Equivalence classes of permutations modulo descents and left-to-right maxima

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Abstract

In a recent paper [2], the authors provide enumerating results for equivalence classes of permutations modulo excedances. In this paper we investigate two other equivalence relations based on descents and left-to-right maxima. Enumerating results are presented for permutations, involutions, derangements, cycles and permutations avoiding one pattern of length three.

Keywords: permutation; equivalence class; descent; left-to-right maximum; pattern; Bell, Motzkin, Catalan, Fine.

1 Introduction and notations

Let S_n be the set of permutations of length n, *i.e.*, all one-to-one correspondences from $[n] = \{1, 2, \ldots, n\}$ into itself. The *one-line notation* of a permutation $\pi \in S_n$ is $\pi_1\pi_2 \cdots \pi_n$ where $\pi_i = \pi(i)$ for $i \in [n]$. The *graphical representation* of $\pi \in S_n$ is the set of points in the plane at coordinates (i, π_i) for $i \in [n]$. A *cycle* in S_n is an n-length permutation π such that there exist some indices $i_1, i_2, \ldots, i_n$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_{n-1}) = i_n$ and $\pi(i_n) = i_1$. A cycle will also be denoted by its *cyclic notation* $\pi = \langle i_1, i_2, \ldots, i_n\rangle$. Let $C_n \subset S_n$ be the set of all cycles of length n. We denote by I_n the set of involutions of length n, *i.e.*, permutations π such that $\pi^2 = Id$ where Id is the identity permutation.

Let π be a permutation in S_n. A *fixed point* of π is a position $i \in [n]$ where $\pi(i) = i$. The set of n-length permutations with no fixed points (called *derangements*) will be denoted D_n. An *excedance* of π is a position $i \in [n-1]$, such that $\pi(i) > i$. The set of excedances of π will be denoted $E(\pi)$. A *descent* of π is a position $i \in [n-1]$, such that $\pi(i) > \pi(i+1)$. Let $D(\pi)$ be the set of descents in π, and $DD(\pi)$ be the set of pairs $(\pi(i), \pi(i+1))$ for $i \in D(\pi)$. By abuse of language, we also use the term *descent* for such a pair. A *left-to-right maximum* is a position $i \in [n]$, such that $\pi(i) > \pi(j)$ for all $j < i$. The set of left-to-right maxima of π will be denoted $L(\pi)$. For instance, if $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ then $E(\pi) = \{2, 4, 7\}$, $D(\pi) = \{2, 4, 5, 7\}$, $DD(\pi) = \{(4, 2), (7, 5), (5, 3), (8, 6)\}$ and $L(\pi) = \{1, 2, 4, 7\}$.

In [2], the authors consider the equivalence relation on S_n in which two permutations π and σ are equivalent if they coincide on their excedance sets, *i.e.*, $E(\pi) = E(\sigma)$ and $\pi(i) = \sigma(i)$ for $i \in E(\pi)$. In this paper we investigate the counterpart of this equivalence relation for descents and left-to-right maxima. More precisely, we define the ℓ-equivalence relation $\sim_\ell$ where $\pi \sim_\ell \sigma$ if and only if π and σ coincide on their left-to-right maximum sets, *i.e.*, $L(\pi) = L(\sigma)$ and $\pi(i) = \sigma(i)$ for $i \in L(\pi)$. Also, we define the d-equivalence relation $\sim_d$ where two permutations π and σ are equivalent if $DD(\pi) = DD(\sigma)$. The motivation for studying this d-equivalence relation is that two permutations π and σ are equivalent under excedance ([2]) if and only if $\phi(\pi)$ and $\phi(\sigma)$ are d-equivalent, where ϕ is the Foata's first transformation [5] (see Theorem 6). All these definitions remain available for subsets of S_n. For instance, the permutation $32541 \in S_5$ is ℓ-equivalent to $32514, 31524, 31542$ and d-equivalent to 54132. The set of ℓ-equivalence (resp. d-equivalence) classes in S_n is denoted $S_n^{\sim_\ell}$ (resp. $S_n^{\sim_d}$).

In this paper we propose to compute the number of ℓ- and d-equivalence classes for several subsets of permutations.

