Catalan number. From the definition of $F(z,q)$ we have

$$H = \partial_q(F(qz,q))|_{q=1} = \sum_{n\geq 0} 2nC_{n+1}z^{2n} + G.$$

Since we have

$$\sum_{n\geq 0} 2nC_{n+1}z^{2n} = 2\frac{1-3z^2-(1-z^2)\sqrt{1-4z^2}}{z^4\sqrt{1-4z^2}},$$

we can substitute this expression in (3). Solving for G, we obtain the desired result. □

The first terms of the series expansion are

$$2z^2 + 14z^4 + 74z^6 + 352z^8 + 1588z^{10} + 6946z^{12} + 29786z^{14} + 126008z^{16} + \cdots,$$

which corresponds to A138156 in [27], that also counts the sum of the path lengths of all incomplete binary trees with n edges. It would be interesting to investigate the link between the area in these paths and the path length in these trees.

3. Paths with a given number of steps

In this part, we count paths in $\mathcal{P}$ with a given number of steps (recall that a step is a move $N = (0,1)$, $S = (0,-1)$, $E \in \{(1,0),(2,0)\}$ connecting two adjacent corners in a wall. With the similar arguments used in the previous section, we easily obtain the following recurrence relations.

$$\left\{\begin{array}{ll} f_0^0 = 1, \text{ and } f_k^0 = 0, & k \geq 1, \\ f_0^1 = 0, f_1^1 = z + zh_0^0, \text{ and } f_k^1 = zh_{k-1}^0, & k \geq 2, \end{array}\right. , \left\{\begin{array}{ll} g_k^0 = zh_{k+1}^1, & k \geq 0, \\ g_k^1 = 0, & k \geq 0, \end{array}\right. \text{ and}$$

$$\left\{\begin{array}{ll} h_0^0 = z + z(h_0^0 + g_0^0), \text{ and } h_k^0 = z(h_k^1 + f_k^1), & k \geq 1, \\ h_0^1 = 0, \text{ and } h_k^1 = z(h_k^0 + g_k^0), & k \geq 1. \end{array}\right. \tag{4}$$

Summing the recursions in (4), we have:

$$\left\{\begin{array}{ll} F^0(u) & = 1, \\ F^1(u) & = zu + z\sum_{k\geq 1} u^k h_{k-1}^0 = zu + zuH^0(u), \end{array}\right. \quad \left\{\begin{array}{ll} G^0(u) & = z\sum_{k\geq 0} u^k h_{k+1}^1 = \frac{z}{u}H^1(u), \\ G^1(u) & = 0, \end{array}\right.$$

$$\left\{\begin{array}{ll} H^0(u) & = h_0^0 + z\sum_{k\geq 1} u^k (f_k^1 + h_k^1) \\ & = h_0^0 + z(F^1(u) + H^1(u)), \\ H^1(u) & = z\sum_{k\geq 0} u^k (g_k^0 + h_k^0) - z(h_0^0 + g_0^0) \\ & = z(H^0(u) + G^0(u)) - h_0^0 + z. \end{array}\right.$$

Solving these functional equations, we deduce

$$F^0(u) = 1 \text{ and } F^1(u) = \frac{uz\left(h_0^0\, uz + h_0^0\, z^2 - h_0^0\, u + z^2 - u\right)}{-uz^4 + u^2z^2 + uz^2 + z^2 - u},$$

$$G^1(u) = 0 \text{ and } G^0(u) = -\frac{z\left(h_0^0\, uz^2 + h_0^0\, z - h_0^0 + z\right)}{-uz^4 + u^2z^2 + uz^2 + z^2 - u},$$

$$H^0(u) = \frac{uz^4 - (u^2 - h_0^0 + u)z^2 + h_0^0 uz - h_0^0 u}{-uz^4 + (u^2 + u + 1)z^2 - u} \text{ and } H^1(u) = -\frac{u\left(h_0^0\, uz^2 + h_0^0\, z - h_0^0 + z\right)}{-uz^4 + u^2z^2 + uz^2 + z^2 - u}.$$