A permutation $\pi \in S_n$ *avoids* the pattern $\tau \in S_k$ if and only if there does not exist any sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ is order-isomorphic to τ (see [13, 14]). We denote by $S_n(\tau)$ the set of permutations of S_n avoiding the pattern τ. For example, if $\tau = 123$ then $52143 \in S_5(\tau)$ while $21534 \notin S_5(\tau)$. Many classical sequences in combinatorics appear as the cardinality of pattern-avoiding permutation classes. A large number of these results were firstly obtained by West and Knuth [8, 12, 13, 14, 15, 16] (see books of Kitaev [7] and Mansour [11]).

In Section 2, we investigate the equivalence relation based on the set of left-to-right maxima. We enumerate ℓ-equivalence classes for S_n, C_n, I_n, D_n and several sets of pattern avoiding permutations. In Section 3, we study equivalence relation for descents and also provide enumerating results for some restricted sets of permutations. See Table 1, 2 and 3 for an overview of these results.

2 Enumeration of classes under ℓ-equivalence relation

Throughout this section two permutations π and σ belong to a same class whenever they coincide on their sets of left-to-right maxima, *i.e.*, $L(\pi) = L(\sigma)$ and $\pi(i) = \sigma(i)$ for $i \in L(\pi)$.

A *Dyck path* of semilength n, $n \geq 0$, will be a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. Let $\mathcal{P}_n$ be the set of all Dyck paths of semilength n. A *peak* of *height* $h \geq 0$ in a Dyck path is a point of ordinate h which is both at the end of an up step and at the beginning of a down step.

From a permutation $\pi \in S_n$, we consider the path on the graphical representation of π with up and right steps along the edges of the squares that goes from the lower-left corner to the upper-right corner and leaving all the points (i, π_i), $i \in [n]$, to the right and remaining always as close to the diagonal $y = x$ as possible (the path can possibly reach the diagonal but never crosses it). Let us define the Dyck path of length $2n$ (called *Dyck path associated with* π) obtained from this lattice path by reading an up-step U every time the path moves up, and a down-step D every time the path moves to the right. It is crucial to notice that only the points (i, π_i) with $i \in L(\pi)$ involve in this construction. See Figure 1 for an illustration of this classical construction.

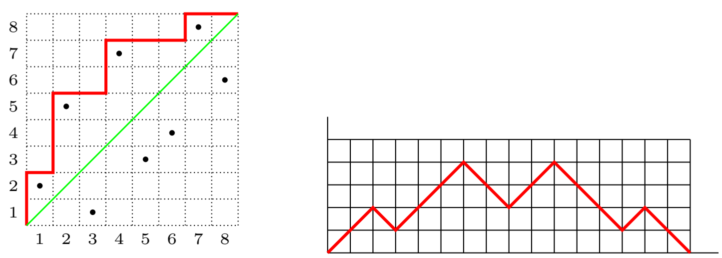

Figure 1: Permutation $\sigma = 2\,5\,1\,7\,3\,4\,8\,6$

Using this construction, all permutations of a same class provide the same Dyck path. Moreover, any Dyck path in $\mathcal{P}_n$ can be obtained from a permutation in S_n. Indeed, we define the sequence $\ell = \ell_1\ell_2 \ldots \ell_r$, (resp. $k = k_1k_2 \ldots k_r$), $r \geq 1$ where ℓ_i (resp. k_i) is the number (resp. the number plus one) of up steps U (resp. down steps D) before the i-th peak. Since P is a Dyck path, we have $k_i \leq \ell_i$ for $i \leq r$. So, we define the permutation $\pi = \ell_1 A_1 \ell_2 A_2 \ldots \ell_r A_r$ where each ℓ_i, $i \leq r$, is at position k_i, and such that the concatenated block $A_1 A_2 \ldots A_r$ consists of the increasing sequence of values in $[n]\backslash\{\ell_1, \ell_2, \ldots, \ell_r\}$. Therefore, ℓ_{i+1} is greater than all elements in A_i which means that the set of left-to-right maxima of π is $L(\pi) = \{k_1, k_2, \ldots, k_r\}$ with $\pi(k_i) = \ell_i$ for $i \leq r$. By construction, π avoids the pattern 321 and P is its associated Dyck path which gives a bijection from $S_n^{\sim_\ell}$ to $\mathcal{P}_n$ that induces Theorem 1. In the following, the permutation π will be called the *associated permutation* of P. Notice that a similar construction already exists in the literature (see the Krattenthaler bijection Ψ defined in [9], Section 4).

Theorem 1 *The sets $S_n^{\sim_\ell}$ (resp. $S_n(321)^{\sim_\ell}$), $n \geq 1$, are enumerated by the Catalan numbers (sequence A000108 in the on-line Encyclopedia of Integer Sequences [18]).*

2.1 Equivalence classes for classical subsets of permutations

In this part we give several enumerating results for classical subsets of S_n (see Table 1).