In order to compute h_0^0, we use the kernel method on $F^1(u)$ (see Section 2 for more details). We factorize the denominator $-uz^4 + u^2z^2 + uz^2 + z^2 - u = z^2(u-r)(u-s)$ with

$$r = \frac{1 - z^2 + z^4 - \sqrt{\left(z^2+z+1\right)\left(z^2-z+1\right)\left(z^2+z-1\right)\left(z^2-z-1\right)}}{2z^2} \quad \text{and}$$

$$s = \frac{1 - z^2 + z^4 + \sqrt{\left(z^2+z+1\right)\left(z^2-z+1\right)\left(z^2+z-1\right)\left(z^2-z-1\right)}}{2z^2}.$$

Cancelling the numerator of $F^1(u)$ by substituting u by r, we obtain

$$h_0^0 = \frac{z}{1 - z - rz^2}.$$

The first terms of the series expansion of h_0^0 are

$$z + z^2 + z^3 + z^4 + 2\,z^5 + 3\,z^6 + 5\,z^7 + 7\,z^8 + 11\,z^9 + \cdots,$$

and this sequence does not appear in [27]. However, if we set $h(n) := [z^n]h_0^0$ then the sequence of odd powers corresponds to A051286, and we have

$$h(2n+1) = \sum_{k=0}^{n} \binom{n-k}{k}^2.$$

Moreover, the sequence of even powers corresponds to A203611, and we have

$$h(2n) = \sum_{k=0}^{n} \binom{k-1}{2k-1-n}\binom{k}{2k-n}.$$

Corollary 5. *The generating function for the number of paths of a given number of steps in $\mathcal{P}$ and ending on the x-axis is*

$$\frac{h_0^0}{z} = \frac{2}{1 - 2z + z^2 - z^4 + \sqrt{1 - 2z^2 - z^4 - 2z^6 + z^8}},$$

and the n-th coefficient of the series expansion is given by $h(n+1)$.

Now, substituting h_0^0 by its value in the previous generating functions, and simplifying by $(u-r)$ in the numerators and denominators, we obtain the following:

Theorem 6. *We have*

$$F^0(u) = 1, \quad F^1(u) = \frac{u(1 + h_0^0 - h_0^0 z)}{z(s-u)}, \quad G^0(u) = \frac{z h_0^0}{s-u}, \quad G^1(u) = 0, \text{ and}$$

$$H^0(u) = \frac{ru + h_0^0}{r(s-u)}, \quad H^1(u) = \frac{u h_0^0}{s-u},$$

where

$$h_0^0 = \frac{z}{1 - z - rz^2}.$$

Finally, the bivariate generating function $S(z,u)$, where the coefficient of $z^n u^k$ is the number of paths with n steps ending at ordinate k, satisfies

$$S(z,u) = 1 + \frac{ru + zru + h_0^0(ru + z^2 r + z)}{zr(s-u)}.$$

The first terms of the series expansion of $S(z,u)$ are

$$1 + (u+1)z + (1+2u)z^2 + (u^2+3u+1)z^3 + (2u^2+4u+2)z^4 + (u^3+4u^2+5u+3)z^5 + \\ + (2u^3+6u^2+8u+5)z^6 + (u^4+5u^3+9u^2+12u+7)z^7 + \\ + (2u^4+8u^3+14u^2+20u+11)z^8 + \cdots$$

Theorem 7. *We have*

$$[u^k]F^0(u) = [k=0],\ k \geq 0, \quad [u^k]F^1(u) = \frac{1+h_0^0(1-z)}{z} \cdot r^k,\ k \geq 1,$$
$$[u^k]G^0(u) = zh_0^0 r^{k+1},\ k \geq 0,$$
$$[u^k]H^0(u) = (1+h_0^0)r^k - [k=0],\ k \geq 0, \quad [u^k]H^1(u) = h_0^0 r^k,\ k \geq 1,$$
$$S(z,0) = h_0^0/z, \quad [u^k]S(z,u) = \frac{1+2h_0^0}{z} \cdot r^k,\ k \geq 1,\ \textit{and}$$

all other coefficients are equal to zero.