Theorem 2 *The sets $D_n^{\sim_\ell}$, $n \geq 1$, are enumerated by the Fine numbers (A000108 in [18]).*

Proof. Using the above construction, we construct a Dyck path P of length $2n$ from $\pi \in D_n$. Since π does not contain any fixed point, P does not contain any peak of height one. Conversely, let P be a Dyck path with no peak of height one and $\pi \in S_n(321)$ be its associated permutation of P (see the above construction). Since P does not contain any peak of height one, this implies that there does not exist $i \in L(\pi)$ such that $\pi_i = i$. Now, for a contradiction, let us assume that there is $j \notin L(\pi)$ such that $\pi_j = j$. Since $j \notin L(\pi)$, there is $i < j$ such that $\pi_i > j = \pi_j$. So, there are at most $j-2$ values $\pi_k < \pi_j$ for $k < j$, or equivalently there is at least one value $\pi_k < \pi_j$ for $k > j$ which contradicts the fact that π avoids the pattern 321. Finally, the result follows because the set of Dyck paths with no peak of height one is enumerated by the Fine numbers (see [4]). $\square$

Theorem 3 *Let Irr_n be the set of permutations $\pi \in S_n$ such that $\pi_i \neq \pi_{i+1} - 1$, $1 \leq i \leq n-1$. The sets $Irr_n^{\sim_\ell}$, $n \geq 1$, are enumerated by the sequence A078481 in [18].*

Proof. Let π be a permutation in Irr_n. Then the Dyck path associated with π does not contain any consecutive steps of the form $UDUD$. Conversely, let P be a Dyck path which does not contain any occurence of consecutive $UDUD$, and $\pi \in S_n(321)$ be its associated permutation. Since P does not contain any occurrence of $UDUD$, this implies that there does not exist $i \in L(\pi)$ such that $\pi_i = \pi_{i+1} - 1$. In the case where there is j, $j \notin L(\pi)$ such that $\pi_j = \pi_{j+1} - 1$, we define the blocks of maximal length $J_1, J_2, \ldots, J_s$ of the form $a, a+1, \ldots, b$ such that $a \leq b$ and $\pi_a = \pi_{a+1} - 1$, $\pi_{a+1} = \pi_{a+2} - 1$, $\ldots$, $\pi_{b-1} = \pi_b - 1$ where $a \notin L(\pi)$. We consider the permutation σ obtained from π by the following process: for each block $J_k = a, a+1, \ldots, b$, $1 \leq k \leq s$, in the one-line notation of π we replace the block $\pi(J_k)$ with its mirror $\pi(J_k')$ where $J_k' = b, b-1, \ldots, a$. So, σ and π belong to the same class, and $\sigma \in Irr_n$. For instance, the Dyck path $UUUUDDDD$ would produce $\pi = 4123 \notin Irr_4$, then applying the described process, the permutation $\sigma = 4321 \in Irr_4$ is obtained. Finally, the result is obtained since the set of Dyck paths with no occurrence of $UDUD$ is enumerated by the sequence A078481 in [18] (see [17]). □

Theorem 4 *The sets $I_n^{\sim_\ell}$, $n \geq 1$, are enumerated by the Motzkin numbers (A001006 in [18]).*

Proof. We will show that each equivalence class contains a unique involution that avoids the pattern 4321 (see A001006 in [18] and [6] for the enumeration of $I_n(4321)$ by Motzkin numbers). Let π be an involution in I_n. If there exists a position i, $i < \pi_i$, such that i is not a left-to-right maximum, then there is $j \in [n]$ such that $j < i < \pi_i < \pi_j$ which means that π contains the pattern 4321. So, we define the involution σ satisfying $L(\pi) = L(\sigma)$ and verifying the additional conditions $\sigma_i = i$ whenever $i \notin L(\pi)$. By construction, σ avoids the pattern 4321 and belongs to the same class of π. Conversely, let σ be an involution avoiding the pattern 4321. Then, the inequality $j < i < \sigma_i < \sigma_j$, $i, j \in [n]$ does not occur. Therefore, if $i \notin L(\sigma)$, then i is necessarily a fixed point. Therefore, there is a unique involution $\sigma \in I_n(4321)$ having $L(\sigma)$ as set of left-to-right maxima. □

Theorem 5 *The sets $C_n^{\sim_\ell}$, $n \geq 1$, are enumerated by the Catalan numbers (A000108 in [18]).*

Proof. Any permutation $\pi \in S_{n-1}$ can uniquely be decomposed as a product of transpositions

$$\pi = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_{n-1}, n-1\rangle$$

where p_i are some integers such that $1 \leq p_i \leq i \leq n-1$ (see for instance [1]).

Let ϕ be the map from S_{n-1} to S_n defined, for every $\pi \in S_{n-1}$, by

$$\phi(\pi) = \langle 1, 1\rangle \cdot \langle p_1, 2\rangle \cdots \langle p_{n-1}, n\rangle$$

where $\pi = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_{n-1}, n-1\rangle$.

Using Corollary 1 in [1], ϕ is a bijection from S_{n-1} to C_n satisfying $L(\pi) = L(\phi(\pi))$ for any $\pi \in S_n$ and such that $\phi(\pi)(k) = \pi(k) + 1$ for $k \in L(\pi)$. Therefore, ϕ induces a bijection from $S_{n-1}^{\sim_\ell}$ to $C_n^{\sim_\ell}$. With Theorem 1, the cardinality of $C_n^{\sim_\ell}$ is the $(n-1)$-th Catalan number. $\square$

Set	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$S_n^{\sim_\ell}$	Catalan	*A*000108	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$C_n^{\sim_\ell}$	Catalan	*A*000108	$1, 1, 2, 5, 14, 42, 132, 429, 1430$
$I_n^{\sim_\ell}$	Motzkin	*A*001006	$1, 2, 4, 9, 21, 51, 127, 323, 835$
$D_n^{\sim_\ell}$	Fine	*A*000957	$0, 1, 2, 6, 18, 57, 186, 622, 2120$
$Irr_n^{\sim_\ell}$	Dyck with no UDUD	*A*078481	$1, 1, 3, 7, 19, 53, 153, 453, 1367$

Table 1: Number of equivalence classes for classical subsets of permutations.

2.2 Equivalence classes for $S_n(\alpha)^{\sim_\ell}$ with $\alpha \in S_3$

In this part we give several enumerating results for the sets $S_n(\alpha)^{\sim_\ell}$ where the pattern α lies in S_3 (see Table 2).

Theorem 1 proves that $S_n(321)^{\sim_\ell}$ is enumerated by the nth Catalan number. Let ϕ be the bijection from $S_n(321)$ to $S_n(312)$ described (modulo a basic symmetry) in [3] (Lemma 4.3, page 148). It has the property to leave all left-to-right maxima fixed. Therefore, it induces a bijection from $S_n(321)^{\sim_\ell}$ to $S_n(312)^{\sim_\ell}$.

Now, let us examine the cases where the pattern α belongs to $\{123, 132, 213, 231\}$.

Theorem 6 *The sets $S_n(123)^{\sim_\ell}$, $n \geq 1$, are enumerated by the central polygonal numbers $1 + \frac{n(n-1)}{2}$ (A000124 in [18]).*

Proof. Let π be a permutation in $S_n(123)$. It is straightforward to see that the left-to-right maxima of π are 1 and i where $\pi_i = n$ for some i, $1 \leq i \leq n$. We necessarily have $i - 1 \leq \pi_1$ because the condition $\pi_1 < \pi_j < n$ implies $j < i$. Since the values i and j, $1 \leq i, j \leq n$, characterize a class in $S_n(123)^{\sim_\ell}$, it follows that the cardinality of $S_n(123)^{\sim_\ell}$ is given by $1 + \sum_{i=2}^{n} \sum_{j=i-1}^{n-1} 1 = 1 + \frac{n(n-1)}{2}$. $\square$

Theorem 7 *For $\alpha \in \{132, 213, 231\}$, the sets $S_n(\alpha)^{\sim_\ell}$, $n \geq 1$, are enumerated by the binary numbers 2^{n-1}.*

Proof. Let π be a permutation in $S_n(231)$. It can be written $\pi = \sigma n \gamma$ where $\sigma \in S_k(231)$ for some k, $0 \leq k \leq n-1$, and γ is obtained from a permutation in $S_{n-k-1}(231)$ by adding k on all these entries. Therefore, the set $L(\pi)$ of left-to-right maxima of π is the union of $\{k+1\}$ with the set $L(\sigma)$ of left-to-right maxima of σ. For $n \geq 1$, let a_n be the cardinality of $S_n(231)^{\sim_\ell}$. Varying k from 0 to $n-1$, we have $a_n = 1 + \sum_{k=1}^{n-1} a_k$ anchored with $a_1 = 1$. Thus, we deduce $a_n = 2^{n-1}$ for $n \geq 1$.