In order to provide a closed form for the coefficient of z^n of all these previous quantities, we need to provide a closed form of $r(n,k) := [z^n]r^k$. We set $r' = r(\sqrt{z})$. Then r' is the generating function for the generalized Catalan numbers, and using the comment of Barry in [27] (see A004148) giving the general term of r'^k, we can easily obtain:

$$r(2k,k) := [z]^{2k} r^k = 1, \quad k \geq 0$$
$$r(2k+2\ell,k) := [z]^{2k+2\ell} r^k = k \sum_{i=\lceil \frac{\ell+1}{2} \rceil}^{\ell} \frac{1}{i} \binom{i}{\ell-i} \binom{i+k-1}{\ell+k-i}, \quad k \geq 1, \ell \geq 1,$$
$$r(2n+1,k) := [z^{2n+1}] r^k = 0, \quad \text{otherwise.}$$

Corollary 6. *The number $s(n,k)$ of paths with n steps in $\mathcal{P}$ ending at ordinate k is given by*

$$s(n,k) = r(n+1,k) + 2\sum_{i \geq 0}^{n+1} h(i) r(n+1-i,k),$$

where $r(n,k)$ and $h(n)$ are defined previously.

Corollary 7. *The generating function for the number of paths of a given number of steps in $\mathcal{P}$ is*

$$S(z,1) = \frac{1+z}{1-z-z^2},$$

and the n-th coefficient $s(n)$ in the series expansion is given by a shift of the Fibonacci sequence A000045 in [27], defined by $s(n) = s(n-1) + s(n-2)$ anchored with $s(0) = 1$ and $s(1) = 2$.

The first terms of the series expansion are

$$1 + 2z + 3z^2 + 5z^3 + 8z^4 + 13z^5 + 21z^6 + 34z^7 + 55z^8 + 89z^9 + \cdots$$

Figure 6 shows the eight paths with four steps. Note that there is a simple bijection ψ between the set of paths in $\mathcal{P}$ with n steps and the set of binary words of length n that do not contain two adjacent ones: reading the path from left to right, we replace each vertical step by 1, and each horizontal step by 0. For instance, the image of the path $ENENEESEEN$ is 0101001001.

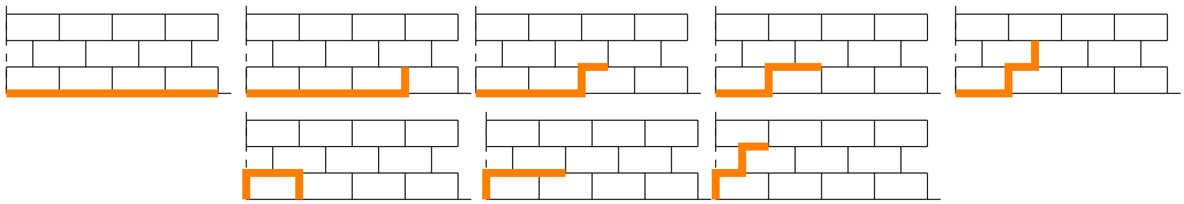

FIGURE 6. The eight paths with four steps (Fibonacci).

By calculating $\partial_u(S(z,u))|_{u=1}$ and using classical methods [11, 23] for an asymptotic approximation of the coefficient of z^n, we obtain the following.

Corollary 8. *The expected ordinate of the last point in all paths in $\mathcal{P}$ with n steps is asymptotically equivalent to*

$$\frac{\sqrt{-15+7\sqrt{5}}\left(5+3\sqrt{5}\right)\sqrt{n}}{10\sqrt{\pi}} \approx 0.5335775634\sqrt{n}.$$

4. Paths with a given length

In this part, we count paths in $\mathcal{P}$ with a given length. We easily obtain the following recurrence relations.

$$\begin{cases} f_0^0 = 1 \text{ and } f_k^0 = 0, & k \geq 1, \\ f_0^1 = 0, f_1^1 = z + zh_0^0, \text{ and } f_k^1 = zh_{k-1}^0, & k \geq 2, \end{cases}, \quad \begin{cases} g_k^0 = zh_{k+1}^1, & k \geq 0, \\ g_k^1 = 0, & k \geq 0, \end{cases} \text{ and}$$

$$\begin{cases} h_0^0 = z^2 + z^2(h_0^0 + g_0^0) \text{ and } h_k^0 = z(h_k^1 + f_k^1), & k \geq 1, \\ h_0^1 = 0 \text{ and } h_k^1 = z(h_k^0 + g_k^0), & k \geq 1. \end{cases} \tag{5}$$