Basic symmetries on permutations allow to obtain the result whenever α lies in $\{132, 312\}$. □

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$\{123\}$	Central polygonal	*A*000124	$1, 2, 4, 7, 11, 16, 22, 29, 37$
$\{312\}, \{321\}$	Catalan	*A*000108	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$\{132\}, \{213\}, \{231\}$	Binary	*A*000079	$1, 2, 4, 8, 16, 32, 64, 128, 256$

Table 2: Number of equivalence classes for permutations avoiding one pattern in $\mathcal{S}_3$.

3 Enumeration of classes under d-equivalence relation

In this section two permutations π and σ belong to a same class whenever $DD(\pi) = DD(\sigma)$, *i.e.*, if the set of pairs (π_i, π_{i+1}) for $i \in D(\pi)$ is equal to the set of pairs (σ_i, σ_{i+1}) for $i \in D(\sigma)$.

A *partition* Π of $[n]$ is any collection of non-empty pairwise disjoint subsets, called *blocks*, whose union is $[n]$. The *standard form* of Π is $\Pi = B_1/B_2/\ldots$, where the blocks B_i are arranged so that their smallest elements are in increasing order. For convenience, we assume also that elements in a same block are arranged in decreasing order. From a permutation $\pi \in S_n$, we associate the unique partition Π defined as follows. Two elements $x > y$ belong to the same block in Π if and only if there exist i and j, $i < j$, such that $\pi_i = x > \pi_{i+1} > \cdots > \pi_{j-1} > \pi_j = y$. Conversely, any partition $\Pi = B_1/B_2/\ldots/B_k$, $k \geq 1$, is the associated to the permutation $B_1 B_2 \ldots B_k$. Theorem 8 becomes a straightforward consequence.

Theorem 8 *The sets $S_n^{\sim_d}$, $n \geq 1$, are enumerated by the Bell numbers (A000110 in [18]).*

Theorem 9 *The sets $S_n(321)^{\sim_d}$, $n \geq 1$, are enumerated by the Motzkin numbers (A001006 in [18]).*

Proof. Let $\pi \in S_n(321)$ and $DD(\pi) = \{(M_1, m_1), (M_2, m_2), \ldots, (M_r, m_r)\}$, $r \geq 0$, be the set of pairs (π_i, π_{i+1}) where i is a descent of π. Since π avoids 321, $DD(\pi)$ does not contain two pairs of the form (π_i, π_{i+1}) and (π_{i+1}, π_{i+2}). Then, we define the involution $\sigma \in I_n$ as follows: $\sigma(M_i) = m_i$, $\sigma(m_i) = M_i$ for $1 \leq i \leq r$ and $\sigma(k) = k$ if k does not appear in any pair of $DD(\pi)$. For a contradiction, let us assume that σ contains a pattern 4321. Then there exist two pairs (M_i, m_i) and (M_j, m_j) in $DD(\pi)$ such that $M_i > M_j > m_j > m_i$. If the descent (M_i, m_i) is on the left (in π) of the descent (M_j, m_j), then the subsequence $M_i M_j m_j$ is a pattern 321 of π; otherwise, the subsequence $M_j m_j m_i$ also is a 321-pattern. In the two cases we obtain a contradiction, which ensures that the involution σ avoids the pattern 4321.

Conversely, let σ be an involution avoiding the pattern 4321. There exists a sequence of pairs $(M_1, m_1), \ldots, (M_r, m_r)$, $r \geq 0$, such that $M_i < M_{i+1}$, $m_i < m_{i+1}$ for $i \leq r-1$ and such that $M_i > m_i$, $\sigma(M_i) = m_i$ and $\sigma(m_i) = M_i$ for $i \leq r$ and $\sigma(k) = k$ for $k \in [n] \backslash \{M_1, \ldots, M_r, m_1, \ldots, m_r\}$. We define the permutation π with the following process. We start with the sequence $M_1 m_1 M_2 m_2 \ldots M_r m_r$; we insert in increasing order all other values k satisfying $\sigma(k) = k$ as follows: if $m_{i-1} < k < m_i$ then we insert k between m_{i-1} and M_i; if $k < m_1$ then we insert k before M_1; and if $k \geq m_r$ then we insert k after m_r. Obviously, this construction induces that π avoids 321. Moreover, σ can be obtained from π by the construction of the beginning of this proof. Thus, there is a bijection between $S_n(321)^{\sim_d}$ and $I_n(4321)$ which is enumerated by the Motzkin numbers (see A001006 in [18] and [6]). □

Lemma 1 *Let π and π' be two permutations in $S_n(132)$ belonging to the same d-equivalence class. If we have $\pi_1 = \pi'_1$ then $\pi = \pi'$.*

Proof. We proceed by induction on n. A simple observation gives the result for $n \leq 3$. Now, let us assume that Lemma 1 is true for $k \leq n-1$. Let π and π' be two permutations in $S_n(132)$ such that $\pi_1 = \pi'_1$. We can write $\pi = \alpha n \beta$ (resp. $\pi' = \alpha' n \beta'$) where $\beta \in S_k(132)$ for some k, $0 \leq k \leq n-1$ (resp. $\beta' \in S_{k'}(132)$ for some k', $0 \leq k' \leq n-1$), and α (resp. α') is obtained from a permutation in $S_{n-k-1}(132)$ (resp. $S_{n-k'-1}(132)$) by adding k (resp. k') on all these entries. Let m (resp. m') be the minimal value of α (resp. α').

Without loss of generality, we assume that $m' \leq m \leq \pi_1 = \pi'_1$. For a contradiction, assume that $m' < m$. So, there is two consecutive entries a and m' in α' such that $a > m'$. As $m' < m$, the descent (a, m') does not appear in α. Thus, (a, m') appears in β. Let α_1 be the first value of α; the subsequence $\alpha_1 a m'$ is necessarily a pattern 132 which is a contradiction. Thus, we have $m = m'$ and then $k = k'$. We deduce α and α' (resp. β and β') are d-equivalent and the recurrence hypothesis gives $\alpha = \alpha'$. Moreover the descent (n, β_1) is equal to the descent (n, β'_1) and then $\beta_1 = \beta'_1$. Using the recurrence hypothesis we conclude $\beta = \beta'$ and then, $\pi = \pi'$. $\square$

Theorem 10 *The sets $S_n(132)^{\sim_d}$, $n \geq 1$, are enumerated by $c_n - c_{n-1} + 1$ where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.*

Proof. Let a_n be the cardinality of $S_n(132)^{\sim_d}$. We distinguish three kinds of classes: (1) classes with a representative π satisfying $\pi_1 = n$; (2) classes with a representative π satisfying $\pi_n = n$; (3) the remaining classes.
Case (1). Such a class contains a permutation π such that $\pi_1 = n$, *i.e.*, $\pi = n\pi'$ with $\pi' \in S_{n-1}(132)$. Using Lemma 1, there is a unique $\sigma = \pi' \in S_{n-1}(132)$ such that $n\sigma$ and π belong to the same class. Thus, the number of classes in this case is also the cardinality of $S_{n-1}(132)$, that is the $(n-1)$-th Catalan number c_{n-1}.
Case (2). Such a class contains a permutation π such that $\pi_n = n$. So, the number of classes in this case is also the number of elements in $S_{n-1}(132)^{\sim_d}$, that is a_{n-1}.
Case (3). Now we consider the classes that do not lie in the two previous cases. Any permutation π of such a class satisfies $\pi_i = n$ for some $i \in [2, n-1]$, and since π avoids 132, π can be written $\pi = \alpha n \beta$ where $\beta \in S_{n-i}(132)$ and α is obtained by adding $(n-i)$ on all entries of a permutation in $S_{i-1}(132)$.

Let us consider j, $1 \leq j \leq i-1$, the position where α reaches its minimum m.

If $j = 1$ then $\pi = (n-i+1)\dots(n-1)n\beta$ and this permutation lies in the same class of $n\beta(n-i+1)\dots(n-1)$ that satisfies Case (1). Then, $j = 1$ does not occur.

Now we assume $j \geq 2$ and let σ be a permutation in $S_n(132)$ lying in the same class of π. Then, σ must contain the two descents (n, π_{i+1}) and (π_{j-1}, m). These two descents necessarily appear in the same order as in π (otherwise, a pattern 132 would be created with $\pi_{i+1}\pi_{j-1}m$). Thus, the minimum m' of values on the left of n in σ is necessarily less or equal to m. For a contradiction, let us assume that $m' < m$. The value m' appears necessarily on the right of the descent (π_{j-1}, m) in σ (otherwise, $m'\pi_{j-1}m$ would be a pattern 132). Therefore, a descent of the form (a, b), $a \geq m$ and $b < m$ would necessarily exists in σ, which is not possible because such a descent cannot belong in π.