Summing the recursions in (5), we have:

$$\begin{cases} F^0(u) &= 1, \\ F^1(u) &= zu + z\sum\limits_{k\geq 1} u^k h_{k-1}^0 = zu + zuH^0(u), \end{cases} \quad \begin{cases} G^0(u) &= z\sum\limits_{k\geq 0} u^k h_{k+1}^1 = \frac{z}{u}H^1(u), \\ G^1(u) &= 0, \end{cases}$$

$$\begin{cases} H^0(u) &= h_0^0 + z\sum\limits_{k\geq 1} u^k(f_k^1 + h_k^1) \\ &= h_0^0 + z(F^1(u) + H^1(u)), \\ H^1(u) &= z\sum\limits_{k\geq 0} u^k(g_k^0 + h_k^0) - z(h_0^0 + g_0^0) \\ &= z(H^0(u) + G^0(u)) - \frac{h_0^0 - z^2}{z}. \end{cases}$$

Solving the above functional equations, we deduce

$$F^0(u) = 1 \text{ and } F^1(u) = \frac{uz\left(h_0^0\, z^2 + z^2 - u\right)}{-uz^4 + u^2z^2 + uz^2 + z^2 - u},$$

$$G^1(u) = 0 \text{ and } G^0(u) = -\frac{h_0^0\, uz^2 + h_0^0\, z^2 + z^2 - h_0^0}{-uz^4 + u^2z^2 + uz^2 + z^2 - u},$$

$$H^0(u) = \frac{z^2\left(uz^2 - u^2 + h_0^0 - u\right)}{-uz^4 + u^2z^2 + uz^2 + z^2 - u} \text{ and } H^1(u) = -\frac{u\left(h_0^0\, uz^2 + h_0^0\, z^2 + z^2 - h_0^0\right)}{\left(-uz^4 + u^2z^2 + uz^2 + z^2 - u\right) z}.$$

In order to compute h_0^0, we use the kernel method on $F^1(u)$ (see Section 2 for more details). We factorize the denominator $-uz^4+u^2z^2+uz^2+z^2-u = z^2(u-r)(u-s)$ where r and s are the same as those defined in Section 3.

Cancelling the numerator of $F^1(u)$ by substituting u with r, we obtain

$$h_0^0 = \frac{r - z^2}{z^2}.$$

Now, substituting h_0^0 by its value in the previous generating functions, and simplifying by $(u-r)$ in the numerators and denominators, we obtain the following:

Theorem 8. *We have*

$$F^0(u) = 1, \quad F^1(u) = \frac{u}{z(s-u)}, \quad G^0(u) = \frac{r - z^2}{z^2(s-u)}, \quad G^1(u) = 0, \text{ and}$$

$$H^0(u) = \frac{1}{z^2(s-u)} - 1, \quad H^1(u) = \frac{u(r - z^2)}{z^3(s-u)}.$$

Finally, the bivariate generating function $S(z,u)$, where the coefficient of $z^n u^k$ is the number of paths of length n ending at ordinate k, satisfies

$$S(z,u) = \frac{rz + z + ru - z^3}{z^3(s-u)}.$$

The first terms of the series expansion of $S(z,u)$ are

$$1 + uz + (u+1)z^2 + (u^2+2u)z^3 + (u^2+2u+2)z^4 + (u^3+3u^2+3u)z^5 + \\ + (u^3+3u^2+4u+4)z^6 + (u^4+4u^3+6u^2+6u)z^7 + \\ + (u^4+4u^3+7u^2+9u+8)z^8 + \cdots$$