Thus, we deduce $m = m'$ and σ has the similar decomposition $\sigma = \alpha' n \beta'$ where $\beta' \in S_{n-i}(132)$ and α' is obtained by adding $(n-i)$ on all entries of some permutation in $S_{i-1}(132)$. So, α (resp. β) is equivalent to α' (resp. β'). Hence, Lemma 1 implies that $\beta = \beta'$. Then, for a given $i \in [2, n-1]$, there are exactly $c_{n-i} \cdot (a_{i-1} - 1)$ classes verifying this case (we subtract one to a_{i-1} because we do not consider $\pi = (n-i+1)\dots n\beta$).

So, such classes are enumerated by $\sum_{k=2}^{n-1}(a_{k-1}-1)\cdot c_{n-k}$.

Considering the three cases, the cardinality a_n of $S_n(132)^{\sim d}$ satisfies for $n \geq 2$,

$$a_n = c_{n-1} + a_{n-1} + \sum_{k=2}^{n-1}(a_{k-1}-1)\cdot c_{n-k}.$$

A simple calculation proves that $a_n = c_n - c_{n-1} + 1$ for $n \geq 2$. □

Theorem 11 *The sets $S_n(123)^{\sim d}$, $n \geq 2$, are enumerated by $c_n + n - (n+2)\cdot 2^{n-3} + \frac{(n-2)(n-1)}{2}$ where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.*

Proof. Let π be a permutation in $S_n(123)$. Then π has a unique decomposition into blocks of decreasing sequences, *i.e.*, $\pi = A_1A_2\ldots A_r$, $1 \leq i \leq r$, where blocks A_i consist of sequences of decreasing values (possibly reduced to one value) and such that $\ell_i < f_{i+1}$ for $1 \leq i \leq r-1$, where f_i (resp. ℓ_i) is the first (resp. last) element of A_i.

We distinguish three cases (1) $r = 1$; (2) $r = 2$; and (3) $r \geq 3$.

Case (1). We necessarily have $\pi = n(n-1)\ldots 21$ and its equivalence class contains only one element.

Case (2). We have $\pi = A_1A_2$ with $\ell_1 < f_2$. (i) If $f_1 < \ell_2$ then the class of π contains only one element since the permutation $\sigma = A_2A_1$ is not there. Therefore, there are $(n-1)$ such classes. (ii) If $f_1 > \ell_2$ then the class of π contains exactly two elements π and $\sigma = A_2A_1$. Since the number of permutations of length n with $n-2$ descents is $2^n - (n+1)$, the number of classes for the subcase (ii) is $\frac{2^n-(n+1)-(n-1)}{2} = 2^{n-1} - n$. Finally, there are $2^{n-1} - n + n - 1 = 2^{n-1} - 1$ classes for Case (2).

Case (3). π contains at least three blocks. We decompose $\pi = A_1BA_r$ where $B = A_2\ldots A_{r-1}$, $r \geq 3$, such that $f_2 = n$, $\ell_{r-1} = 1$ with A_1 and A_r possibly empty.

Let σ be a permutation in $S_n(123)$ belonging to the class of π. We will prove that σ is either π or A_rBA_1.

For this, we will prove that the block B also appears in σ. It is obvious whenever B is a decreasing sequence. Now, let us assume that B is the concatenation of at least two blocks, that is $B = A_2\ldots A_{r-1}$ with $r \geq 4$.

(i) For a contradiction, we suppose that there exist i and j, $2 \leq i < j \leq r-1$ such that A_i appears on the right of A_j into σ, *i.e.*, $\sigma = \alpha A_j\beta A_i\gamma$ for some α,β,γ possibly empty. Since σ avoids 123 and $f_j < f_i$, α does not contain any value a such that $a < f_j$ (otherwise af_jf_i would be a pattern 123). Also, α does not contain any value a such that $a > f_j$ (otherwise there would be $b \leq a$ such that (b, f_j) is descent in σ that does not appear in π). Thus α is necessarily empty. By a simple symmetry , γ is also empty.

This implies that all other blocks of π appear between the two blocks A_j and A_i in σ. Thus, all other blocks consist of values a such that $a \in [1, \ell_j - 1] \cup [f_i + 1, n]$ (otherwise $\ell_j a f_i$ would be a pattern 123). Since π contains at least three blocks there is at least one block between A_j and A_i in σ.