Theorem 9. *We have*

$$[u^k]F^0(u) = [k=0],\ k \geq 0, \quad [u^k]F^1(u) = \frac{r^k}{z},\ k \geq 1,$$

$$[u^k]G^0(u) = r^{k+2}/z^2 - r^{k+1},\ k \geq 0,$$

$$[u^k]H^0(u) = r^{k+1}/z^2 - [k=0],\ k \geq 0, \quad [u^k]H^1(u) = r^{k+1}/z^3 - r^k/z,\ k \geq 1,$$

$$S(z,0) = \frac{r - z^2}{z^4}, \qquad [u^k]S(z,u) = \frac{r^{k+1}}{z^2}\left(1 + r + \frac{1}{z} - z^2\right),\ k \geq 1, \text{ and}$$

all other coefficients are equal to zero.

Using the closed form of $[z^n]r^k$ obtained in the previous section, we deduce the following.

Corollary 9. *The number $s(n,k)$ of paths of length n in $\mathcal{P}$ ending at ordinate k is given by*

$$s(2,0)=1, s(n,0)=r(n+4,1),\ n\geq 4,\ \textit{and}$$
$$s(n,k)=r(n+2,k+1)+r(n+3,k+1)-r(n,k+1)+r(n+2,k+2),$$

where $r(n,k):=[z^n]r^k$ is given before Corollary 6

Corollary 10. *The generating function for the number of paths of a given length in $\mathcal{P}$ is*

$$S(z,1)=\frac{z^3-rz-z-r}{z^3(1-s)}=\frac{-1+2z^2+2z^5+z^6+(1-z^2)\sqrt{1-2z^2-z^4-2z^6+z^8}}{2z^5(1-z-z^2)}.$$

The sequences of coefficients of z^{2n+1} corresponds to A003440 and the sequences of coefficients of z^{2n} does appear in [27]

Corollary 11. *The generating function for the number of paths of a given length in $\mathcal{P}$ ending on the x-axis is $\frac{r-z^2}{z^4}$ and the n-th coefficient of the series expansion is given by $r(2n,0)$, which corresponds to the n-th generalized Catalan number (see A004148).*

Corollary 12. *An asymptotic for the expected ordinate of the last point in all paths in $\mathcal{P}$ of a given length is*

$$\frac{\left(3+\sqrt{5}\right)\sqrt{-15+7\sqrt{5}}}{2\sqrt{\pi n}}\approx 1.193115703\cdot\frac{1}{\sqrt{n}}.$$

Figure 7 shows the eight paths of length 8 ending on the x-axis.

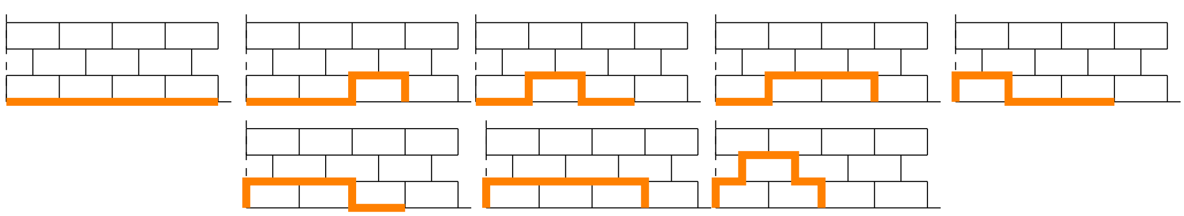

Figure 7. The eight paths of length 8 ending on the x-axis (generalized Catalan)

We end the section by exhibiting a constructive bijection ψ between paths of length $2n$ ending on the x-axis in $\mathcal{P}$ and peakless Motzkin paths with $n+1$ steps (i.e., Motzkin paths with no occurrence of UD). Recall that $E_1=(1,0)$ and $E_2=(2,0)$, and that $\mathcal{P}^1$ is the set of paths in the wall starting at $(1,1)$, ending at ordinate 1, and never going to the x-axis. If $Q\in\mathcal{P}^1$, then we recall that $\bar{Q}$ is the path in $\mathcal{P}$ obtained from Q after the translation by the vector $(-1,-1)$. These definitions are already given in Section 2 just before Proposition 1.