The case $\ell_j = 1$ does not occur. Indeed, this would mean that all blocks between A_j and A_i contain values x greater than f_i which creates a descent of the form (x, f_i) that does not appear in π. A similar argument proves that $f_i = n$ does not occur.

Let us consider the case where n and 1 do not appear in A_i and A_j. Since B contains at least two blocks, n and 1 do not appear in the same block in B. Let R (resp. S) be the block containing n (resp. 1). In σ, the last element $\ell(R)$ of R is necessarily less than ℓ_j (otherwise σ would contain a pattern 123, that is $\ell_j \ell(R) x$ where x is the value just after $\ell(R)$ in σ). A same argument shows that the first element $f(S)$ of S is greater than f_i. In π, this would mean that $\ell(R)\ell_i f(S)$ is a pattern 123 which is a contradiction. Finally, all blocks of B appear in σ in the same order as π.

(ii) Now we will prove that A_1 does not appear between A_2 and A_{r-1} in σ. For a contradiction, let us assume that A_1 appears between A_2 and A_{r-1} in σ. Let a be the value of σ just after the block A_1. If $a < \ell_1$ then σ contains the descent (ℓ_1, a) that does not appear in π; otherwise, $\ell_2 \ell_1 a$ would be a pattern 123 in σ which gives a contradiction. A same argument proves that A_r does not appear between A_2 and A_{r-1} in σ.

Therefore, we deduce that either $\sigma = \pi$ or $\sigma = A_r B A_1$.

Now we will enumerate permutations $\pi \in S_n(123)$ of Case (3) such that there is $\sigma \in S_n(123)$, $\sigma \neq \pi$, belonging to the same class of π, *i.e.*, $A_1 B A_r$ and $A_r B A_1$ do not contain any pattern 123. This case is characterized by the fact that there is no value a in the block B such that $\min\{\ell_1, \ell_r\} < a < \max\{f_1, f_r\}$ (otherwise, one of the two permutations $A_1 B A_r$, $A_r B A_1$ would contain the pattern 123). See Figure 1 for a graphical representation of such a permutation.

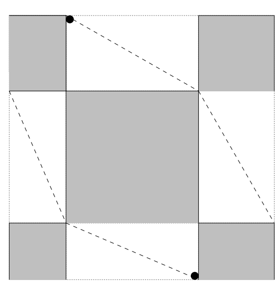

Figure 2: Illustration of $\pi = A_1 B A_r \in S_n(123)$ having two elements in its class.

If B contains only one block, A_1 and A_r are non-empty blocks. Varying the size k of B from 2 to $n-2$, and the size ℓ of A_1 from 1 to $n-k-1$, the number of permutations having two elements in its class is $a_n = \sum_{k=2}^{n-2}(k-1)\cdot\sum_{\ell=1}^{n-k-1}\binom{n-k}{\ell}$.

If B contains at least two blocks, A_1 and A_r are blocks (possibly empty). Varying the size k of B from 4 to $n-1$ and varying the size ℓ of A_1 from 0 to $n-k$, the number of permutations having two elements in its class is $b_n = \sum_{k=4}^{n-1}(2^{k-2}-(k-1))\cdot\sum_{\ell=0}^{n-k}\binom{n-k}{\ell}$.

Finally, the number of classes in $S_n(123)$ is obtained from c_n by subtracting the number of classes having two elements, that is $c_n - \frac{1}{2}(2^n - 2n + a_n + b_n) = c_n - (n+2)\cdot 2^{n-3} + \frac{n(n-1)}{2} + 1$. $\square$

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$\{\}$	Bell	$A000110$	$1, 2, 5, 15, 52, 203, 877, 4140, 21147$
$\{231\}, \{312\}$	Catalan	$A000108$	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$\{321\}$	Motzkin	$A001006$	$1, 2, 4, 9, 21, 51, 127, 323, 835$
$\{132\}, \{213\}$	$c_n - c_{n-1} + 1$	New	$1, 2, 4, 10, 29, 91, 298, 1002, 3433$
$\{123\}$	$c_n - (n+2)\cdot 2^{n-3} + \frac{n(n-1)}{2} + 1$	New	$1, 2, 4, 9, 25, 84, 307, 1139, 4195$

Table 3: Number of equivalence classes for permutations avoiding at most one pattern of $\mathcal{S}_3$.

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