Proposition 2. *Let us consider the map ψ recursively defined from $\mathcal{P}$ to the set $\mathcal{PM}$ of peakless Motzkin paths as follows. For $P \in \mathcal{P}$, we set:*

$$\psi(P) = \begin{cases} H & \text{if } P = \epsilon, \\ H\psi(Q) & \text{if } P = E_2\ Q \text{ with } Q \in \mathcal{P}, \\ U\psi(\bar{Q})D & \text{if } P = NE_1QE_1S \in \mathcal{P}, \text{ and } Q \in \mathcal{P}^1, \\ U\psi(\bar{Q})D\psi(R) & \text{if } P = NE_1QE_1SE_2R \in \mathcal{P}, \text{ with } R \in \mathcal{P} \text{ and } Q \in \mathcal{P}^1. \end{cases}$$

Due to the recursive definition, the image by ψ of a path ending on the x-axis of length $2n$ in $\mathcal{P}$ is a peakless Motzkin path with $n+1$ steps, and it is easy to see that ψ is a bijection. For instance, the image of $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1E_1S$ (of length 16) is the peakless Motzkin path $UHUHDDUHD$ (of length 9), see Figure 8. Note that the definition of ψ is basically the same as in Proposition 1 since we use the same decomposition of a path (the only change is that we use horizontal steps instead of peaks UD).

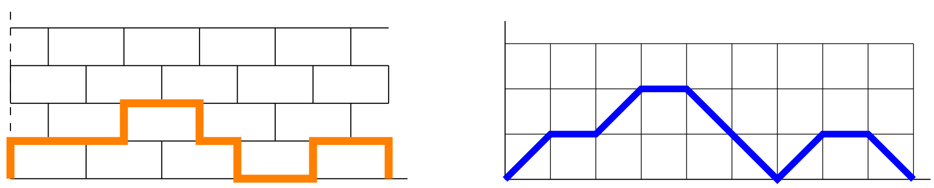

FIGURE 8. The path $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1E_1S$ of length 16 and its image by ψ, the peakless Motzkin path $UHUHDDUHD$ of length 9.

5. Conclusions

In this work, we enumerate lattice paths in the wall with respect to their width, number of steps, length, and area. Can we identify the underlying limit law (see [11])? The main tools are multivariate generating functions and the kernel method. We can extend our results by considering additional statistics, for example the number of *turns* of the path, i.e., the number of occurrences of the subpaths EN, ES, NE, and SE. For example, it is possible to give an expression for the multivariate generating function $S(z,u,w)$, where the coefficient of $z^n u^k w^\ell$ is the number of paths with n steps, ℓ turns, and ending at ordinate k. The expression is too large, however, the first few terms of the Taylor expansion are

$$\begin{aligned} S(z,u,w) = 1 + (1+u)z + (1+2uw)z^2 + (1+2uw+uw^2+u^2w^2)z^3 \\ + (1+2uw+w^2+2uw^2+2u^2w^3)z^4 \\ + (1+2uw+3uw^2+u^2w^2+2w^3+2u^2w^3+u^2w^4+u^3w^4)z^5 + \cdots \end{aligned}$$

Moreover, the bivariate generating function $S(z,1,w)$, where the coefficient of $z^n w^\ell$ is the number of paths with n steps and ℓ turns is given by

$$S(z,1,w) = \frac{(w^2-2w+1)z^2 - z - 1}{w^2z^2 + z - 1}.$$

Therefore, the generating function for the total number of turns in all paths of a given width is

$$\frac{2z^2(1+z)}{(1-z-z^2)^2} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 60z^6 + 116z^7 + 218z^8 + 402z^9 + \cdots,$$

and the n-th coefficient corresponds to twice the convolution of the Fibonacci sequence (see A023610 in [27]), that is,

$$2 \cdot \sum_{i=2}^{n+1} F_i F_{n+1-i}.$$

Can we obtain enumerative results for paths with other boundary conditions? Can we generalize our study if we drop the non-overlapping constraint and allow the step $W = (-1, 0)$? Finally, the paths studied in the last section are in one-to-one correspondence with paths of a given length in the honeycomb lattice, which suggests to explore in more detail the link between the paths in the wall and the honeycomb lattice.

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